

CAS CS 131 - Combinatorial Structures

Spring 2011

SOME REVIEW PROBLEMS FOR EXAM 1

OUT: TUESDAY, FEBRUARY 8

1. A self-proclaimed “great logician” has invented a new quantifier, on par with $\exists$ (“there exists”) and $\forall$ (“for all”). The new quantifier is symbolized by U and read “there exists a unique”. The proposition $UxP(x)$ is true iff there is *exactly one* x for which $P(x)$ is true. The logician has noted, “There used to be two quantifiers, but now there are three! I have extended the whole field of mathematics by 50%!”

- (a) Write a proposition equivalent to $UxP(x)$ using only the $\exists$ quantifier, $=$, and logical connectives.
- (b) Write a proposition equivalent to $UxP(x)$ using only the $\forall$ quantifier, $=$, and logical connectives.

2. A media tycoon has an idea for an all-news television network called LNN: The Logic News Network. Each segment will begin with the definition of some relevant sets and predicates. The day’s happenings can then be communicated concisely in logic notation. For example, a broadcast might begin as follows:

THIS IS LNN. Let S be the set $\{\text{Bill, Monica, Ken, Linda, Betty}\}$. Let $D(x)$ be a predicate that is true if x is deceitful. Let $L(x, y)$ be a predicate that is true if x likes y . Let $G(x, y)$ be a predicate that is true if x gave gifts to y .

Complete the broadcast by translating the following statements into logic notation.

- (a) If neither Monica nor Linda is deceitful, then Bill and Monica like each other.
 - (b) Everyone except for Ken likes Betty, and no one except Linda likes Ken.
 - (c) If Ken is not deceitful, then Bill gave gifts to Monica, and Monica gave gifts to someone.
 - (d) Everyone likes someone and dislikes someone else.
3. Write down the negation of each of the following statements in clear and concise English. Do not use the expression “It is not the case” in your answers.
 - (a) Either $a^2 > 0$ or a is not a real number.
 - (b) For every real number x , there is an integer n such that $n > x$.

4. Determine whether or not each of the following arguments is valid. Support your answer with a proof.

If I stay up late at night,
then I will be tired in the morning.

I am tired this morning.

I stayed up late last night.

5. Let n be a positive integer ($n \geq 1$). Prove that $\log_2 n$ is rational if and only if n is a power of 2.
6. Let x and y be nonnegative real numbers. The *arithmetic mean* of x and y is defined to be $(x + y)/2$, and the *geometric mean* is defined to be $\sqrt{xy}$. Prove that the arithmetic mean is equal to the geometric mean if and only if $x = y$.
7. Prove by induction that for all positive integers, $n \geq 1$:

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n(n + 1) = \frac{n(n+1)(n+2)}{3}.$$
8. Use induction to prove that for all real values $r \neq 1$:

$$1 + r + r^2 + r^3 + \cdots + r^n = \frac{1-r^{n+1}}{1-r}$$
9. (a) Prove by mathematical induction that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for any positive integer $n \geq 1$.
 (b) Prove by mathematical induction that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$ for any positive integer $n \geq 1$.
 (c) Use the results of (a) and (b) to establish that $(1 + 2 + 3 + \cdots + n)^2 = 1^3 + 2^3 + 3^3 + \cdots + n^3$ for every positive integer $n \geq 1$.
10. Prove that, for any natural number $n \geq 1$, $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.
11. Prove that $n! > 2^n$ for all $n \geq 4$. (Recall $n! = n(n-1)(n-2) \cdots \times 3 \times 2 \times 1$.)
12. Prove that it is possible to fill an order for $n \geq 32$ pounds of fish given bottomless wheelbarrows full of 5-pound and 9-pound fish.
13. Use strong induction to prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.
14. Suppose we want to divide a class of n students into groups each containing either 4 or 5 students. Provide a strong induction proof that a class with $n \geq 12$ students can be divided into groups of 4 or 5.
15. The Fibonacci numbers are defined as follows:

$F_1 = 1, F_2 = 1$, and for all $k \geq 3, F_k = F_{k-1} + F_{k-2}$.

The first few terms of the Fibonacci sequence are:

1, 1, 2, 3, 5, 8, 13, 21, $\cdots$

We can't find every single number in the Fibonacci sequence – for instance, 4 is not a number in the sequence. But can we express every $n \geq 1$ as the sum of distinct terms in the Fibonacci sequence? Indeed, we can!

Use strong induction to prove the following:

Theorem. Every $n \geq 1$ can be expressed as the sum of distinct terms in the Fibonacci sequence.

16. Consider a variation of the Unstacking game demonstrated in lecture. As before, the player is presented with a stack of $n \geq 1$ bricks. Through a sequence of moves, she must reduce this to n single-brick stacks while scoring as many points as possible. A move consists of dividing a single stack of $(a + b)$ bricks (where $a, b > 0$) into two stacks with heights a and b . Suppose that this move is worth $a + b$ points. Find the best strategy and use strong induction to prove that there is no better strategy.