Link Analysis Ranking
How do search engines decide how to rank your query results?

• Guess why Google ranks the query results the way it does

• How would you do it?
Naïve ranking of query results

• Given query $q$
• Rank the web pages $p$ in the index based on $\text{sim}(p,q)$

• Scenarios where this is not such a good idea?
Why Link Analysis?

• First generation search engines
  – view documents as flat text files
  – could not cope with size, spamming, user needs
    • Example: Honda website, keywords: automobile manufacturer

• Second generation search engines
  – Ranking becomes critical
  – use of Web specific data: Link Analysis
  – shift from relevance to authoritativenssness
  – a success story for the network analysis
Link Analysis: Intuition

• A link from page $p$ to page $q$ denotes endorsement
  – page $p$ considers page $q$ an authority on a subject
  – mine the web graph of recommendations
  – assign an authority value to every page
**Link Analysis Ranking Algorithms**

- Start with a collection of web pages
- Extract the underlying hyperlink graph
- Run the LAR algorithm on the graph
- Output: an *authority weight* for each node
Algorithm input

- **Query dependent**: rank a small subset of pages related to a specific query
  - HITS (Kleinberg 98) was proposed as query dependent

- **Query independent**: rank the whole Web
  - PageRank (Brin and Page 98) was proposed as query independent
Query-dependent LAR

• Given a query $q$, find a subset of web pages $S$ that are related to $S$
• Rank the pages in $S$ based on some ranking criterion
Query-dependent input

Root Set
Query-dependent input
Query dependent input

IN
Root Set
OUT
Query dependent input

Base Set

IN

Root Set

OUT
Properties of a good seed set $S$

- $S$ is relatively small.
- $S$ is rich in relevant pages.
- $S$ contains most (or many) of the strongest authorities.
How to construct a good seed set $S$

- For query $q$ first collect the $t$ highest-ranked pages for $q$ from a text-based search engine to form set $\Gamma$
- $S = \Gamma$
- Add to $S$ all the pages pointing to $\Gamma$
- Add to $S$ all the pages that pages from $\Gamma$ point to
Link Filtering

• Navigational links: serve the purpose of moving within a site (or to related sites)
  • www.espn.com → www.espn.com/nba
  • www.yahoo.com → www.yahoo.it
  • www.espn.com → www.msn.com

• Filter out navigational links
  – same domain name
  – same IP address
How do we rank the pages in seed set $S$?

- In degree?
- Intuition
- Problems
Hubs and Authorities [K98]

- Authority is not necessarily transferred directly between authorities
- Pages have double identity
  - hub identity
  - authority identity
- Good hubs point to good authorities
- Good authorities are pointed by good hubs
HITS Algorithm

• Initialize all weights to 1.
• Repeat until convergence
  – $O$ operation: hubs collect the weight of the authorities
    \[ h_i = \sum_{j:i \rightarrow j} a_j \]
  – $I$ operation: authorities collect the weight of the hubs
    \[ a_i = \sum_{j:j \rightarrow i} h_j \]
  – Normalize weights under some norm
HITS and eigenvectors

- The HITS algorithm is a power-method eigenvector computation
  - in vector terms $a^t = A^T h^{t-1}$ and $h^t = A a^{t-1}$
  - so $a^t = A^T A a^{t-1}$ and $h^t = A A^T h^{t-1}$
  - The authority weight vector $a$ is the eigenvector of $A^T A$ and the hub weight vector $h$ is the eigenvector of $A A^T$
  - Why do we need normalization?

- The vectors $a$ and $h$ are singular vectors of the matrix $A$
Singular Value Decomposition

\[ A = U \Sigma V^T = [\tilde{u}_1 \quad \tilde{u}_2 \quad \cdots \quad \tilde{u}_r] \begin{bmatrix} \sigma_1 & \ & \ & \ \\ & \sigma_2 & \ & \ \\ & & \ddots & \ \\ & & & \sigma_r \end{bmatrix} \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \vdots \\ \tilde{v}_r \end{bmatrix} \]

- **r**: rank of matrix A
- **\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r**: singular values (square roots of eig-vals of \( A^TA \) and \( A^T \))
- \( \tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_r \): left singular vectors (eig-vectors of \( A^TA \))
- \( \tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_r \): right singular vectors (eig-vectors of \( A^T \))

\[ A = \sigma_1 \tilde{u}_1 \tilde{v}_1^T + \sigma_2 \tilde{u}_2 \tilde{v}_2^T + \cdots + \sigma_r \tilde{u}_r \tilde{v}_r^T \]
Singular Value Decomposition

- **Linear trend** $\mathbf{v}$ in matrix $\mathbf{A}$:
  - the tendency of the row vectors of $\mathbf{A}$ to align with vector $\mathbf{v}$
  - strength of the linear trend: $\mathbf{A} \mathbf{v}$
- SVD discovers the linear trends in the data
- $\mathbf{u}_i, \mathbf{v}_i$: the $i$-th strongest linear trends
- $\sigma_i$: the strength of the $i$-th strongest linear trend

- HITS discovers the **strongest linear trend** in the authority space
HITS and the TKC effect

- The HITS algorithm favors the most dense community of hubs and authorities
  - Tightly Knit Community (TKC) effect
HITS and the TKC effect

• The HITS algorithm favors the most dense community of hubs and authorities
  – Tightly Knit Community (TKC) effect
HITS and the TKC effect

- The HITS algorithm favors the most dense community of hubs and authorities
  - Tightly Knit Community (TKC) effect
HITS and the TKC effect

• The HITS algorithm favors the most dense community of hubs and authorities
  – Tightly Knit Community (TKC) effect
HITS and the TKC effect

• The HITS algorithm favors the most dense community of hubs and authorities
  – Tightly Knit Community (TKC) effect
HITS and the TKC effect

• The HITS algorithm favors the most dense community of hubs and authorities
  – Tightly Knit Community (TKC) effect
HITS and the TKC effect

- The HITS algorithm favors the most dense community of hubs and authorities
  - Tightly Knit Community (TKC) effect

weight of node \( p \) is proportional to the number of \((BF)^n\) paths that leave node \( p \)
HITS and the TKC effect

• The HITS algorithm favors the most dense community of hubs and authorities
  – Tightly Knit Community (TKC) effect

![Diagram of HITS algorithm with normalized values](attachment:image.png)
Query-independent LAR

- Have an a-priori ordering of the web pages

- **Q**: Set of pages that contain the keywords in the query \( q \)

- Present the pages in Q ordered according to order \( \pi \)

- **What are the advantages of such an approach?**
InDegree algorithm

- Rank pages according to in-degree
  \[ w_i = |B(i)| \]
PageRank algorithm [BP98]

- Good authorities should be pointed by good authorities
- Random walk on the web graph
  - pick a page at random
  - with probability $1 - \alpha$ jump to a random page
  - with probability $\alpha$ follow a random outgoing link
- Rank according to the stationary distribution

$$PR(p) = \alpha \sum_{q \rightarrow p} \frac{PR(q)}{|F(q)|} + (1 - \alpha) \frac{1}{n}$$
Markov chains

• A Markov chain describes a discrete time stochastic process over a set of states

\[ S = \{ s_1, s_2, \ldots, s_n \} \]

according to a transition probability matrix

\[ P = \{ P_{ij} \} \]

– \( P_{ij} \) = probability of moving to state \( j \) when at state \( i \)
  • \( \sum_j P_{ij} = 1 \) (stochastic matrix)

• Memorylessness property: The next state of the chain depends only at the current state and not on the past of the process (first order MC)
  – higher order MCs are also possible
Random walks

- Random walks on graphs correspond to Markov Chains
  - The set of states $S$ is the set of nodes of the graph $G$
  - The transition probability matrix is the probability that we follow an edge from one node to another
An example

\[
A = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
P = \begin{bmatrix}
0 & 1/2 & 1/2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
1/3 & 1/3 & 1/3 & 0 & 0 \\
1/2 & 0 & 0 & 0 & 1/2 \\
\end{bmatrix}
\]
State probability vector

• The vector $q^t = (q^t_1, q^t_2, \ldots, q^t_n)$ that stores the probability of being at state $i$ at time $t$
  \[ q^0_i \] the probability of starting from state $i$

\[ q^t = q^{t-1} P \]
An example

\[ P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix} \]

\[ q_{t+1}^1 = \frac{1}{3} q_t^4 + \frac{1}{2} q_t^5 \]
\[ q_{t+1}^2 = \frac{1}{2} q_t^1 + q_t^3 + \frac{1}{3} q_t^4 \]
\[ q_{t+1}^3 = \frac{1}{2} q_t^1 + \frac{1}{3} q_t^4 \]
\[ q_{t+1}^4 = \frac{1}{2} q_t^5 \]
\[ q_{t+1}^5 = q_t^2 \]
Stationary distribution

- A stationary distribution for a MC with transition matrix $P$, is a probability distribution $\pi$, such that $\pi = \pi P$

- A MC has a unique stationary distribution if
  - it is irreducible
    - the underlying graph is strongly connected
  - it is aperiodic
    - for random walks, the underlying graph is not bipartite

- The probability $\pi_i$ is the fraction of times that we visited state $i$ as $t \to \infty$

- The stationary distribution is an eigenvector of matrix $P$
  - the principal left eigenvector of $P$ – stochastic matrices have maximum eigenvalue 1
Computing the stationary distribution

• The Power Method
  – Initialize to some distribution \( q^0 \)
  – Iteratively compute \( q^t = q^{t-1}P \)
  – After enough iterations \( q^t \approx \pi \)
  – Power method because it computes \( q^t = q^0P^t \)

• Why does it converge?
  – follows from the fact that any vector can be written as a linear combination of the eigenvectors
    • \( q^0 = v_1 + c_2v_2 + \ldots c_nv_n \)

• Rate of convergence
  – determined by \( \lambda_2^t \)
The PageRank random walk

- Vanilla random walk
  - make the adjacency matrix stochastic and run a random walk

\[
P = \begin{bmatrix}
0 & 1/2 & 1/2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
1/3 & 1/3 & 1/3 & 0 & 0 \\
1/2 & 0 & 0 & 1/2 & 0
\end{bmatrix}
\]
The PageRank random walk

• What about sink nodes?
  – what happens when the random walk moves to a node without any outgoing inks?

\[
P = \begin{bmatrix}
0 & 1/2 & 1/2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1/3 & 1/3 & 1/3 & 0 & 0 \\
1/2 & 0 & 0 & 1/2 & 0 \\
\end{bmatrix}
\]
The PageRank random walk

- Replace these row vectors with a vector $v$
  - typically, the uniform vector

\[
P' = \begin{bmatrix}
0 & 1/2 & 1/2 & 0 & 0 \\
1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\
0 & 1 & 0 & 0 & 0 \\
1/3 & 1/3 & 1/3 & 0 & 0 \\
1/2 & 0 & 0 & 1/2 & 0 \\
\end{bmatrix}
\]

$$P' = P + dv^T$$

\[
d = \begin{cases}
1 & \text{if } i \text{ is sink} \\
0 & \text{otherwise}
\end{cases}
\]
The PageRank random walk

• How do we guarantee irreducibility?
  – add a random jump to vector $v$ with prob $\alpha$
• typically, to a uniform vector

$P'' = \alpha \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{bmatrix} + (1 - \alpha) \begin{bmatrix} 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{bmatrix}$

$P'' = \alpha P' + (1-\alpha)uv^T$, where $u$ is the vector of all 1s
Effects of random jump

• Guarantees irreducibility
• Motivated by the concept of random surfer
• Offers additional flexibility
  – personalization
  – anti-spam
• Controls the rate of convergence
  – the second eigenvalue of matrix $P''$ is $\alpha$
A PageRank algorithm

• Performing vanilla power method is now too expensive – the matrix is not sparse

\[ q^0 = v \]
\[ t = 1 \]

repeat
\[ q^t = (P')^T v^t \]
\[ \delta = \| q^t - q^{t-1} \| \]
\[ t = t + 1 \]
until \( \delta < \epsilon \)

Efficient computation of \( y = (P'')^T x \)

\[ y = \alpha P'^T x \]
\[ \beta = \| x \|_1 - \| y \|_1 \]
\[ y = y + \beta v \]
Random walks on undirected graphs

• In the stationary distribution of a random walk on an undirected graph, the probability of being at node $i$ is proportional to the (weighted) degree of the vertex

• Random walks on undirected graphs are not “interesting”
Research on PageRank

• Specialized PageRank
  – personalization [BP98]
    • instead of picking a node uniformly at random favor specific nodes that are related to the user
  – topic sensitive PageRank [H02]
    • compute many PageRank vectors, one for each topic
    • estimate relevance of query with each topic
    • produce final PageRank as a weighted combination

• Updating PageRank [Chien et al 2002]
• Fast computation of PageRank
  – numerical analysis tricks
  – node aggregation techniques
  – dealing with the “Web frontier”
Previous work

• The problem of identifying the most important nodes in a network has been studied before in social networks and bibliometrics

• The idea is similar
  – A link from node p to node q denotes endorsement
  – mine the network at hand
  – assign an centrality/importance/standing value to every node
Social network analysis

- Evaluate the **centrality** of individuals in social networks
  - **degree centrality**
    - the (weighted) degree of a node
  - **distance centrality**
    - the average (weighted) distance of a node to the rest in the graph
      \[
      D_c(v) = \frac{1}{\sum_{u \neq v} d(v, u)}
      \]
  - **betweenness centrality**
    - the average number of (weighted) shortest paths that use node \( v \)
      \[
      B_c(v) = \sum_{s \neq v \neq t} \frac{\sigma_{st}(v)}{\sigma_{st}}
      \]
Counting paths – Katz 53

- The importance of a node is measured by the weighted sum of paths that lead to this node
- \( A^m[i,j] \) = number of paths of length \( m \) from \( i \) to \( j \)
- Compute
  \[
P = bA + b^2A^2 + \ldots + b^mA^m + \ldots = (I - bA)^{-1} - I\]
- converges when \( b < \lambda_1(A) \)
- Rank nodes according to the column sums of the matrix \( P \)
Bibliometrics

• Impact factor (E. Garfield 72)
  – counts the number of citations received for papers of the journal in the previous two years

• Pinsky-Narin 76
  – perform a random walk on the set of journals
  – $P_{ij}$ = the fraction of citations from journal i that are directed to journal j