Practice Problem Set 7: Continuous Random Variables and Distributions

Reading: The textbooks do not cover the material, thus your class notes are the definite reference. Chapter 8 in MU provides a good introduction to continuous random variables and distributions, but it does not cover all the material we saw in class. Chapter 6 in Schaum's has an introduction to the Normal distribution, including a table with CDF values for the standard Normal distribution.

Optional extra practice. Solve some of the exercises on the Normal distribution in Schaum's Chapter 6.

Exercise 1. Alice goes to the post office to mail a letter, and she is equally likely to find either 0 or 1 customers ahead of her. The service time for the customer ahead, if present, is exponentially distributed with parameter λ. Find the CDF of Alice's waiting time.

Exercise 2. Bob's favorite programming language has a library that provides a subroutine **UniformSample** that returns a value sampled uniformly at random from the interval $(0, 1)$. Bob decides to use this subroutine to sample values according to his favorite distribution as follows.

Bob's favorite distribution has a continuous CDF F that is strictly increasing over the values $\{x : 0 < F(x) < 1\}$. Thus for every value $p \in (0, 1)$, there is a unique x that satisfies $F(x) = u$. Bob uses the following algorithm to sample a value x: call **UniformSample** to obtain a probability $p \in (0, 1)$, and return the value x such that $F(x) = p$.

(a) Show that Bob's algorithm is correct. That is, let X be a random variable equal to the value returned by Bob's algorithm. Show that the CDF of X is F.

(b) Describe how to implement Bob's algorithm to sample a value from the distribution Exponential(λ).

Exercise 3. Let X be a random variable with the following PDF:

$$f_X(x) = \begin{cases} p\lambda e^{-\lambda x} & \text{if } x \geq 0 \\ (1-p)\lambda e^{\lambda x} & \text{if } x < 0 \end{cases}$$

where λ and p are constants such that $\lambda > 0$ and $p \in [0, 1]$. Find the expectation and variance of X in two different ways:

(a) Use straightforward calculation starting from the definitions of expectation and variance.

(b) Use the law of total expectation together with the expectation and variance of the Exponential distribution that we found in class.

Exercise 4. Charlie is a dedicated Easter egg hunter, and this year he decided to use the following algorithm for his egg hunting bonanza. Charlie first chooses a number $N > 0$ according to his favorite discrete distribution. He then searches for eggs in N locations. In each location i, he spends T_i time looking for eggs, where every T_i is distributed according to his favorite continuous distribution with expectation μ and variance σ^2. Let $T = \sum_{i=1}^{N} T_i$ denote the total time that Charlie spends looking for eggs. Show that, if N and all the T_i's are mutually independent, then

$$\mathbf{Ex}(T) = \mu\, \mathbf{Ex}(N) \text{ and } \mathbf{Var}(T) = \sigma^2\, \mathbf{Ex}(N) + \mu^2\, \mathbf{Var}(N).$$

Exercise 5. Let $X \sim \text{Normal}(0, 1)$ and $Y \sim \text{Normal}(1, 4)$.

(*a*) Find $\Pr(X \leq 1.5)$, $\Pr(X \leq -1)$, and $\Pr(-1 \leq Y \leq 1)$.

(*b*) Find the PDF of the random variable $\frac{1}{2}(Y-1)$.

Exercise 6. The temperature of ProbabilityVille can be modeled as a normal random variable with mean and standard deviation both equal to 50 degrees Fahrenheit. Find the probability that the temperature at a randomly chosen time is at most 59 degrees Fahrenheit.