Practice Problem Set 1: Review (sets, sequences, etc.), Counting

Material covered: Lectures of weeks 2 and 3
Reading: Schaum's Chapters 1-2, LLM Chapter 15 (Cardinality Rules).

Problems from Schaums. Pick a subset of these problems (try to cover all the different topics, with an emphasis on the topics you find more difficult): 1.47 – 1.50, 1.54 – 1.57, 1.59, 1.62, 1.63, 1.68, 1.69, 1.71, 1.74, 1.78, 2.41, 2.45 – 2.50, 2.51 – 2.56, 2.59 – 2.68, 2.69 – 2.74.

Exercise 1. Four men (Andre, Justin, Usain, Yohan) and three women (Elaine, Tori, Shelly-Ann) run a race. How many possible outcomes to the race can have:

1. A man in the first place, a man in the second place, and a woman in the third place?
2. The women finishing before all the men?
3. Exactly one woman finishing in top three?
4. At least one woman finishing in the top three?
5. Elaine finishing before Usain?

Exercise 2. An independent dormitory has 11 residents. Each resident is assigned exactly one task, such that: 3 wash pots, 2 clean the kitchen, 3 clean the bathrooms, 1 cleans the common room, and 2 cook dinner. In how many ways can this be done? (Hint. Use the bookkeeper rule!)

Exercise 3. Two identical decks of cards are mixed together. In how many ways can the cards in this double deck be arranged?

Exercise 4. Here are some letters: T E R G E G S Z. How may possible ways can you order these letters while making sure that the ordered sequence of letters does not contain the words: TREE, TREES, EGG, EGGS.

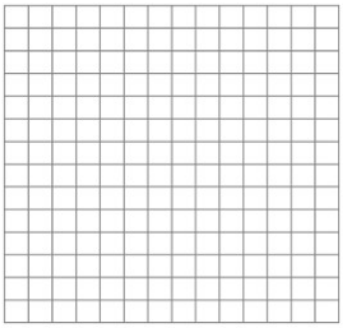
(For example, TREEGGSZ is not a good ordering, since the first four letters spell out the word "TREE" and the fourth-to-seventh letters spell out the word EGGS. Meanwhile, GTREZEGS is a good ordering.)

Exercise 5. A hash function h maps bit strings of length $n = 2000$ to bit strings of length $m = 120$. That is $h : \{0,1\}^n \to \{0,1\}^m$.

- How many bit strings of length n can we have? Of length m?
- Prove that a *collision* must exist, i.e. that there are at least two inputs x, x' such that $x \neq x'$ but $h(x) = h(x')$.
- Define the size of a collision as the number of inputs that map to the same output (e.g. the collision with x, x' has size 2.) Prove that there is at least one collision of size 100.

(Hint: pigeons.)

Exercise 6.



Consider the 14x14 grid shown here. You need to walk from the right-bottom-most corner of the grid to the top left-most corner of grid. Each "step" in your walk involves either moving one unit to the left, OR one unit to the up. You can step LEFT or UP in any order you want.

1. How many steps are in a particular walk?
2. How many possible walks can you take? (Hint: Try to map this problem to counting bit-strings.)
3. How many possible walks can you take if the grid is 10x15?
4. How many possible walks can you take if the grid is still 14x14 but now you are also allowed to take AT MOST 2 steps DOWN?