CAS CS 131 - Combinatorial Structures

Spring 2011

Problem Set #3 (Sums)
Out: Thursday, February 17
Due: Thursday, February 24

NO LATE SUBMISSIONS WILL BE ACCEPTED

To be completed individually.

1. You had previously proved by induction that, for any natural number $n \geq 1$, $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

   Use this result to evaluate the following sum.

$$\sum_{k=n+1}^{2n} k^2$$

2. You had previously proved by induction that, for all real values $r \neq 1$: $1 + r + r^2 + r^3 + \cdots + r^n = \frac{1-r^{n+1}}{1-r}$, for any natural number $n \geq 0$.

   Use this result to evaluate the following sum.

$$\sum_{i=0}^{n} \sum_{j=0}^{m} 3^{i+j}$$

3. Tommy the Monster is a financial service provider who offers loans on the following terms.

   - Tommy loans a client m dollars in the morning. This puts the client m dollars in debt to Tommy.
   - Each evening, Tommy first charges a "service fee", which increases the client's debt by f dollars, and then Tommy charges interest, which multiplies the debt by a factor of p. For example, if Tommy's interest rate were a modest 5% per day, then p would be 1.05.

   (a) What is the client's debt at the end of the first day?
   (b) What is the client's debt at the end of the second day?
   (c) Write a formula for the client's debt after d days and find an equivalent closed form.

4. You have seen this so-called "perturbation" method to evaluate a geometric sum:

$$\begin{aligned} S &= 1 + z + z^2 + \cdots + z^n \\ zS &= z + z^2 + \cdots + z^n + z^{n+1} \\ S - zS &= 1 - z^{n+1} \\ S &= \frac{1 - z^{n+1}}{1 - z} \end{aligned}$$

Use the same approach to find a closed-form expression for this sum:

$$T = z + 2z^2 + 3z^3 + \cdots + nz^n$$

5. Find a closed-form expression equal to the following sum. Show your work.

$$\sum_{j=1}^{n} \sum_{i=0}^{\infty} j^{5/3} \left(1 - \frac{1}{2j^{1/3}}\right)^i$$