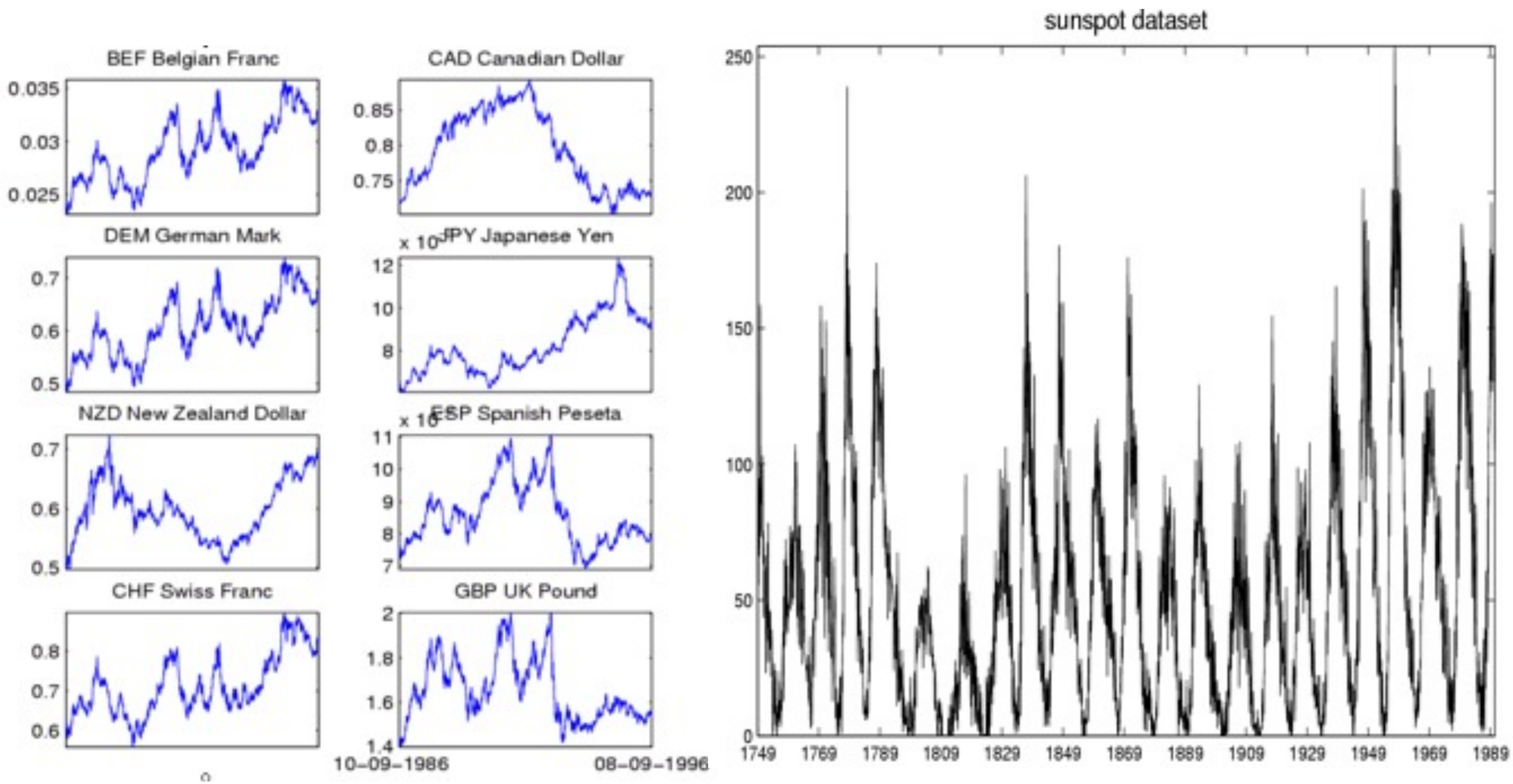
Time-series data analysis

Why deal with sequential data?

- Because all data is sequential ☺
- All data items arrive in the data store in some order
- Examples
  - transaction data
  - documents and words
- In some (or many) cases the order does not matter
- In many cases the order is of interest

Time-series data



- Financial time series, process monitoring…

Questions

- What is the structure of sequential data?
- Can we represent this structure

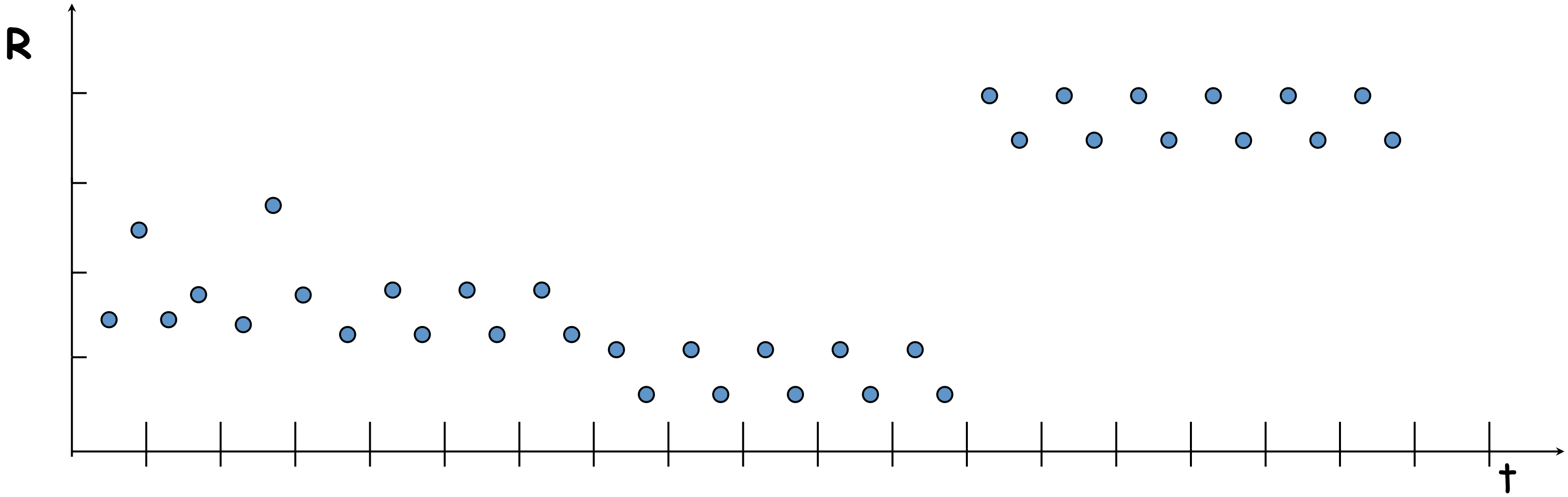
Sequence segmentation

- Gives an accurate representation of the structure of sequential data
- How?
  - By trying to find homogeneous segments
- Segmentation question:
- Can a sequence $T=\{t_1,t_2,\ldots,t_n\}$ be described as a concatenation of subsequences $S_1,S_2,\ldots,S_k$ such that each S_i is in some sense homogeneous?
- The corresponding notion of segmentation in unordered data is clustering

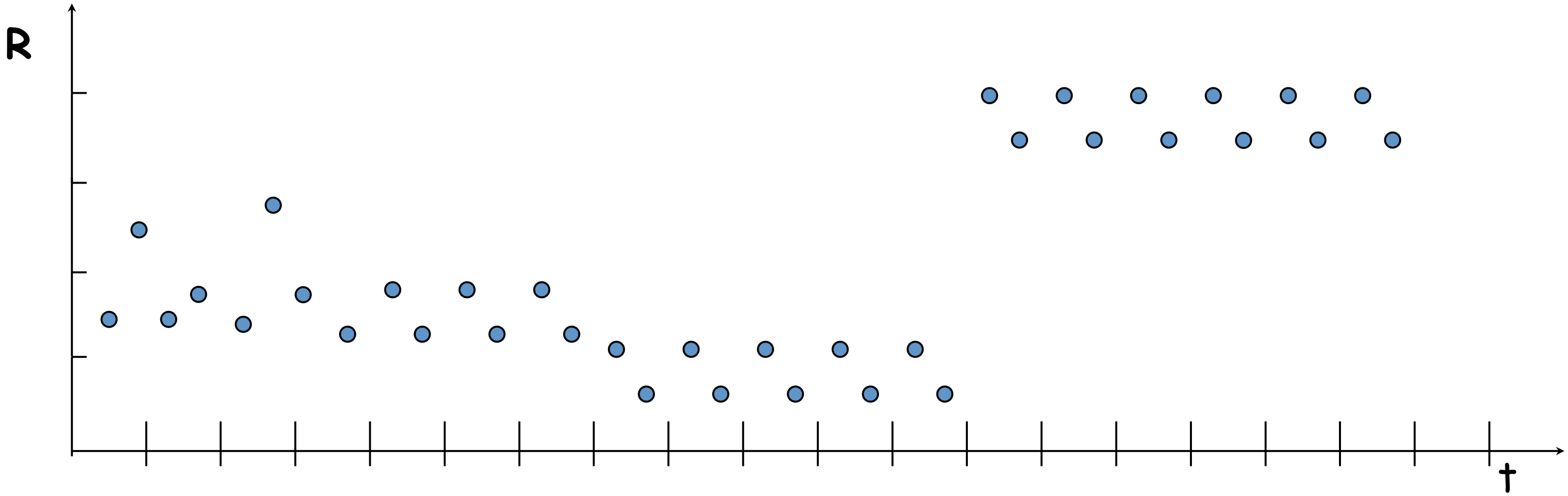
Dynamic-programming algorithm

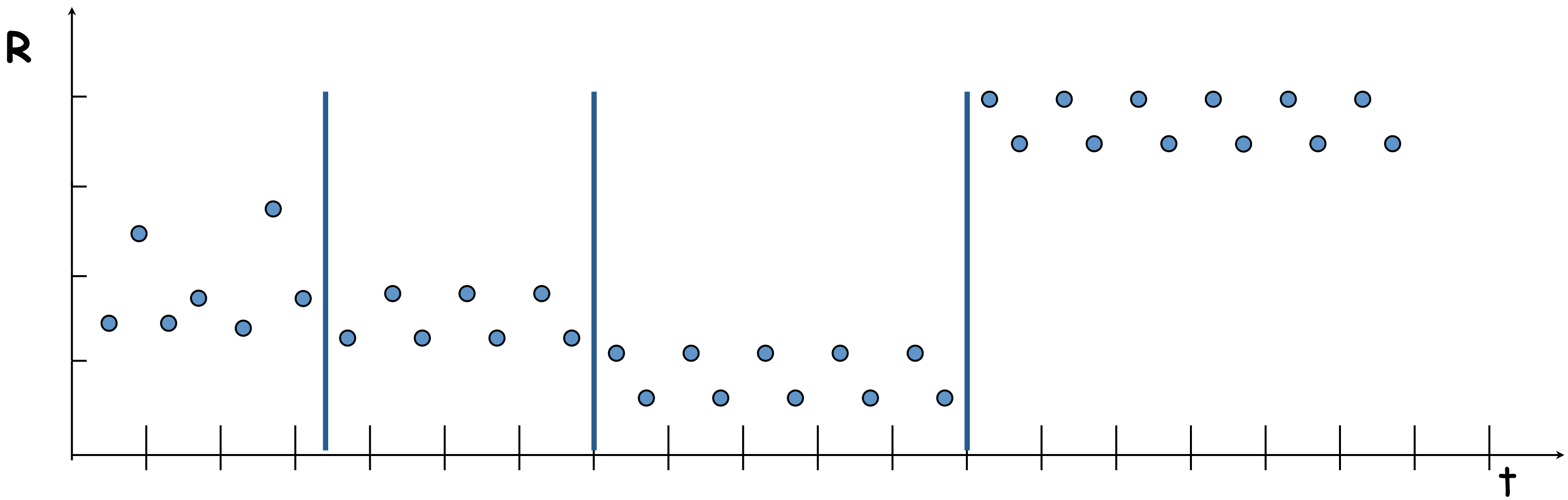
- Sequence T, length n, k segments, cost function $E()$, table M
- For $i=1$ to n
  - Set $M[1,i]=E(T[1\ldots i])$ //Everything in one cluster
- For $j=1$ to k
  - Set $M[j,j] = 0$ //each point in its own cluster
- For $j=2$ to k
  - For $i=j+1$ to n
    - Set $M[j,i] = \min_{i'<i}\{M[j-1,i]+E(T[i'+1\ldots i])\}$
- To recover the actual segmentation (not just the optimal cost) store also the minimizing values i'
- Takes time $O(n^2k)$, space $O(kn)$

Example



Example





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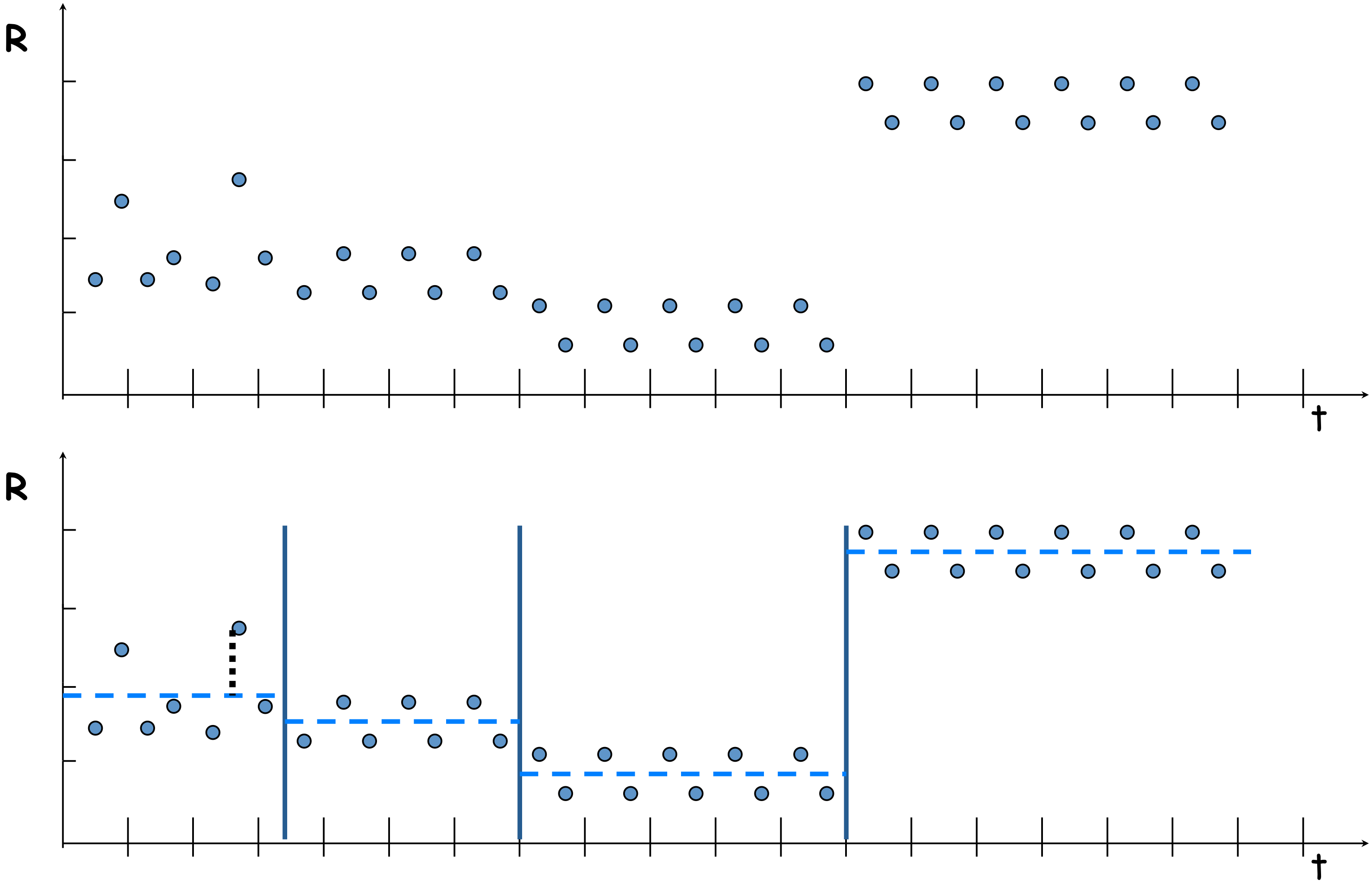


Basic definitions

- Sequence $T = \{t_1, t_2, \ldots, t_n\}$: an ordered set of n d-dimensional real points $t_i \in R^d$

- A k-segmentation S: a partition of T into k contiguous segments $\{s_1, s_2, \ldots, s_k\}$

  – Each segment $s \in S$ is represented by a single value $\mu_s \in R^d$ (the representative of the segment)

- Error $E_p(S)$: The error of replacing individual points with representatives

$$E_p(S) = \left(\sum_{s \in S} \sum_{t \in s} | t - \mu_s |^p \right)^{\frac{1}{p}}$$

The k-segmentation problem

Given a sequence T of length n and a value k, find a k-segmentation $S = \{s_1, s_2, \ldots, s_k\}$ of T such that the

- Common cases for the error function E_p: $p = 1$ and $p = 2$.
  - When $p = 1$, the best μ_s corresponds the median of the points in segment s.
  - When $p = 2$, the best μ_s corresponds to the mean of the points in segment s.

Optimal solution for the k-segmentation problem

- Bellman'61] The k-segmentation problem can be solved optimally using a standard dynamic-programming algorithm

$$
\begin{aligned}
&E_p(S_{\text{opt}}(T\left[1\ldots n\right], k)) = \\
&\quad \min_{j<n} \quad \{E_p\left(S_{\text{opt}}(T\left[1\ldots j\right], k-1)\right) \\
&\qquad\qquad\qquad + E_p\left(S_{\text{opt}}(T\left[j+1,\ldots,n\right], 1)\right)\}
\end{aligned}
$$

- Running time $O(n^2k)$
  - Too expensive for large datasets!

Heuristics

- Bottom-up greedy (BU): O(nlogn)
  - [Keogh and Smyth'97, Keogh and Pazzani'98]
- Top-down greedy (TD): O(nlogn)
  - [Douglas and Peucker'73, Shatkay and Zdonik'96, Lavrenko et. al'00]
- Global Iterative Replacement (GiR): O(nl)
  - [Himberg et. al '01]
- Local Iterative Replacement (LiR): O(nl)
  - [Himberg et. al '01]

Approximation algorithm

- [**Theorem**] The segmentation problem can be approximated within a constant factor of 3 for both E_1 and E_2 error measures. That is,

$$E_p(S_{DnS}) \leq 3E_p(S_{OPT}) \quad p = 1,2$$

- The running time of the approximation algorithm is:

$$O(n^{4/3}k^{5/3})$$

Divide 'n Segment (DnS) algorithm

- **Main idea**
  - Split the sequence arbitrarily into subsequences
  - Solve the k-segmentation problem in each subsequence
  - Combine the results
- **Advantages**
  - Extremely simple
  - High quality results
  - Can be applied to other segmentation problems[Gionis'03, Haiminen'04,Bingham'06]

DnS algorithm – Details

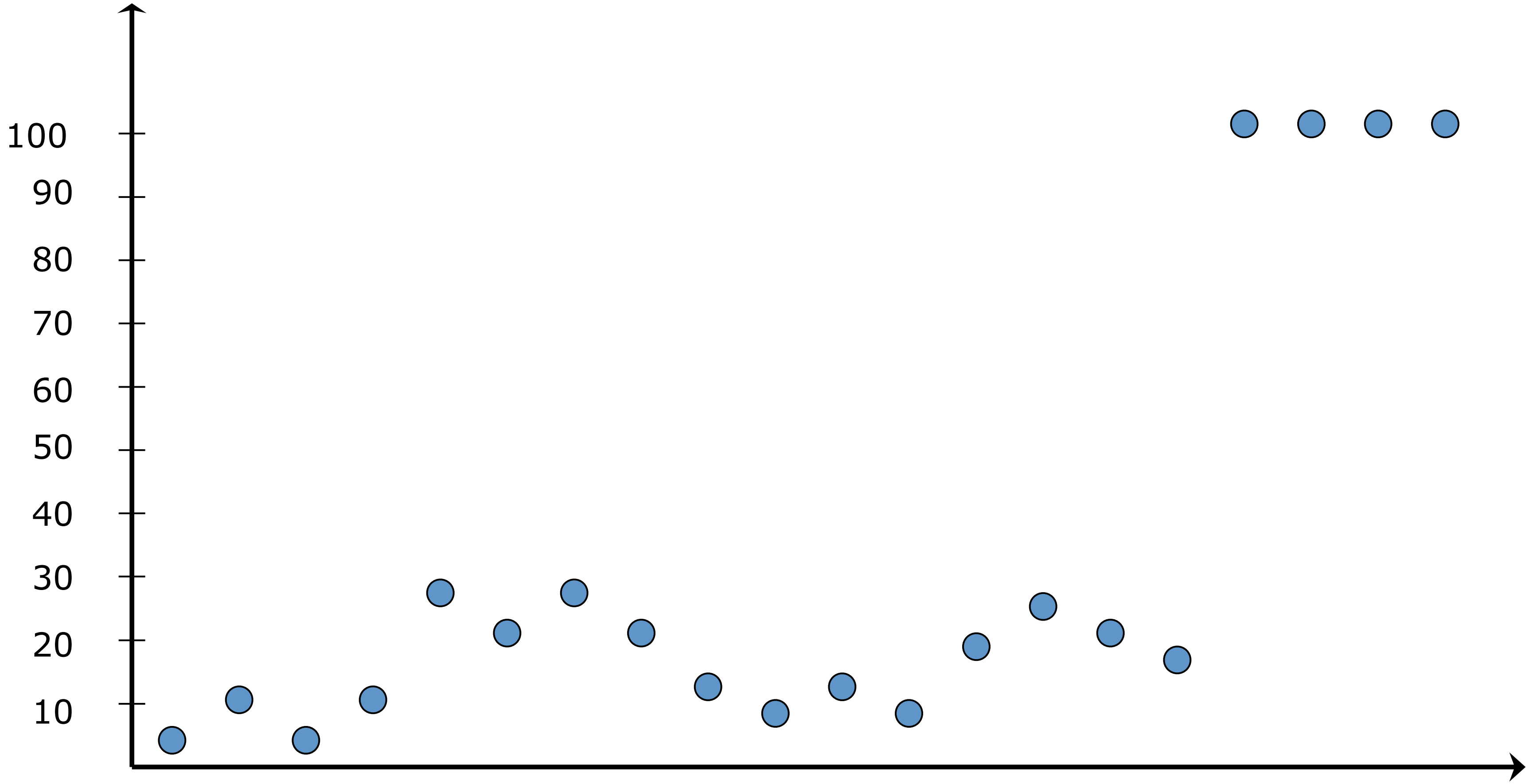
Input: Sequence T, integer k

Output: a k-segmentation of T

1. Partition sequence T arbitrarily into m disjoint intervals $T_1, T_2, \ldots, T_m$
2. For each interval T_i solve optimally the k- segmentation problem using DP algorithm
3. Let T' be the concatenation of mk representatives produced in Step 2. Each representative is weighted with the length of the segment it represents
4. Solve optimally the k-segmentation problem for T' using the DP algorithm and output this segmentation as the final segmentation

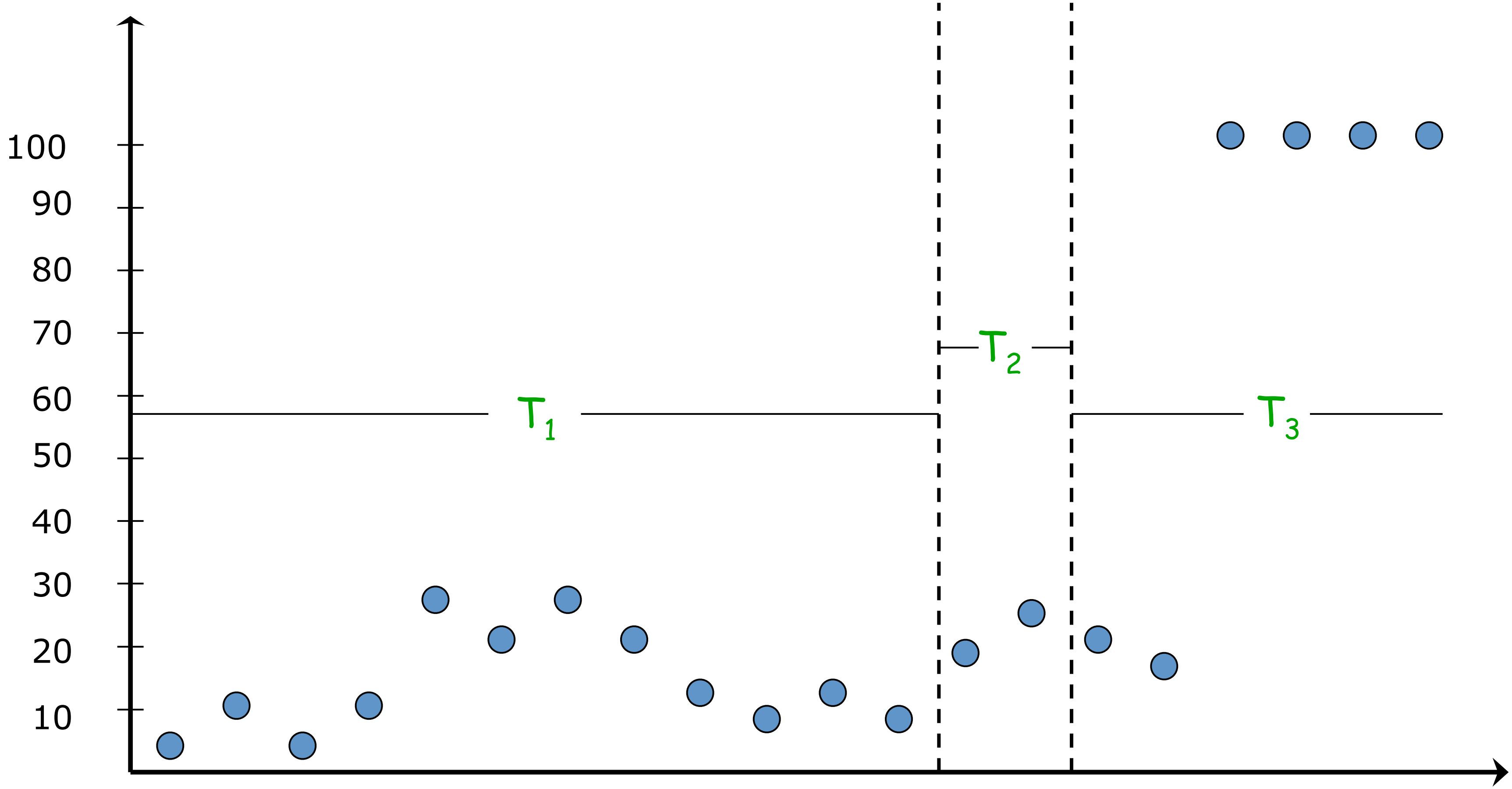
The DnS algorithm

Input sequence T consisting of n=20 points (k=2)



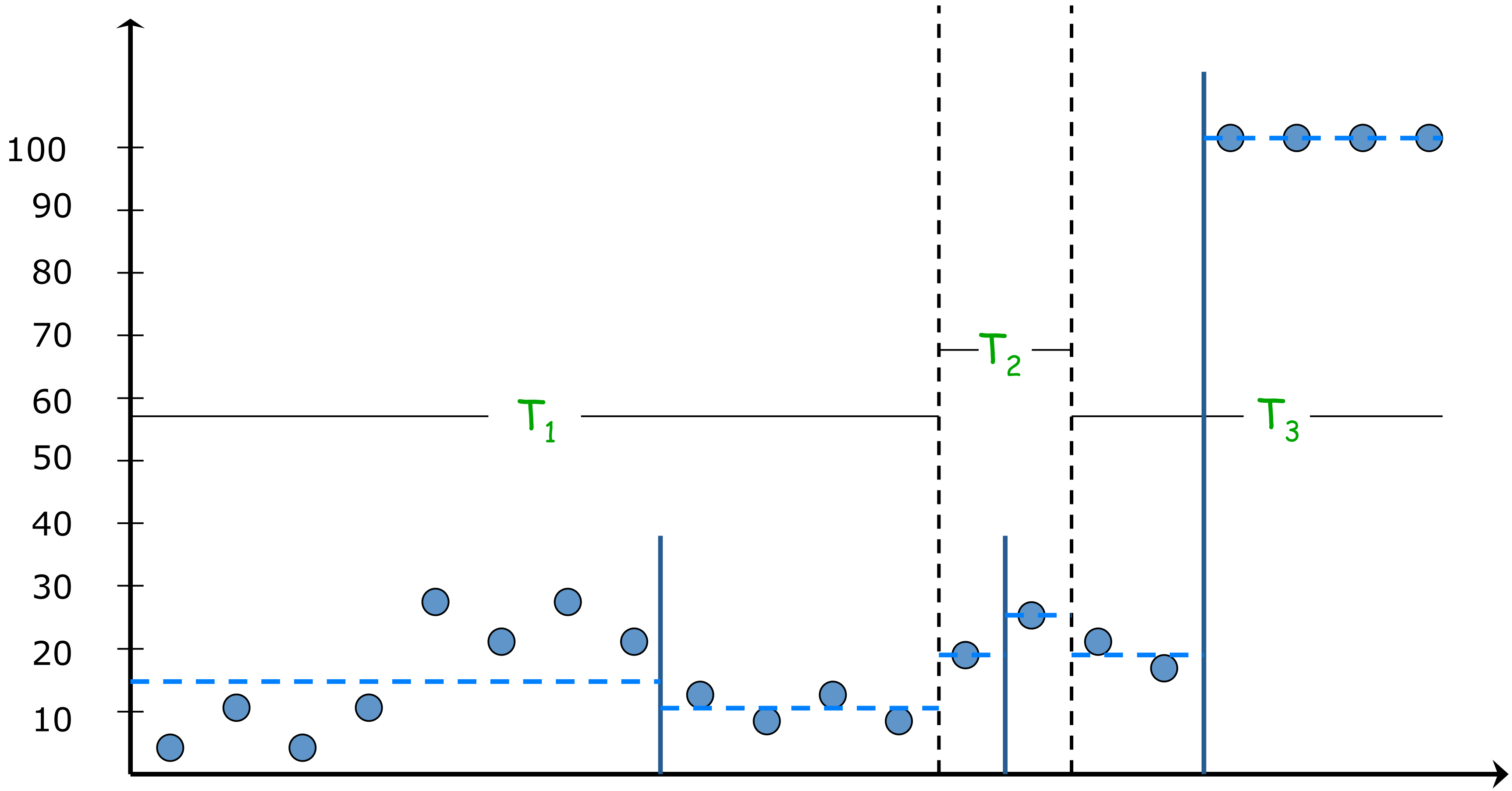
The DnS algorithm – Step 1

Partition the sequence into m=3 disjoint intervals



The DnS algorithm – Step 2

Solve optimally the k-segmentation problem into each partition (k=2)



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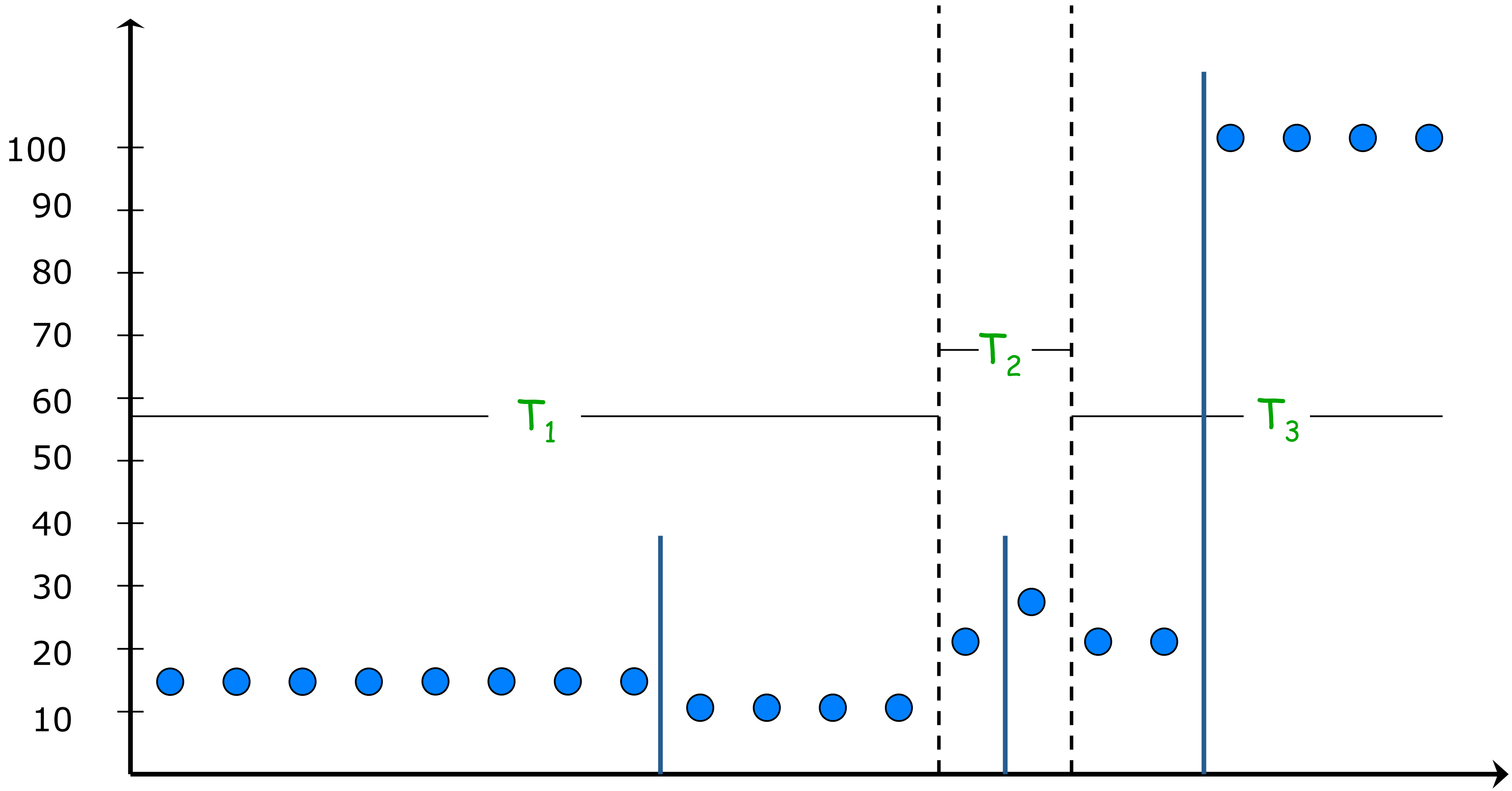
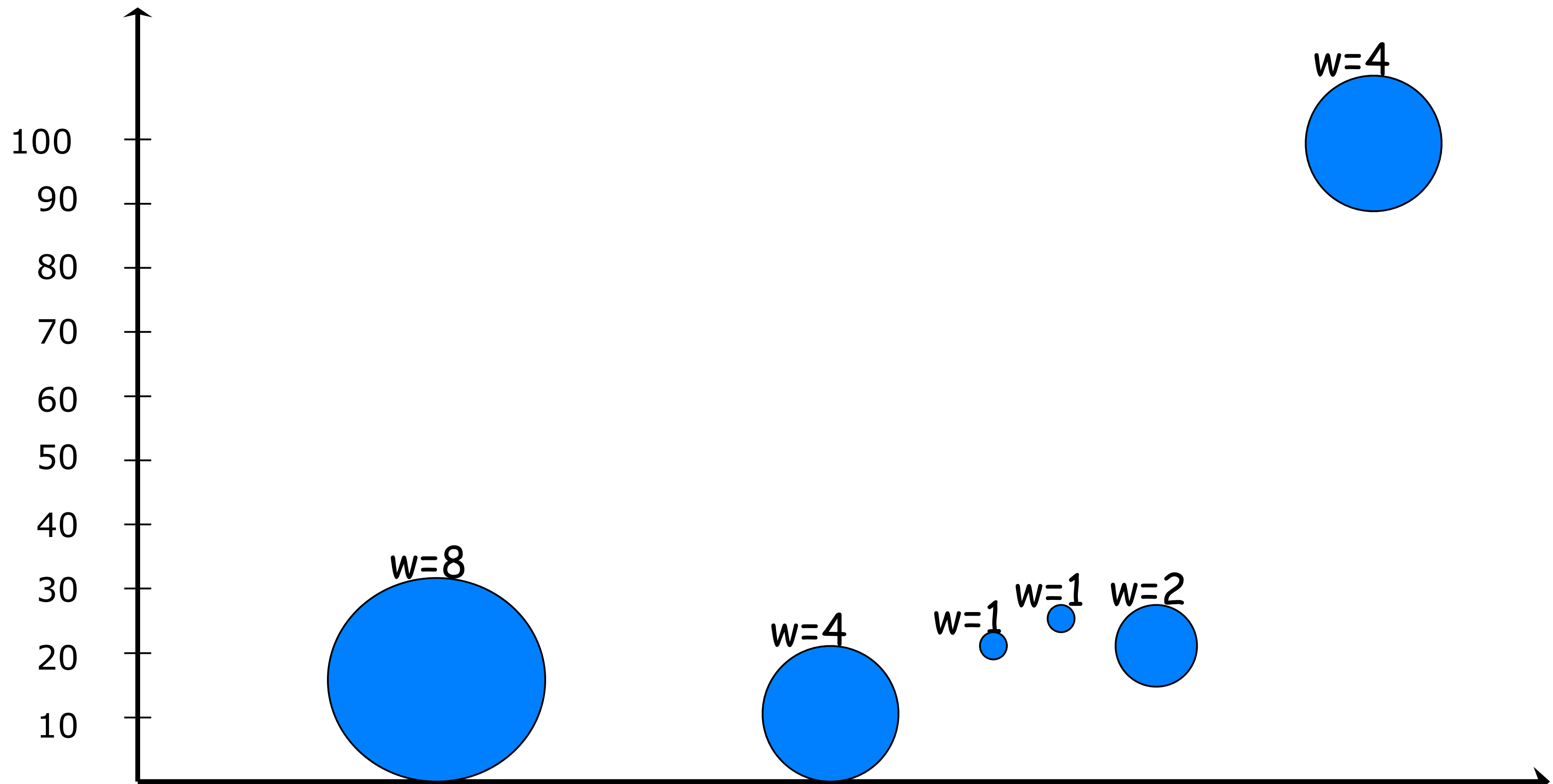


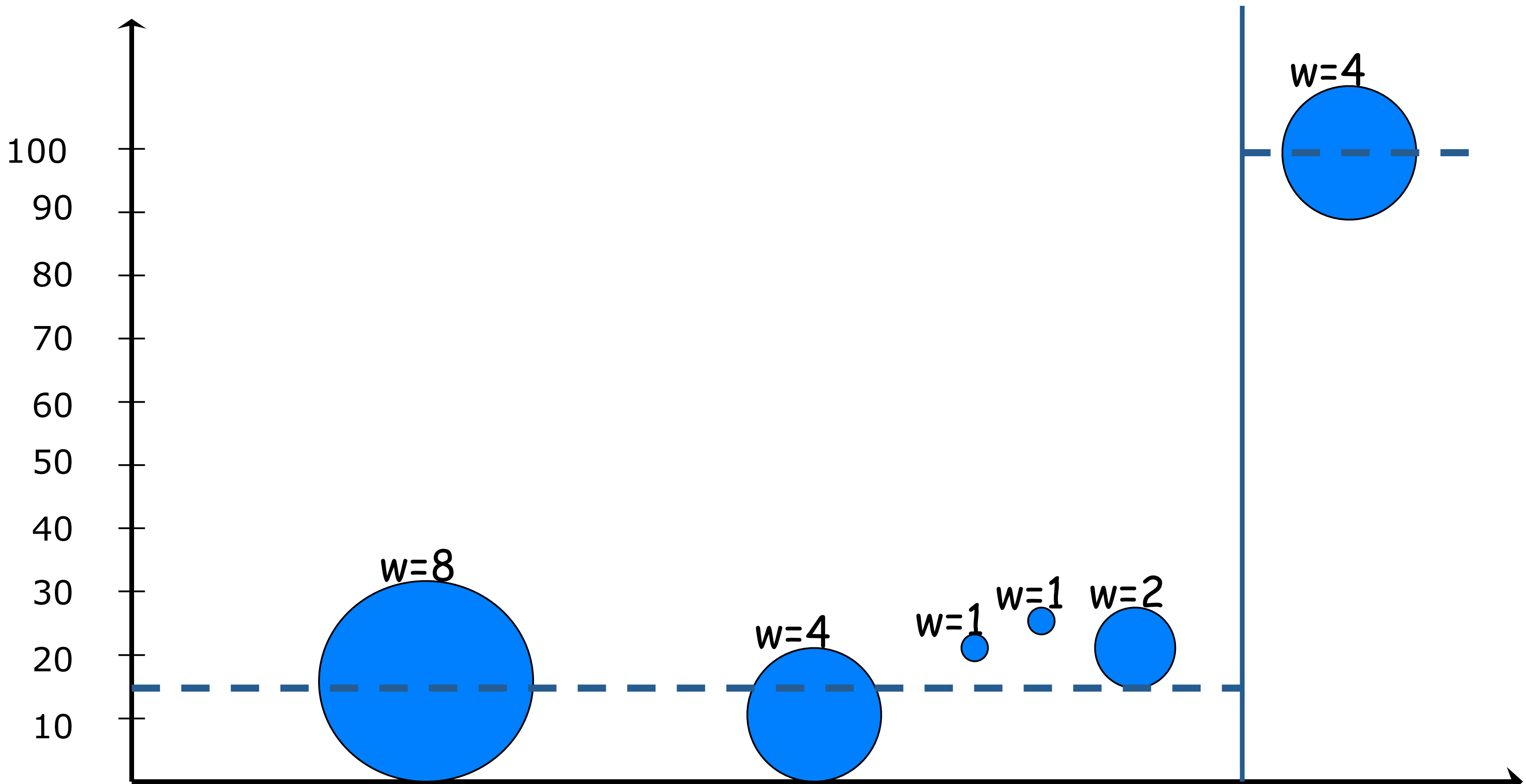
The DnS algorithm – Step 3

Sequence T' consisting of mk=6 representantives



The DnS algorithm – Step 4

Solve k-segmentation on T' (k=2)



Running time

- In the case of equipartition in Step 1, the running time of the algorithm as a function of m is:

$$R(m) = m\left(\frac{n}{m}\right)^2 k + (mk)^2 k$$

- The function R(m) is minimized for

$$m_0 = \left(\frac{n}{k}\right)^{\frac{2}{3}}$$

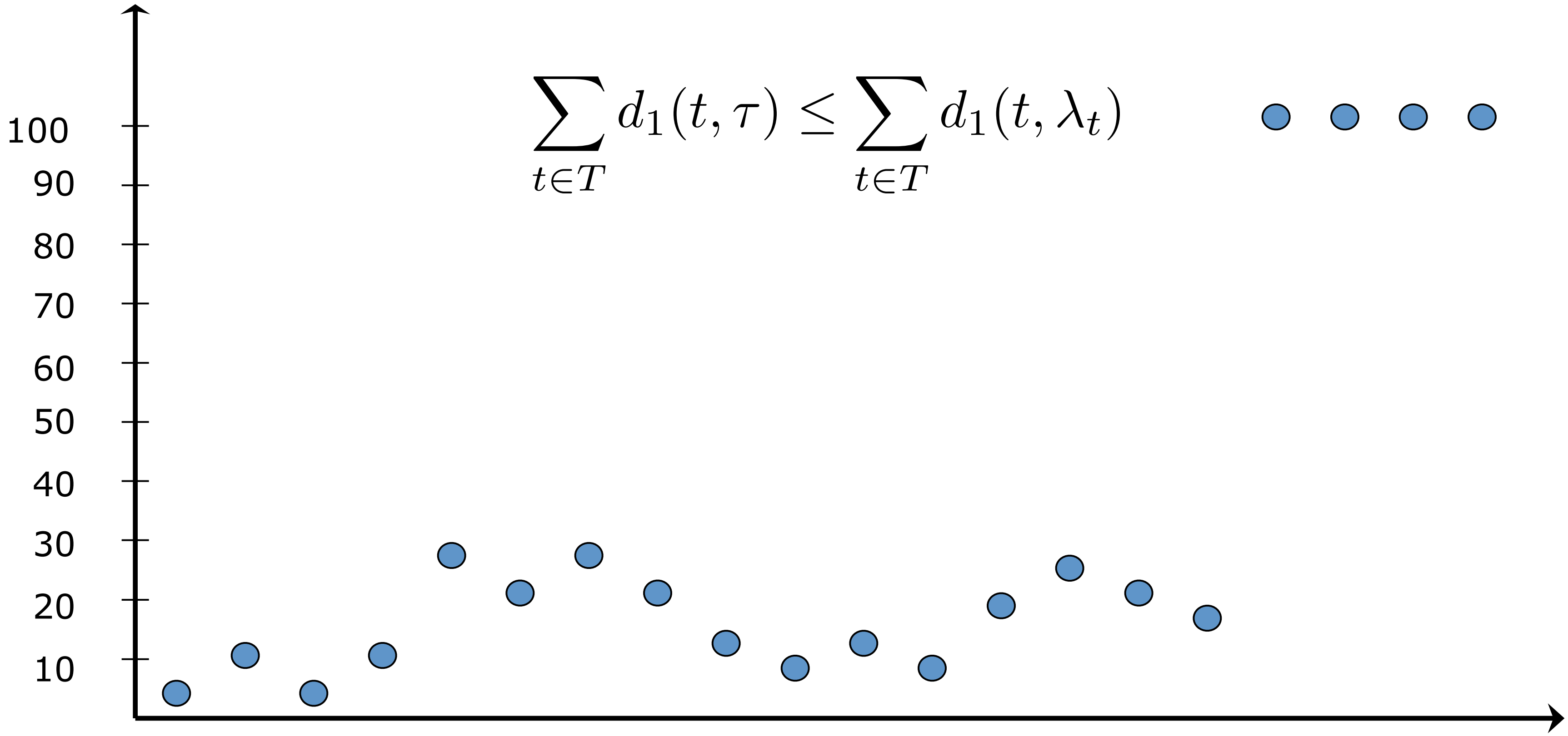
- Running time $R(m_0) = 2n^{4/3}k^{5/3}$

The segmentation error

- [**Theorem**] The segmentation error of the DnS algorithms is at most three times the error of the optimal (DP) algorithm for both E_1 and E_2 error measures. $$E_p(S_{DnS}) \leq 3E_p(S_{OPT}) \quad p = 1,2$$

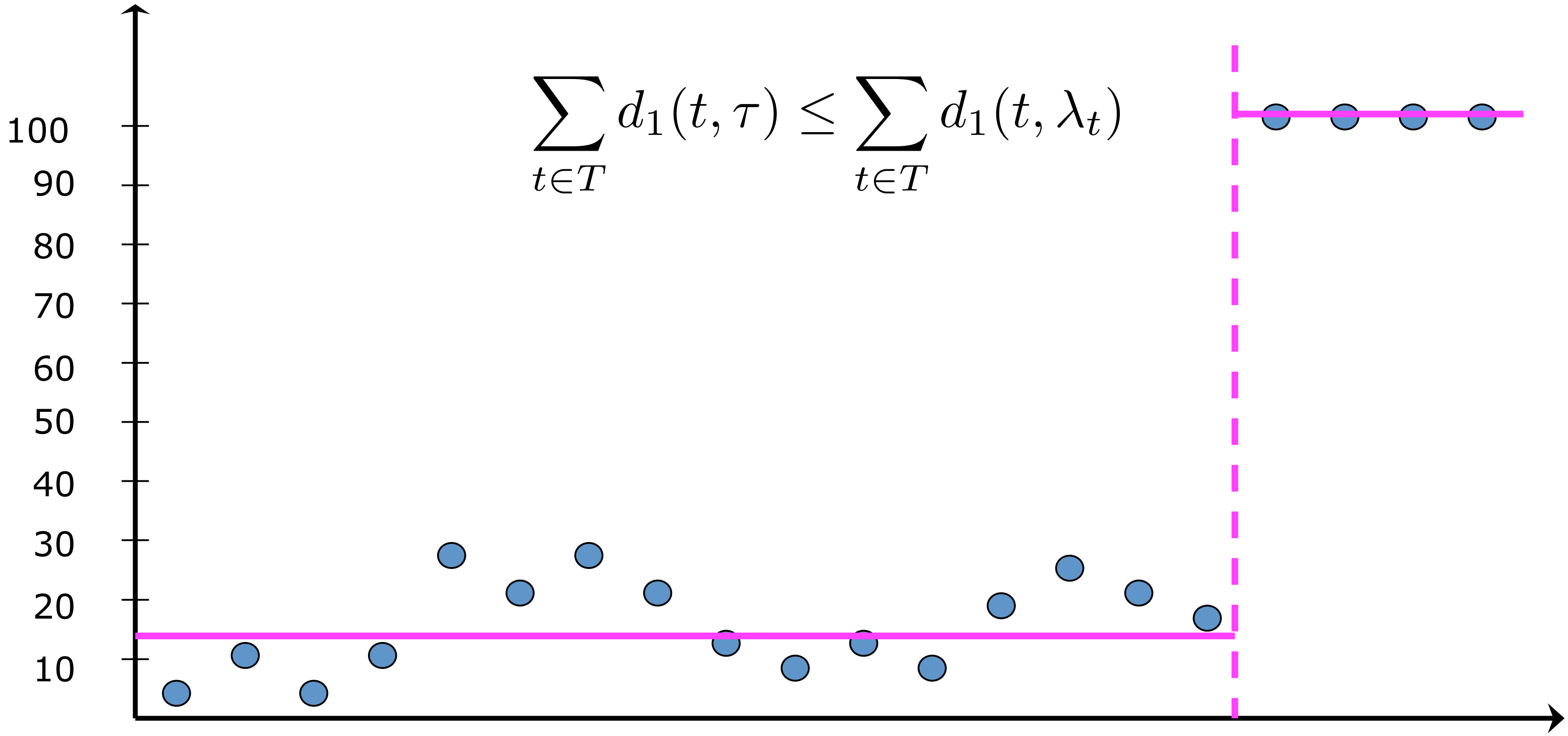
Proof for E_1

$$\sum_{t \in T} d_1(t, \tau) \leq \sum_{t \in T} d_1(t, \lambda_t)$$



Proof for E$_1$

$$\sum_{t \in T} d_1(t, \tau) \leq \sum_{t \in T} d_1(t, \lambda_t)$$



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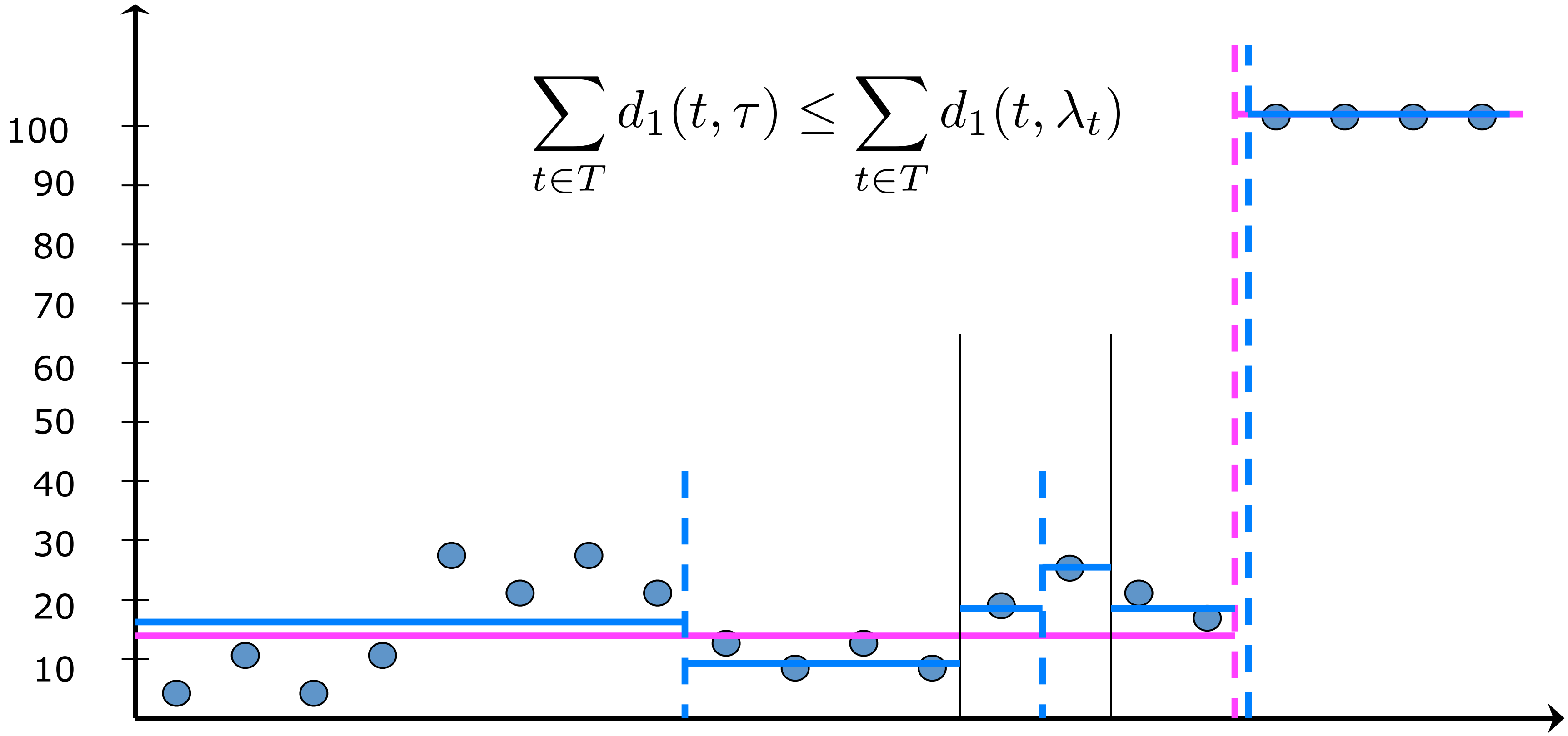
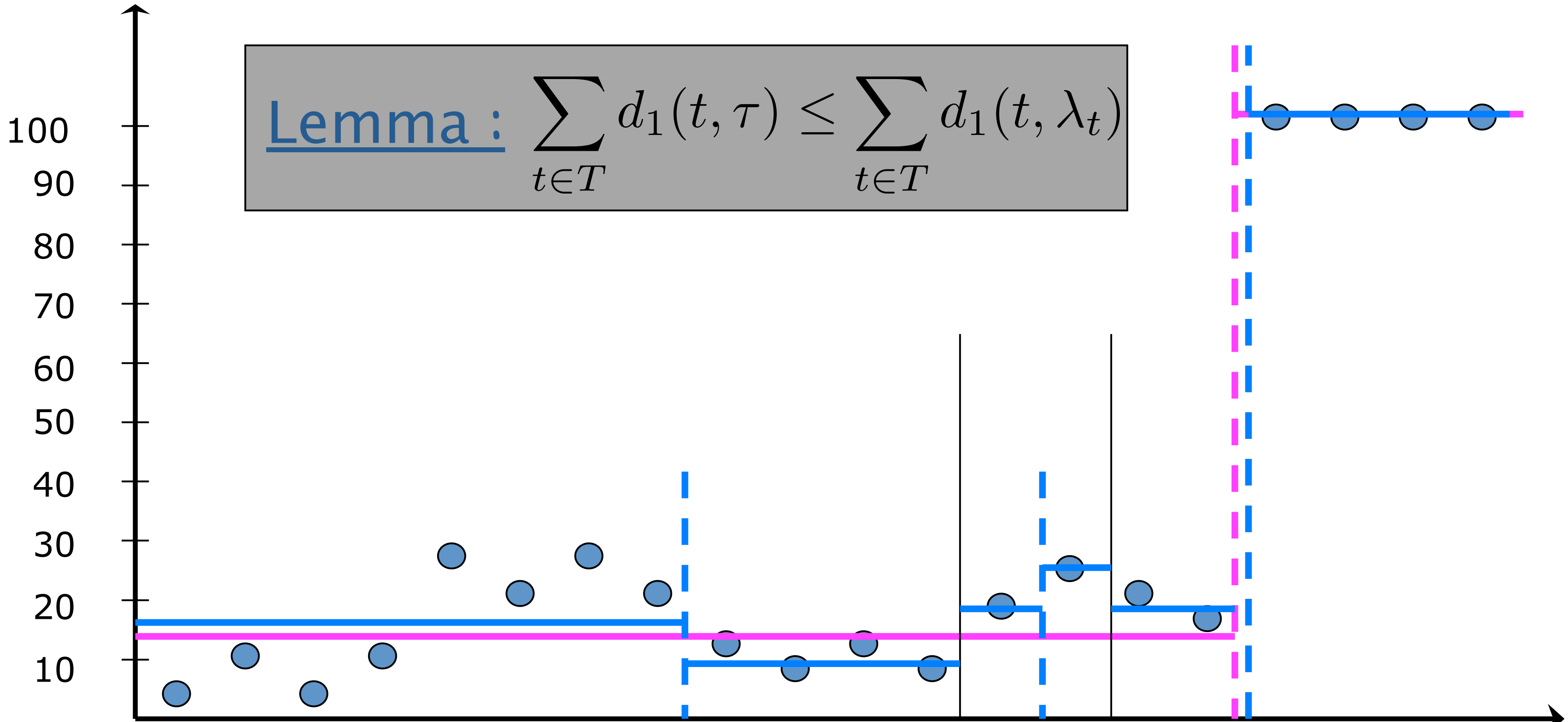


$$\sum_{t \in T} d_1(t, \tau) \leq \sum_{t \in T} d_1(t, \lambda_t)$$
100
90
80
70
60
50
40
30
20
10

Proof for E_1

- λ_t: the representative of point t in the optimal segmentation
- τ: the representative of point t in the segmentation of Step 2



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Lemma : $\sum_{t \in T} d_1(t, \tau) \leq \sum_{t \in T} d_1(t, \lambda_t)$

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\begin{aligned}
E_1(S_{DnS}) &= \sum_{t \in T} d_1(t, \mu_t) \\
&\leq \sum_{t \in T} \left(d_1(t, \tau) + d_1(\tau, \mu_t)\right) \\
&\leq \sum_{t \in T} \left(d_1(t, \tau) + d_1(\tau, \lambda_t)\right) \\
&\leq \sum_{t \in T} \left(d_1(t, \tau) + d_1(\tau, t) + d_1(t, \lambda_t)\right) \\
&\leq 2 \sum_{t \in T} d_1(t, \lambda_t) + \sum_{t \in T} d_1(t, \lambda_t) \\
&= 3E\left(S_{OPT}\right)
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&\leq \sum_{t \in T} \left(d_1(t, \tau) + d_1(\tau, \mu_t)\right) \quad \text{(triangle inequality)} \\
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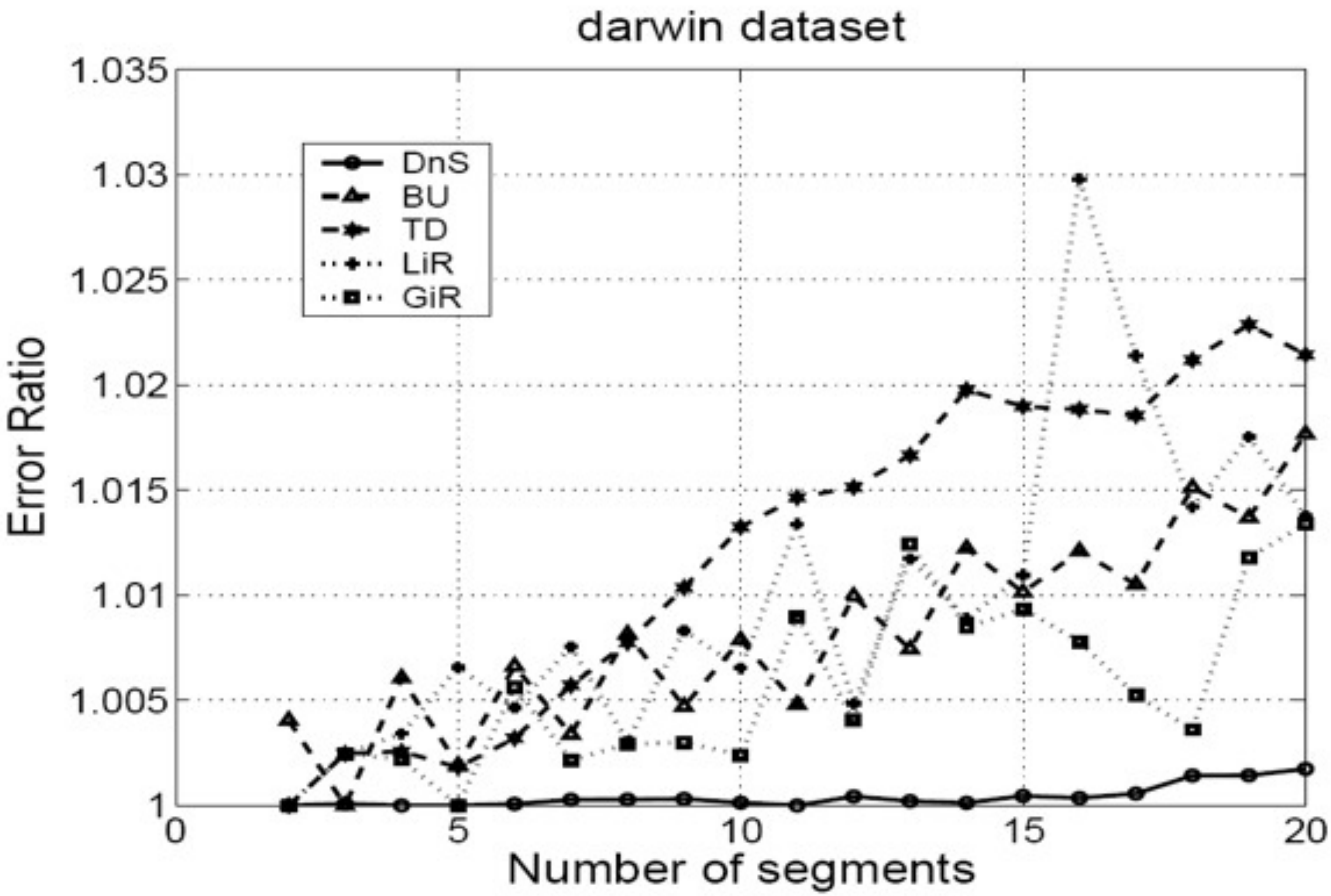
&= 3E(S_{OPT})
\end{aligned}
$$

Trading speed for accuracy

- Recursively divide (into m pieces) and segment
- If $\chi=(n_i)^{1/2}$, where n_i the length of the sequence in the i-th recursive level ($n_1=n$) then
  - running time of the algorithm is $O(n\log\log n)$
  - the segmentation error is at most $O(\log n)$ worse than the optimal
- If χ =const, the running time of the algorithm is $O(n)$, but there are no guarantees for the segmentation error

Real datasets - DnS algorithm



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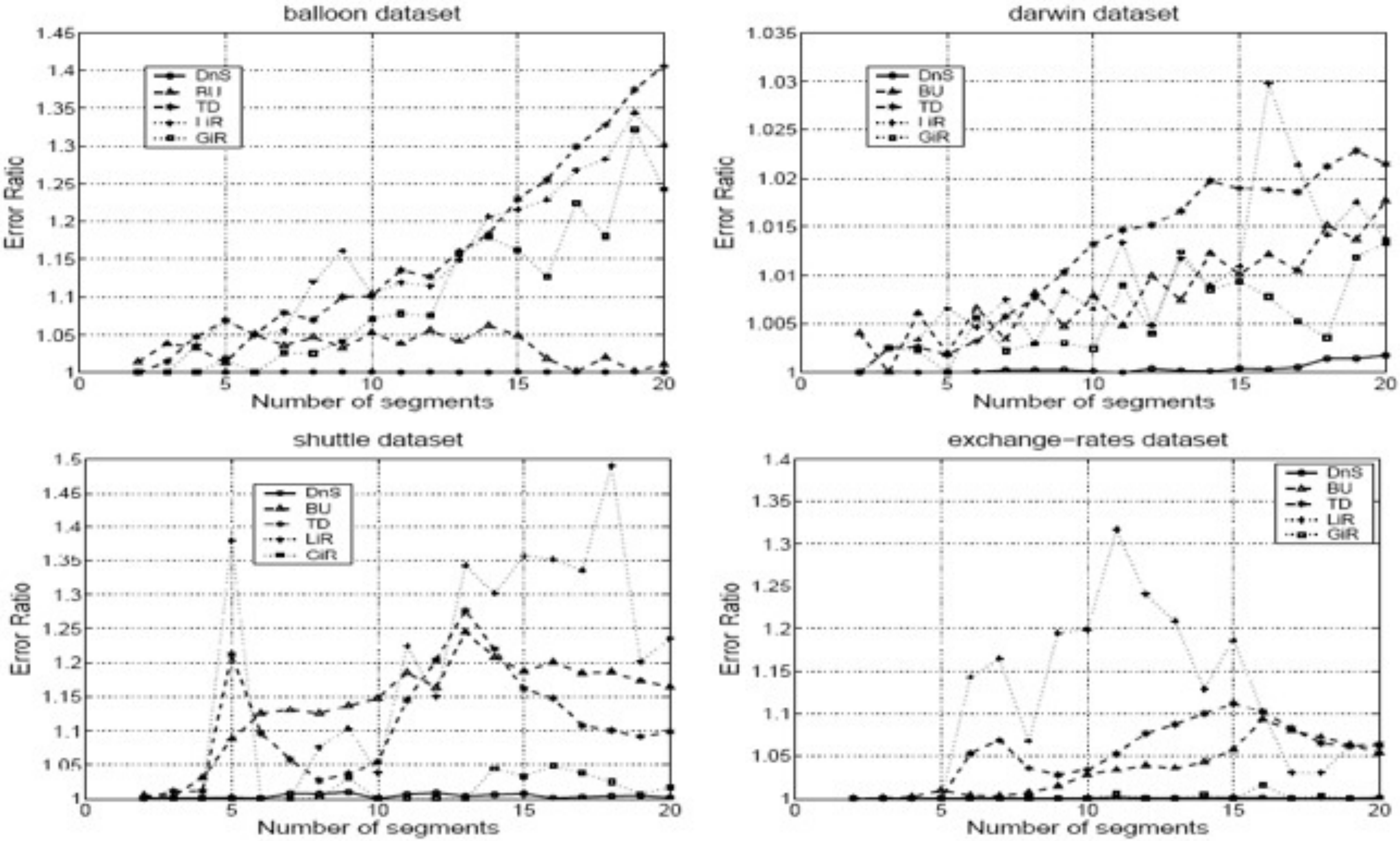

Speed vs. accuracy in practice