Reducing the collection of itemsets: alternative representations and combinatorial problems
Too many frequent itemsets

• If \( \{a_1, \ldots, a_{100}\} \) is a frequent itemset, then there are

\[
\binom{100}{1} + \binom{100}{2} + \ldots + \binom{100}{100} = 2^{100} - 1
\]

\(1.27 \times 10^{30}\) frequent sub-patterns!

• There should be some more \textit{condensed} way to describe the data
Frequent itemsets maybe too many to be helpful

• If there are many and large frequent itemsets enumerating all of them is costly.

• We may be interested in finding the *boundary* frequent patterns.

• **Question:** Is there a good definition of such boundary?
Borders of frequent itemsets

- Itemset $X$ is more *specific* than itemset $Y$ if $X$ superset of $Y$ (notation: $Y < X$). Also, $Y$ is more *general* than $X$ (notation: $X > Y$).

- **The Border:** Let $S$ be a collection of frequent itemsets and $P$ the lattice of itemsets. The *border* $Bd(S)$ of $S$ consists of all itemsets $X$ such that *all more general itemsets* than $X$ are in $S$ and *no pattern more specific* than $X$ is in $S$.

$$Bd(S) = \left\{ X \in P \mid \text{for all } Y \in P \text{ with } Y \prec X \text{ then } Y \in S, \right. \\
\left. \text{and for all } W \in P \text{ with } X \prec W \text{ then } W \notin S \right\}$$
Positive and negative border

\[ Bd(S) = \left\{ X \in P \right\mid \text{for all } Y \in P \text{ with } Y \prec X \text{ then } Y \in S, \text{ and for all } W \in P \text{ with } X \prec W \text{ then } W \not\in S \right\} \]

- **Positive border**: Itemsets in the border that are also frequent (belong in \( S \))
  \[ Bd^+(S) = \left\{ X \in S \right\mid \text{for all } Y \in P \text{ with } X \prec Y \text{ then } Y \not\in S \right\} \]

- **Negative border**: Itemsets in the border that are not frequent (do not belong in \( S \))
  \[ Bd^-(S) = \left\{ X \in P \setminus S \right\mid \text{for all } Y \in P \text{ with } Y \prec X \text{ then } Y \in S \right\} \]
Examples with borders

• Consider a set of items from the alphabet: \{A,B,C,D,E\} and the collection of frequent sets

\[ S = \{\{A\},\{B\},\{C\},\{E\},\{A,B\},\{A,C\},\{A,E\},\{C,E\},\{A,C,E\}\} \]

• The negative border of collection \( S \) is

\[ \text{Bd}^{-}(S) = \{\{D\},\{B,C\},\{B,E\}\} \]

• The positive border of collection \( S \) is

\[ \text{Bd}^{+}(S) = \{\{A,B\},\{A,C,E\}\} \]
Descriptive power of the borders

• **Claim:** A collection of frequent sets $S$ can be *fully described* using only the positive border ($Bd^+(S)$) or only the negative border ($Bd^-(S)$).
Maximal patterns

Frequent patterns without proper frequent super pattern
Maximal Frequent Itemset

An itemset is maximal frequent if none of its immediate supersets is frequent.
Maximal patterns

• The set of maximal patterns is the same as the positive border

• Descriptive power of maximal patterns:
  – Knowing the set of all maximal patterns allows us to reconstruct the set of all frequent itemsets!!

  – We can only reconstruct the set not the actual frequencies
Closed patterns

• An itemset is closed if none of its immediate supersets has the same support as the itemset

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Maximal vs Closed Itemsets

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Transaction Ids

Not supported by any transactions
Maximal vs Closed Frequent Itemsets

Minimum support = 2

Closed and maximal

Closed but not maximal

# Closed = 9
# Maximal = 4
Why are closed patterns interesting?

- \( s(\{A,B\}) = s(A) \), i.e., \( \text{conf}(\{A\} \rightarrow \{B\}) = 1 \)

- We can infer that for every itemset \( X \),
  \[ s(A \cup \{X\}) = s(\{A,B\} \cup X) \]

- No need to count the frequencies of sets \( X \cup \{A,B\} \) from the database!

- If there are lots of rules with confidence 1, then a significant amount of work can be saved
  - Very useful if there are strong correlations between the items and when the transactions in the database are similar
Why closed patterns are interesting?

• Closed patterns and their frequencies alone are sufficient representation for all the frequencies of all frequent patterns

• **Proof:** Assume a frequent itemset $X$:
  
  - $X$ is closed $\Rightarrow s(X)$ is known
  - $X$ is not closed $\Rightarrow$
    
    $s(X) = \max \{s(Y) \mid Y \text{ is closed and } X \text{ subset of } Y\}$
Maximal vs Closed sets

• Knowing all maximal patterns (and their frequencies) allows us to reconstruct the set of frequent patterns.

• Knowing all closed patterns and their frequencies allows us to reconstruct the set of all frequent patterns and their frequencies.
A more algorithmic approach to reducing the collection of frequent itemsets
Prototype problems: Covering problems

- **Setting:**
  - Universe of $N$ elements $U = \{U_1, \ldots, U_N\}$
  - A set of $n$ sets $S = \{s_1, \ldots, s_n\}$
  - Find a collection $C$ of sets in $S$ ($C$ subset of $S$) such that $\bigcup_{c \in C} c$ contains many elements from $U$

- **Example:**
  - $U$: set of documents in a collection
  - $s_i$: set of documents that contain term $t_i$
  - Find a collection of terms that cover most of the documents
Prototype covering problems

- **Set cover problem**: Find a small collection \( C \) of sets from \( S \) such that all elements in the universe \( U \) are covered by some set in \( C \)

- **Best collection problem**: find a collection \( C \) of \( k \) sets from \( S \) such that the collection covers as many elements from the universe \( U \) as possible

- Both problems are NP-hard

- Simple approximation algorithms with provable properties are available and very useful in practice
Set-cover problem

• Universe of \( N \) elements \( U = \{U_1, \ldots, U_N\} \)
• A set of \( n \) sets \( S = \{s_1, \ldots, s_n\} \) such that \( U_i s_i = U \)

• **Question:** Find the smallest number of sets from \( S \) to form collection \( C \) (\( C \) subset of \( S \)) such that \( U \subseteq C \subseteq C = U \)

• The set-cover problem is **NP-hard** (what does this mean?)
**Trivial algorithm**

- Try all subcollections of $S$
- Select the smallest one that covers all the elements in $U$
- The running time of the trivial algorithm is $O(2^{|S|} |U|)$
- This is way too slow
Greedy algorithm for set cover

• Select first the largest-cardinality set $s$ from $S$

• Remove the elements from $s$ from $U$

• Recompute the sizes of the remaining sets in $S$

• Go back to the first step
As an algorithm

1. \( X = U \)
2. \( C = \{\} \)
3. \textbf{while} \( X \) is not empty \textbf{do}
   - For all \( s \in S \) let \( a_s = |s \text{ intersection } X| \)
   - Let \( s \) be such that \( a_s \) is \textit{maximal}
   - \( C = C \cup \{s\} \)
   - \( X = X \setminus s \)
How can this go wrong?

• No global consideration of how good or bad a selected set is going to be
How good is the greedy algorithm?

• Consider a minimization problem
  – In our case we want to minimize the *cardinality* of set \( C \)

• Consider an instance \( I \), and cost \( a^*(I) \) of the optimal solution
  – \( a^*(I) \): is the minimum number of sets in \( C \) that cover all elements in \( U \)

• Let \( a(I) \) be the cost of the approximate solution
  – \( a(I) \): is the number of sets in \( C \) that are picked by the greedy algorithm

• An algorithm for a minimization problem has approximation factor \( F \) if for all instances \( I \) we have that

\[
a(I) \leq F \times a^*(I)
\]

• *Can we prove any approximation bounds for the greedy algorithm for set cover?*
How good is the greedy algorithm for set cover?

• *(Trivial?)* Observation: The greedy algorithm for set cover has approximation factor $b = |s_{\text{max}}|$, where $s_{\text{max}}$ is the set in $S$ with the largest cardinality

• Proof:
  – $a^*(I) \geq N/|s_{\text{max}}|$ or $N \leq |s_{\text{max}}|a^*(I)$
  – $a(I) \leq N \leq |s_{\text{max}}|a^*(I)$
How good is the greedy algorithm for set cover? A tighter bound

• The greedy algorithm for set cover has approximation factor $F = O(\log |s_{\text{max}}|)$

• **Proof**: (From CLR “Introduction to Algorithms”)

Best-collection problem

• Universe of $N$ elements $U = \{U_1, \ldots, U_N\}$
• A set of $n$ sets $S = \{s_1, \ldots, s_n\}$ such that $U_i s_i = U$

• **Question:** Find the collection $C$ consisting of $k$ sets from $S$ such that $f(C) = \left| U_{c \in C} \right|$ is maximized

• The best-collection problem is NP-hard

• Simple approximation algorithm has approximation factor $F = (e-1)/e$
Greedy approximation algorithm for the best-collection problem

- \(C = \emptyset\)
- **for every** set \(s\) in \(S\) and **not** in \(C\) compute the gain of \(s\):
  \[
g(s) = f(C \cup \{s\}) - f(C)
  \]
- Select the set \(s\) with the **maximum** gain
- \(C = C \cup \{s\}\)
- **Repeat until** \(C\) has \(k\) elements
Basic theorem

• The **greedy** algorithm for the best-collection problem has approximation factor $F = (e-1)/e$

• $C^*$: **optimal** collection of cardinality $k$
• $C$: collection output by the **greedy** algorithm
• $f(C) \geq (e-1)/e \times f(C^*)$
Submodular functions and the greedy algorithm

• A function $f$ (defined on sets of some universe) is **submodular** if
  
  – for all sets $S$, $T$ such that $S$ is *subset* of $T$ and $x$ any element in the universe
  
  – $f(S \cup \{x\}) - f(S) \geq f(T \cup \{x\}) - f(T)$

• **Theorem:** For all maximization problems where the optimization function is **submodular**, the **greedy** algorithm has approximation factor

  \[ F = \frac{(e-1)}{e} \]
Again: Can you think of a more algorithmic approach to reducing the collection of frequent itemsets
Approximating a collection of frequent patterns

• Assume a collection of frequent patterns $S$

• Each pattern $X \in S$ is described by the patterns that covers

  $\text{Cov}(X) = \{ Y | Y \in S \text{ and } Y \text{ subset of } X \}$

• **Problem:** Find $k$ patterns from $S$ to form set $C$ such that

  $$|\bigcup_{X \in C} \text{Cov}(X)|$$

  is maximized
Frequent itemsets

Non-frequent itemsets

border

empty set