Practice Problem Set 8: Normal Distribution, Central Limit Theorem

Reading: See notes on the normal distribution, and the normal approximation of the binomial distribution in Schaum's 6.3-6.6. See also the class notes:
`http://cs-people.bu.edu/aene/cs237sp17/handouts/intro-statistics.pdf`

Schaum's Outlines problems. 6.21-6.36, 6.67-6.77.

Exercise 1. (Easy problem.) Let X be normally distributed with mean 10 and variance 400. Determine $\Pr[0 \leq X \leq 40]$ using the Normal table[1].

Exercise 2. A random number generator outputs numbers in the set $\{-1, 0, 1\}$ independently and randomly so that the output 1 comes up with probability $\frac{3}{7}$, -1 comes up with probability $\frac{3}{7}$, and 0 comes up with probability $\frac{1}{7}$. Let X_i be the i^{th} number you get from the random number generator, and let $\overline{X}_n$ be the sample mean of the random numbers you get, *i.e.,*

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

1. What is $\mathbf{Ex}(X_i)$ for each i?
2. What is $\mathbf{Var}(X_i)$ for each i?
3. What is $\mathbf{Ex}(\overline{X}_n)$?
4. What is $\mathbf{Var}(\overline{X}_n)$?
5. Let $n = 2$. What is $\Pr(\overline{X}_n \leq 0)$?
6. Let $n = 100$. Use the central limit theorem (Theorem 6.4 in Schaum's, see also class notes) to estimate $\Pr(\overline{X}_n \leq \frac{1}{n})$.

Exercise 3. We transmit binary sequences over a communication channel. There is noise that interferes with the communication; thus, each bit is received correctly with probability 0.55.

To mitigate the noise, each bit is sent n times, where n is an odd number. (This is a trivial kind of error correction code called a "repetition code".) We decode each 'bit' by taking the majority of the n received bits. For example, if $n = 5$ and we receive the message 01010 we decode the bit as '0', since we see 3 zeros and 2 ones.

Use a normal distribution to estimate how large n should be so that we correctly decode each bit with probability 0.99.

Exercise 4. We flip a fair coin n times. The $Z_i = 1$ if the outcome of the i^{th} coin toss is heads, and 0 otherwise. Let $Z = \frac{1}{n} \sum_{i=1}^{n} Z_i$ be the relative frequency of the appearance of heads.

Use a normal distribution to estimate how large n should be such that Z is within ± 0.02 of $\mathbf{Ex}(Z)$ with probability 0.95.

[1] `http://www.normaltable.com/`