Link Analysis Ranking
How do search engines decide how to rank your query results?

• Guess why Google ranks the query results the way it does

• How would you do it?
Naïve ranking of query results

• Given query $q$
• Rank the web pages $p$ in the index based on $\text{sim}(p,q)$

• Scenarios where this is not such a good idea?
Why Link Analysis?

• First generation search engines
  – view documents as flat text files
  – could not cope with size, spamming, user needs
    • Example: Honda website, keywords: automobile manufacturer

• Second generation search engines
  – Ranking becomes critical
  – use of Web specific data: Link Analysis
  – shift from relevance to authoritativeness
  – a success story for the network analysis
Link Analysis: Intuition

• A link from page $p$ to page $q$ denotes endorsement
  – page $p$ considers page $q$ an authority on a subject
  – mine the web graph of recommendations
  – assign an authority value to every page
Link Analysis Ranking Algorithms

• Start with a collection of web pages

• Extract the underlying hyperlink graph

• Run the LAR algorithm on the graph

• Output: an authority weight for each node
Algorithm input

• **Query dependent**: rank a small subset of pages related to a specific query
  – HITS (Kleinberg 98) was proposed as query dependent

• **Query independent**: rank the whole Web
  – PageRank (Brin and Page 98) was proposed as query independent
Query-dependent LAR

• Given a query $q$, find a subset of web pages $S$ that are related to $S$
• Rank the pages in $S$ based on some ranking criterion
Query–dependent input

Root Set
Query-dependent input

IN

Root Set

OUT
Query dependent input
Query dependent input

Base Set

IN

Root Set

OUT
Properties of a good seed set $S$

- $S$ is relatively small.
- $S$ is rich in relevant pages.
- $S$ contains most (or many) of the strongest authorities.
How to construct a good seed set $S$

• For query $q$ first collect the $t$ highest-ranked pages for $q$ from a text-based search engine to form set $\Gamma$

• $S = \Gamma$

• Add to $S$ all the pages pointing to $\Gamma$

• Add to $S$ all the pages that pages from $\Gamma$ point to
Link Filtering

• Navigational links: serve the purpose of moving within a site (or to related sites)
  • www.espn.com → www.espn.com/nba
  • www.yahoo.com → www.yahoo.it
  • www.espn.com → www.msn.com

• Filter out navigational links
  – same domain name
  – same IP address
How do we rank the pages in seed set $S$?

• In degree?

• Intuition

• Problems
Hubs and Authorities [K98]

• Authority is not necessarily transferred directly between authorities

• Pages have double identity
  – hub identity
  – authority identity

• Good hubs point to good authorities

• Good authorities are pointed by good hubs
HITS Algorithm

• Initialize all weights to 1.
• Repeat until convergence
  – O operation: hubs collect the weight of the authorities
    \[ h_i = \sum_{j:i \rightarrow j} a_j \]
  – I operation: authorities collect the weight of the hubs
    \[ a_i = \sum_{j:j \rightarrow i} h_j \]
  – Normalize weights under some norm
HITS and eigenvectors

• The HITS algorithm is a power-method eigenvector computation
  – in vector terms $a^t = A^T h^{t-1}$ and $h^t = A a^{t-1}$
  – so $a^t = A^T A a^{t-1}$ and $h^t = A A^T h^{t-1}$
  – The authority weight vector $a$ is the eigenvector of $A^T A$ and the hub weight vector $h$ is the eigenvector of $A A^T$
  – Why do we need normalization?

• The vectors $a$ and $h$ are singular vectors of the matrix $A$
Singular Value Decomposition

\[ A = U \Sigma V^T \]

\[ U = [\vec{u}_1, \ldots, \vec{u}_r] \quad V = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r] \]

\[ \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_r) \]

- **r**: rank of matrix \( A \)

- \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r \): singular values (sq. roots of eig-vals \( AA^T, A^TA \))

- \( \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r \): left singular vectors (eig-vectors of \( AA^T \))

- \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r \): right singular vectors (eig-vectors of \( A^TA \))

\[ A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \ldots + \sigma_r \vec{u}_r \vec{v}_r^T \]
Singular Value Decomposition

• Linear trend $v$ in matrix $A$:
  – the tendency of the row vectors of $A$ to align with vector $v$
  – strength of the linear trend: $Av$

• SVD discovers the linear trends in the data

• $u_i, v_i$: the $i$–th strongest linear trends

• $\sigma_i$: the strength of the $i$–th strongest linear trend

• HITS discovers the strongest linear trend in the authority space
HITS and the TKC effect

• The HITS algorithm favors the most dense community of hubs and authorities
  – Tightly Knit Community (TKC) effect
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weight of node $p$ is proportional to the number of $(BF)^n$ paths that leave node $p$
HITS and the TKC effect

• The HITS algorithm favors the most dense community of hubs and authorities
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![Diagram](image-url)

after normalization with the max element as $n \to \infty$
Query–independent LAR

- Have an a–priori ordering of the web pages

- $Q$: Set of pages that contain the keywords in the query $q$

- Present the pages in $Q$ ordered according to order $\pi$

- **What are the advantages of such an approach?**
InDegree algorithm

• Rank pages according to in-degree
  \[ w_i = |B(i)| \]
PageRank algorithm [BP98]

- Good authorities should be pointed by good authorities
- Random walk on the web graph
  - pick a page at random
  - with probability $1 - \alpha$ jump to a random page
  - with probability $\alpha$ follow a random outgoing link
- Rank according to the stationary distribution

$$
PR(p) = \alpha \sum_{q \rightarrow p} \frac{PR(q)}{|F(q)|} + (1 - \alpha) \frac{1}{n}
$$

1. Red Page
2. Purple Page
3. Yellow Page
4. Blue Page
5. Green Page
Markov chains

• A Markov chain describes a discrete time stochastic process over a set of states
  \[ S = \{s_1, s_2, \ldots, s_n\} \]
  according to a transition probability matrix
  \[ P = \{P_{ij}\} \]
  – \( P_{ij} \) = probability of moving to state \( j \) when at state \( i \)
  • \( \sum_j P_{ij} = 1 \) (stochastic matrix)

• Memorylessness property: The next state of the chain depends only at the current state and not on the past of the process (first order MC)
  – higher order MCs are also possible
Random walks

• Random walks on graphs correspond to Markov Chains
  – The set of states $S$ is the set of nodes of the graph $G$
  – The transition probability matrix is the probability that we follow an edge from one node to another
An example

\[ A = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1
\end{bmatrix} \]

\[ P = \begin{bmatrix}
0 & 1/2 & 1/2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
1/3 & 1/3 & 1/3 & 0 & 0 \\
1/2 & 0 & 0 & 0 & 1/2
\end{bmatrix} \]
State probability vector

- The vector $q^t = (q_{t1}^t, q_{t2}^t, \ldots, q_{tn}^t)$ that stores the probability of being at state $i$ at time $t$

  $q_{0i}^0 = \text{the probability of starting from state } i$

  $q^t = q^{t-1} P$
An example

\[
P = \begin{bmatrix}
0 & 1/2 & 1/2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
1/3 & 1/3 & 1/3 & 0 & 0 \\
1/2 & 0 & 0 & 1/2 & 0 \\
\end{bmatrix}
\]

\[
q^{t+1}_1 = \frac{1}{3} q^t_4 + \frac{1}{2} q^t_5
\]

\[
q^{t+1}_2 = \frac{1}{2} q^t_1 + q^t_3 + \frac{1}{3} q^t_4
\]

\[
q^{t+1}_3 = \frac{1}{2} q^t_1 + \frac{1}{3} q^t_4
\]

\[
q^{t+1}_4 = \frac{1}{2} q^t_5
\]

\[
q^{t+1}_5 = q^t_2
\]
Stationary distribution

• A stationary distribution for a MC with transition matrix \( P \), is a probability distribution \( \pi \), such that \( \pi = \pi P \).

• A MC has a unique stationary distribution if
  – it is irreducible
    • the underlying graph is strongly connected
  – it is aperiodic
    • for random walks, the underlying graph is not bipartite
• The probability \( \pi_i \) is the fraction of times that we visited state \( i \) as \( t \to \infty \).
• The stationary distribution is an eigenvector of matrix \( P \)
  – the principal left eigenvector of \( P \) – stochastic matrices have maximum eigenvalue 1.
Computing the stationary distribution

• The Power Method
  – Initialize to some distribution $q^0$
  – Iteratively compute $q^t = q^{t-1}P$
  – After enough iterations $q^t \approx \pi$
  – Power method because it computes $q^t = q^0P^t$

• Why does it converge?
  – follows from the fact that any vector can be written as a linear combination of the eigenvectors
    • $q^0 = v_1 + c_2v_2 + \ldots + c_nv_n$

• Rate of convergence
  – determined by $\lambda_2^t$
The PageRank random walk

• Vanilla random walk
  – make the adjacency matrix stochastic and run a random walk

\[
P = \begin{bmatrix}
0 & 1/2 & 1/2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\
1/2 & 0 & 0 & 1/2 & 0 & 0 \\
\end{bmatrix}
\]
The PageRank random walk

• What about **sink** nodes?
  – what happens when the random walk moves to a node without any outgoing inks?

\[
P = \begin{bmatrix}
0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0
\end{bmatrix}
\]
The PageRank random walk

• Replace these row vectors with a vector $v$ – typically, the uniform vector

\[
P' = \begin{bmatrix}
0 & 1/2 & 1/2 & 0 & 0 \\
1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\
0 & 1 & 0 & 0 & 0 \\
1/3 & 1/3 & 1/3 & 0 & 0 \\
1/2 & 0 & 0 & 1/2 & 0 \\
\end{bmatrix}
\]

\[
P' = P + dv^T \quad d = \begin{cases} 1 & \text{if } i \text{ is sink} \\ 0 & \text{otherwise} \end{cases}
\]
The PageRank random walk

• How do we guarantee irreducibility?
  – add a random jump to vector \( \mathbf{v} \) with prob \( \alpha \)
  • typically, to a uniform vector

\[
P'' = \alpha \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{bmatrix} + (1-\alpha) \begin{bmatrix} 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{bmatrix}
\]

\[P'' = \alpha P' + (1-\alpha) \mathbf{uv}^T, \text{ where } \mathbf{u} \text{ is the vector of all } 1s\]
Effects of random jump

• Guarantees irreducibility
• Motivated by the concept of random surfer
• Offers additional flexibility
  – personalization
  – anti-spam
• Controls the rate of convergence
  – the second eigenvalue of matrix $P''$ is $\alpha$
A PageRank algorithm

- Performing vanilla power method is now too expensive – the matrix is not sparse

\[
q^0 = \nu \\
t = 1 \\
\text{repeat} \\
q^t = (P'')^T q^{t-1} \\
\delta = ||q^t - q^{t-1}|| \\
t = t + 1 \\
\text{until } \delta < \varepsilon
\]

Efficient computation of
\[
q^t = (P'')^T q^{t-1}
\]

\[
q^t = a P'T q^{t-1} \\
\beta = ||q^{t-1}||_1 - ||q^t||_1 \\
q^t = q^t + \beta \nu
\]
Random walks on undirected graphs

- In the stationary distribution of a random walk on an undirected graph, the probability of being at node $i$ is proportional to the (weighted) degree of the vertex

- Random walks on undirected graphs are not “interesting”
Research on PageRank

• Specialized PageRank
  – personalization [BP98]
    • instead of picking a node uniformly at random favor specific nodes that are related to the user
  – topic sensitive PageRank [H02]
    • compute many PageRank vectors, one for each topic
    • estimate relevance of query with each topic
    • produce final PageRank as a weighted combination

• Updating PageRank [Chien et al 2002]
• Fast computation of PageRank
  – numerical analysis tricks
  – node aggregation techniques
  – dealing with the “Web frontier”