Problem Set #4 (Asymptotic Notation and Recurrences)  
Out: Thursday, March 1  
Due: Thursday, March 8  

NO LATE SUBMISSIONS WILL BE ACCEPTED  

To be completed individually.

1. Find a closed-form expression equal to the following product. Show your work.
\[ \prod_{i=1}^{n} 2^{i^2 + i} \]

2. Find a closed-form expression equal to the following summation. Show your work.
\[ \sum_{i=3}^{\infty} \left( -\frac{1}{2} \right)^{i} \]

3. Prove that the following relationships hold.
   (a) \( 3x^2 + 2x + 1 = O(x^2) \).
   (b) \( \ln x = O(\log_b x) \), where the base \( b \) is any positive real number other than 1.

4. Prove that if \( f(x) = O(g(x)) \), and \( g(x) = O(h(x)) \), then \( f(x) = O(h(x)) \).

5. Show that the following asymptotic relationships hold:
   (a) \( 2n + \log n = \Theta(n) \)
   (b) \( n^2 = o(1.01^n) \)
   (c) \( \log n = o(n) \)
   (d) \( 2^{n/2} = o(2^n) \)
   (e) \( \frac{n^2 + 2n - 3}{n^2 - n} \sim 1 \)
   (f) \( \sum_{i=0}^{n} 2^{2i+1} = \Theta(4^n) \).

6. Solve the recurrence equation \( a_n = a_{n-1} + 2^n \) for \( n \geq 1 \), given \( a_0 = 5 \).

7. Suppose we modify the traditional rules for the Towers of Hanoi Problem (that we have seen in class) by requiring that one moves discs only to an adjacent peg.
   (a) Solve the puzzle when there are \( n = 2 \) discs and show your moves in a diagram of three columns representing the three pegs, and discs numbered 1, 2, \( \cdots \) representing smallest to larger discs.
(b) Give a recurrence relation for $T_n$, the number of moves required to transfer $n$ discs from one peg to another.

(c) Find an explicit formula for $T_n$.

(d) Suppose we can move a disc a second. Estimate the time required to transfer the discs if $n = 8, 16, 32, 64$. 