Practice Problem Set 2: Basic Probability

Reading: Schaum's Chapter 3, LLM Chapter 14 (Events and Probability Spaces).

Optional Extra Practice. Any of the problems in Schaums's Chapter 3, but especially 3.6-3.15 and 3.41-3.55 and 3.64.

Exercise 1. The axioms of probability are as follows:

Let S be a sample space (*i.e.,* the set of all possible outcomes), and let $\mathcal{F}$ be the class of all events (where an event from $\mathcal{F}$ is a set of outcomes in S). The Pr is probability function if

1. For all events $A \in \mathcal{F}$, then $\Pr[A] \geq 0$
2. $\Pr[S] = 1$
3. For disjoint events A, B (*i.e.,* $A \cap B = \emptyset$), then $\Pr[A \cup B] = \Pr[A] + \Pr[B]$.

Using these axioms and elementary set theory, prove that:

1. For every event $A \in F$, it follows that $\Pr[A] \leq 1$.
2. For any events $A, B \in F$, it follows that $\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$.
3. Let events $A_i \in \mathcal{F}$ for $i = 1...n$ be **disjoint** events. (That is, $A_i \cap A_j = \emptyset$ for every $i \neq j$.)
   (a) Let event $D = \cup_{i=1}^{n} A_i$. (Notice that the A_i form a *partition* of D.) Prove that
   $$\Pr[D] = \sum_{i=1}^{n} \Pr[A_i]$$
   (b) Prove that $\sum_{i=1}^{n} \Pr[A_i] \leq 1$.
4. For events $A_i \in F$ for $i = 1...n$, and event $D = \cup_{i=1}^{n} A_i$, prove that
   $$\Pr[D] \leq \sum_{i=1}^{n} \Pr[A_i]$$
   (This is called the "union bound").

Exercise 2. A bin contains 10 white balls and 8 black balls. We draw 6 balls at random. Find the probability that among the 6 balls we draw, we have exactly 4 white ones.

Exercise 3. You and your friend are among 10 people sitting at a round table. What is the probability that you sit next to each other?

Exercise 4. Twenty people, numbered from {1,2,3,..,20} have to form two teams, each containing 10 people. People are randomly arranged into teams.

1. What is the probability that one of the teams consists of players 1,2,3,4,5,6,7,8,9, and 10?

2. What is the probability that players 1, 2, and 3 are on the same team?

3. What is the probability that both players 1 and 2 are not on the same team as players 3 and 4?

Exercise 5. We play a game with three weird, but fair dice. (That is, each die lands on any of its six faces with probability $\frac{1}{6}$.) The game goes like this: I choose a die. You choose a die. We roll. The person who rolls the higher number wins.

- The first die has faces numbered $1, 1, 5, 5, 6, 6$.
- The second die has faces numbered $2, 2, 7, 7, 8, 8$.
- The third die has faces numbered $3, 3, 3, 9, 9, 9$.

I choose the first die. Which die should you choose to roll so that you maximize your probability of beating me?

Exercise 6. The US senate consists of 2 senators from each of 50 states. In one session, there are 50 senators in the room. What is the probability that

- One of the MA senators is in the room?
- All states are represented in the room?