CAS CS 131 - Combinatorial Structures

Spring 2012

Some Review Problems for Exam 1
Out: Tuesday, February 7

1. A self-proclaimed "great logician" has invented a new quantifier, on par with $\exists$ ("there exists") and $\forall$ ("for all"). The new quantifier is symbolized by U and read "there exists a unique". The proposition $UxP(x)$ is true iff there is *exactly one* x for which $P(x)$ is true. The logician has noted, "There used to be two quantifiers, but now there are three! I have extended the whole field of mathematics by 50%!"
   (a) Write a proposition equivalent to $UxP(x)$ using only the $\exists$ quantifier, =, and logical connectives.
   (b) Write a proposition equivalent to $UxP(x)$ using only the $\forall$ quantifier, =, and logical connectives.
2. Prove by induction that for all positive integers, $n \geq 1$:
   $1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$.
3. Use induction to prove that for all real values $r \neq 1$:
   $1 + r + r^2 + r^3 + \cdots + r^n = \frac{1-r^{n+1}}{1-r}$
4. (a) Prove by mathematical induction that $1+2+3+\cdots+n = \frac{n(n+1)}{2}$ for any positive integer $n \geq 1$.
   (b) Prove by mathematical induction that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$ for any positive integer $n \geq 1$.
   (c) Use the results of (a) and (b) to establish that $(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3$ for every positive integer $n \geq 1$.
5. Let n be a positive integer ($n \geq 1$). Prove that $\log_2 n$ is rational if and only if n is a power of 2.
6. Let x and y be nonnegative real numbers. The *arithmetic mean* of x and y is defined to be $(x+y)/2$, and the *geometric mean* is defined to be $\sqrt{xy}$. Prove that the arithmetic mean is equal to the geometric mean if and only if $x = y$.
7. Prove that, for any natural number $n \geq 1$, $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.
8. Prove that $n! > 2^n$ for all $n \geq 4$. (Recall $n! = n(n-1)(n-2)\cdots \times 3 \times 2 \times 1$.)
9. Prove that it is possible to fill an order for $n \geq 32$ pounds of fish given bottomless wheelbarrows full of 5-pound and 9-pound fish.
10. The Fibonacci numbers are defined as follows:
    $F_1 = 1, F_2 = 1$, and for all $k \geq 3, F_k = F_{k-1} + F_{k-2}$.

The first few terms of the Fibonacci sequence are:

$1, 1, 2, 3, 5, 8, 13, 21, \cdots$

We can't find every single number in the Fibonacci sequence – for instance, 4 is not a number in the sequence. But can we express every $n \geq 1$ as the sum of distinct terms in the Fibonacci sequence? Indeed, we can!

Use strong induction to prove the following:

Theorem. Every $n \geq 1$ can be expressed as the sum of distinct terms in the Fibonacci sequence.

11. Consider a variation of the Unstacking game demonstrated in lecture. As before, the player is presented with a stack of $n \geq 1$ bricks. Through a sequence of moves, she must reduce this to n single-brick stacks while scoring as many points as possible. A move consists of dividing a single stack of $(a + b)$ bricks (where $a, b > 0$) into two stacks with heights a and b. Suppose that this move is worth $a + b$ points. Find the best strategy and use strong induction to prove that there is no better strategy.