CAS CS 131 - Combinatorial Structures

Spring 2013

Problem Set #6 (Summations and Asymptotic Notation)
Out: Monday, March 25
Due: Tuesday, April 2

NO LATE SUBMISSIONS WILL BE ACCEPTED

To be completed individually.

1. You had previously proved by induction that, for any natural number $n \geq 1$, $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.
   Use this result to evaluate the following sum.

$$\sum_{k=n+1}^{2n} k^2$$

2. You had previously proved by induction that, for all real values $r \neq 1$: $1 + r + r^2 + r^3 + \cdots + r^n = \frac{1-r^{n+1}}{1-r}$, for any natural number $n \geq 0$.
   Use this result to evaluate the following sum.

$$\sum_{i=0}^{n} \sum_{j=0}^{m} 3^{i+j}$$

3. Find a closed-form expression equal to the following product. Show your work.

$$\prod_{i=1}^{n} 2^{i^2+i}$$

4. Find a closed-form expression equal to the following summation. Show your work.

$$\sum_{i=3}^{\infty} (\frac{-1}{2})^i$$

5. You have seen this so-called "perturbation" method to evaluate a geometric sum:

$$\begin{aligned} S &= 1 + z + z^2 + \cdots + z^n \\ zS &= z + z^2 + \cdots + z^n + z^{n+1} \\ S - zS &= 1 - z^{n+1} \\ S &= \frac{1 - z^{n+1}}{1 - z} \end{aligned}$$

   Use the same approach to find a closed-form expression for this sum:
   $T = z + 2z^2 + 3z^3 + \cdots + nz^n$

6. Prove that the following relationships hold.

   (a) $3x^2 + 2x + 1 = O(x^2)$.

   (b) $\ln x = O(\log_b x)$, where the base b is any positive real number other than 1.

7. Prove that if $f(x) = O(g(x))$, and $g(x) = O(h(x))$, then $f(x) = O(h(x))$.

8. Show that the following asymptotic relationships hold:

   (a) $2n + \log n = \Theta(n)$

   (b) $n^2 = o(1.01^n)$

   (c) $\log n = o(n)$

   (d) $2^{n/2} = o(2^n)$

   (e) $\frac{n^2+2n-3}{n^2-7} \sim 1$

   (f) $\sum_{i=0}^{n} 2^{2i+1} = \Theta(4^n)$.