CAS CS 131 - Combinatorial Structures

Spring 2013

Problem Set #9 (More Counting)
Out: Tuesday, April 23
Due: Tuesday, April 30

NO LATE SUBMISSIONS WILL BE ACCEPTED

To be completed individually.

1. In the last problem set, you were asked to find an appropriate bijective mapping between a set of sequences and each of the following sets in question. This time, we are interested to count as well! So first do the mapping to a set of sequences and then count all the sequences in the set.

   (a) In how many ways can k elements be chosen from an $n-$element set $\{x_1, x_2, \cdots, x_n\}$?

   (b) How many different ways are there to select a dozen doughnuts if five varieties are available?

   (c) How many different solutions over the natural numbers are there to the equation: $x_1 + x_2 + x_3 + \cdots + x_8 = 90$? A solution is a specification of the value of each variable x_i. Two solutions are different if different values are specified for some variable x_i.

   (d) An electronic toy displays a 3×3 grid of colored squares. At all times, three are red, three are green, and three are yellow. How many such configurations are possible?

2. In how many different ways we can place two pawns and a rook on a chess board such that no two pieces share a row or a column?

3. From 150 used cars sitting on a lot, 20 are to be selected for a test designed to check certain safety requirements. These cars will then be put back onto the lot and, again, 20 will be selected for a test designed to check antipollution standards.

   (a) In how many ways can the first selection be made?

   (b) In how many ways can the second selection be made?

   (c) In how many ways can both selections be made?

   (d) In how many ways can both selections be made if *exactly* eight cars are to undergo both tests?

4. Find the answer to the following questions:

   - In how many different ways can n people stand in a waiting line?
   - In how many different ways can n can sit around a table?
   - In how many different ways can n people sit around a table with $n+1$ seats? (Of course they have to leave one seat empty).
   - In how many different ways can n people sit around a table with $n+2$ seats? (Of course they have to leave two seats empty).

5. You want to choose a team of m people from a pool of n people for your startup company, and from these m people you want to choose k to be the team managers. You took cs131, so you know you can do this in

   $$\binom{n}{m}\binom{m}{k}$$

   ways. But your CFO, who went to Harvard Business School, comes up with the formula

   $$\binom{n}{k}\binom{n-k}{m-k}$$

   Before doing the reasonable thing–dump on your CFO or Harvard Business School–you decide to check his answer against yours.

   (a) Start by giving an *algebraic* proof that your CFO's formula agrees with yours.

   (b) Now give a *combinatorial* argument proving this same fact.

6. Prove that $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$.

   (a) First, Give a *combinatorial* argument proving that the total number of subsets of set $A = \{x_1, x_2, x_3, \cdots, x_n\}$ is 2^n.

   (b) Then, Give another *combinatorial* argument proving that the total number of subsets of set A is also $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$.

   (c) Use parts (a) and (b) to prove that $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$.