More on Rankings

Query-independent LAR

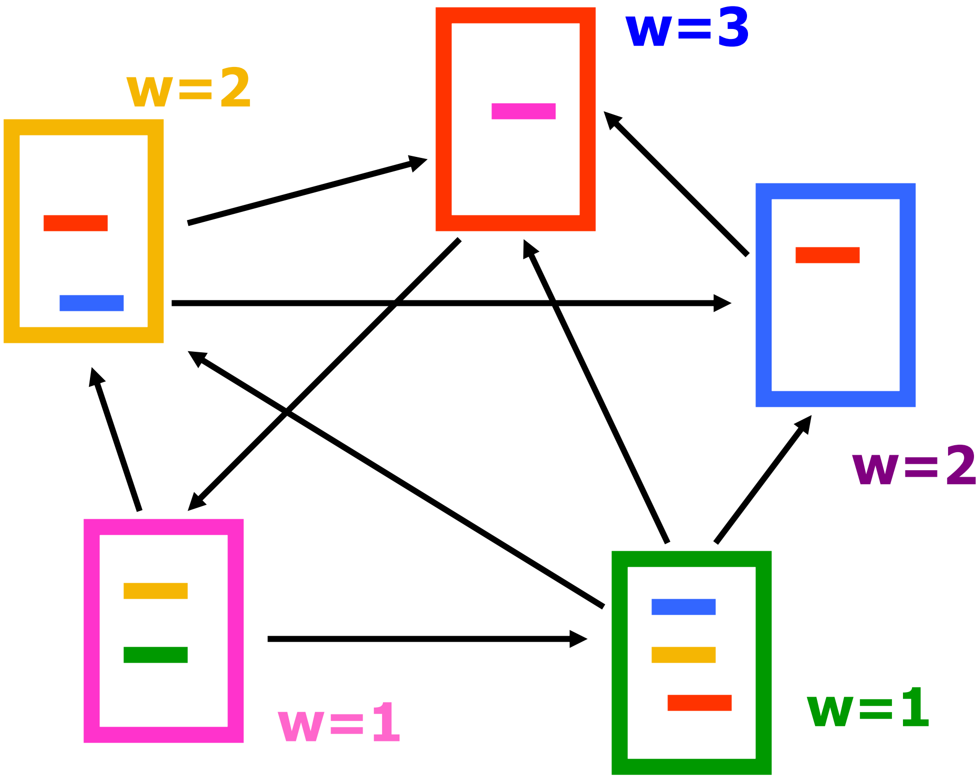
- Have an a-priori ordering of the web pages

- **Q**: Set of pages that contain the keywords in the query **q**
- Present the pages in **Q** ordered according to order **π**

- **What are the advantages of such an approach?**

InDegree algorithm

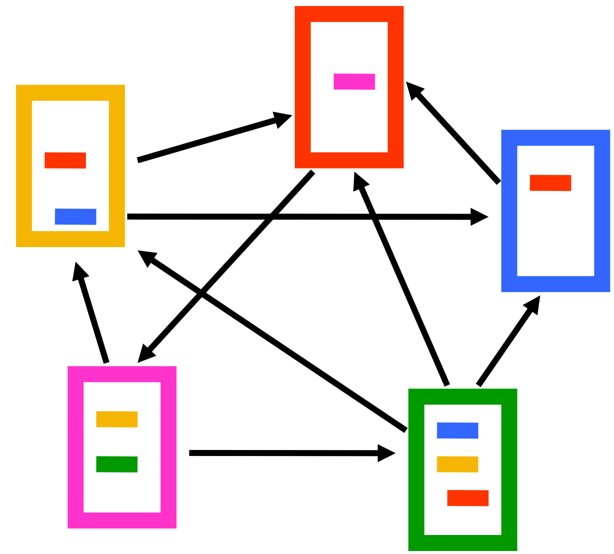
- Rank pages according to in-degree
  - $w_i = |B(i)|$



1. Red Page
2. Yellow Page
3. Blue Page
4. Purple Page
5. Green Page

PageRank algorithm [BP98]

- Good authorities should be pointed by good authorities
- Random walk on the web graph
  - pick a page at random
  - with probability 1- α jump to a random page
  - with probability α follow a random outgoing link
- Rank according to the stationary distribution
- $$PR(p) = \alpha \sum_{q \to p} \frac{PR(q)}{|F(q)|} + (1-\alpha)\frac{1}{n}$$



1. Red Page
2. Purple Page
3. Yellow Page
4. Blue Page
5. Green Page

Markov chains

- A Markov chain describes a discrete time stochastic process over a set of states

$$S = \{s_1, s_2, \dots s_n\}$$

according to a transition probability matrix

$$P = \{P_{ij}\}$$

  - P_{ij} = probability of moving to state j when at state i
    - $\sum_j P_{ij} = 1$ (stochastic matrix)

- Memorylessness property: The next state of the chain depends only at the current state and not on the past of the process (first order MC)
  - higher order MCs are also possible

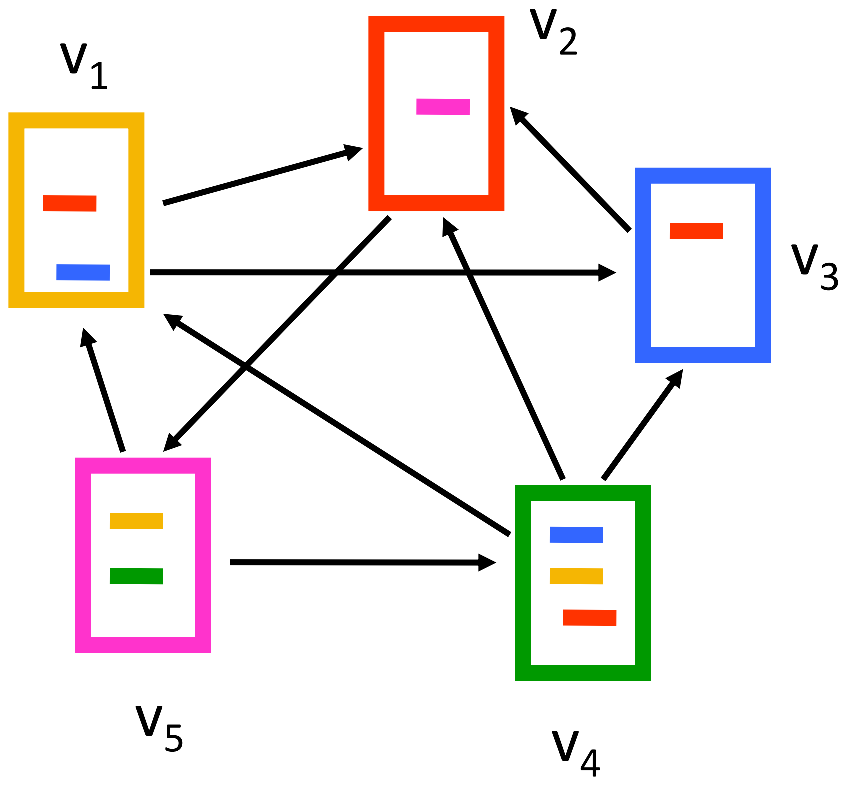
Random walks

- Random walks on graphs correspond to Markov Chains
  - The set of states S is the set of nodes of the graph G
  - The transition probability matrix is the probability that we follow an edge from one node to another

An example

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{bmatrix}$$



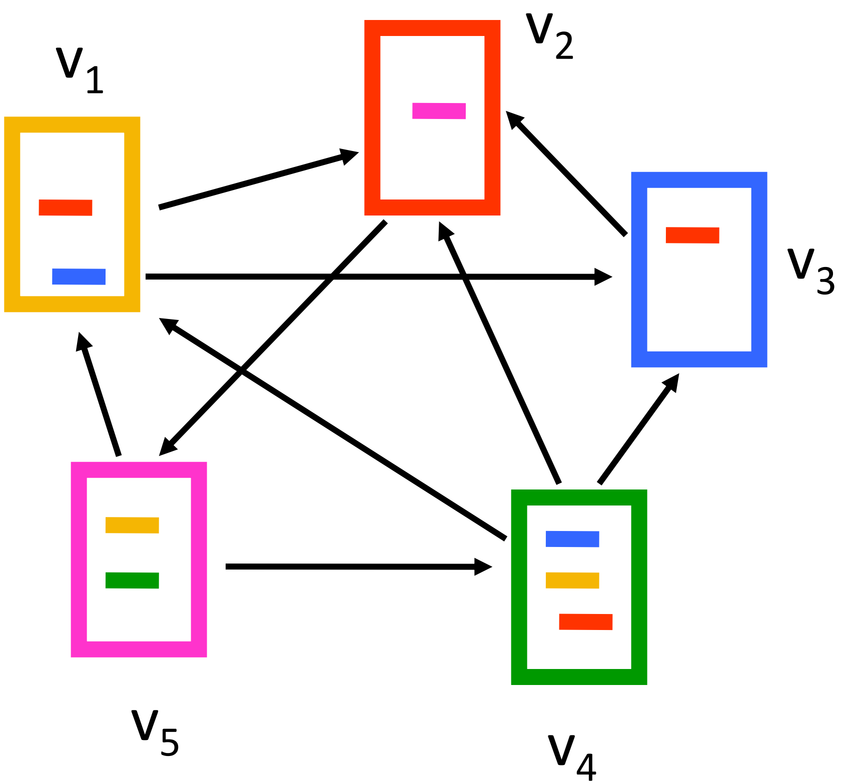
State probability vector

- The vector $q^t = (q^t_1, q^t_2, \dots, q^t_n)$ that stores the probability of being at state i at time t
  - q^0_i = the probability of starting from state i

$$q^t = q^{t-1} P$$

An example

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$



$q^{t+1}_1 = 1/3\ q^t_4 + 1/2\ q^t_5$

$q^{t+1}_2 = 1/2\ q^t_1 + q^t_3 + 1/3\ q^t_4$

$q^{t+1}_3 = 1/2\ q^t_1 + 1/3\ q^t_4$

$q^{t+1}_4 = 1/2\ q^t_5$

$q^{t+1}_5 = q^t_2$

Stationary distribution

- A stationary distribution for a MC with transition matrix P, is a probability distribution π, such that $\pi = \pi P$

- A MC has a unique stationary distribution if
  - it is irreducible
    - the underlying graph is strongly connected
  - it is aperiodic
    - for random walks, the underlying graph is not bipartite
- The probability π_i is the fraction of times that we visited state i as $t \to \infty$
- The stationary distribution is an eigenvector of matrix P
  - the principal left eigenvector of P – stochastic matrices have maximum eigenvalue 1

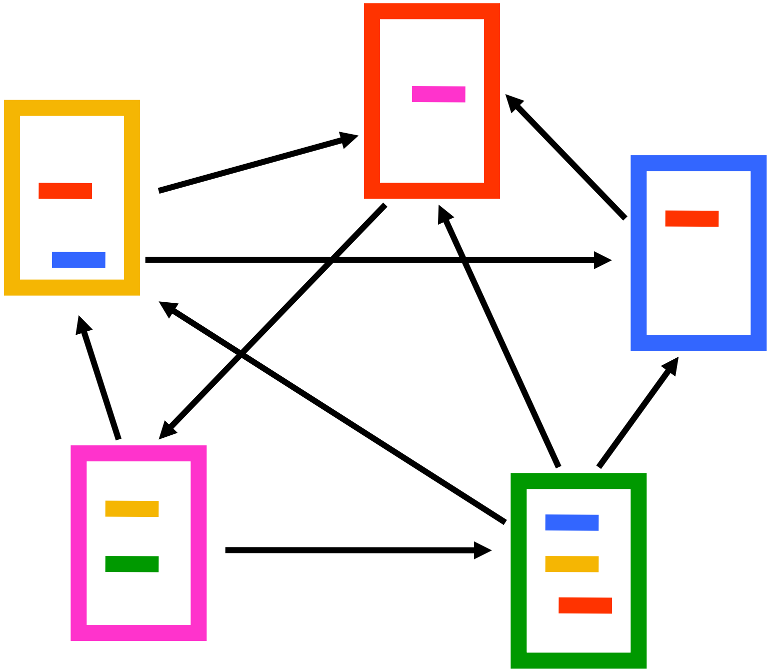
Computing the stationary distribution

- The Power Method
  - Initialize to some distribution q^0
  - Iteratively compute $q^t = q^{t-1}P$
  - After enough iterations $q^t \approx \pi$
  - Power method because it computes $q^t = q^0P^t$
- Rate of convergence
  - determined by λ_2

The PageRank random walk

- Vanilla random walk
  - make the adjacency matrix stochastic and run a random walk

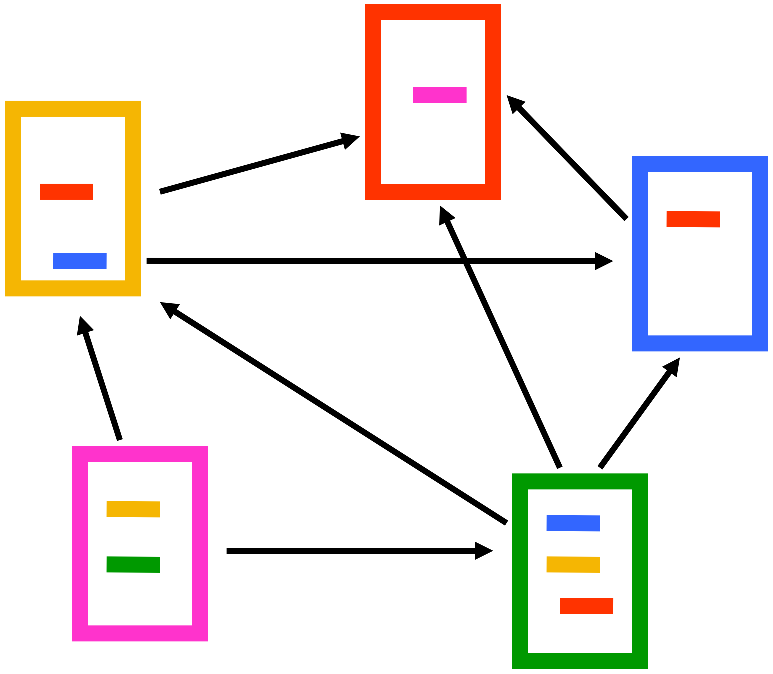
$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$



The PageRank random walk

- What about sink nodes?
  - what happens when the random walk moves to a node without any outgoing inks?

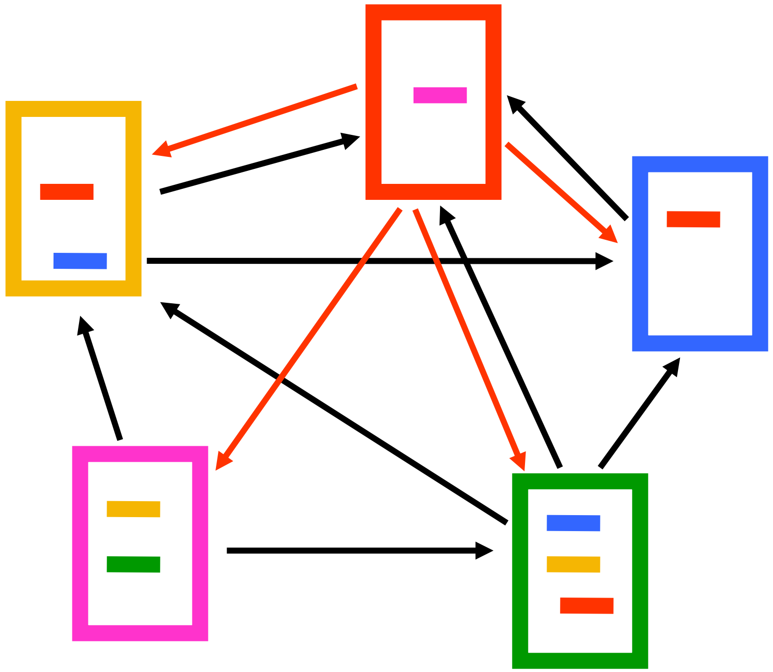
$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$



The PageRank random walk

- Replace these row vectors with a vector v
  - typically, the uniform vector

$$P' = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$



$P' = P + dv^T$

$$d = \begin{cases} 1 & \text{if i is sink} \\ 0 & \text{otherwise} \end{cases}$$

The PageRank random walk

- How do we guarantee irreducibility?
  - add a random jump to vector v with prob α
    - typically, to a uniform vector

$$P'' = \alpha \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{bmatrix} + (1-\alpha) \begin{bmatrix} 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{bmatrix}$$

$P'' = \alpha P' + (1-\alpha)uv^T$, where u is the vector of all 1s

Effects of random jump

- Guarantees irreducibility
- Motivated by the concept of random surfer
- Offers additional flexibility
  - personalization
  - anti-spam
- Controls the rate of convergence
  - the second eigenvalue of matrix P'' is α

A PageRank algorithm

- Performing vanilla power method is now too expensive – the matrix is not sparse

$q^0 = v$

$t = 1$

repeat

$\quad q^t = (P'')^T q^{t-1}$

$\quad \delta = \|q^t - q^{t-1}\|$

$\quad t = t + 1$

until $\delta < \varepsilon$

Efficient computation of $y = (P'')^T x$

$y = \alpha P^T x$

$\beta = \|x\|_1 - \|y\|_1$

$y = y + \beta v$

Random walks on undirected graphs

- In the stationary distribution of a random walk on an undirected graph, the probability of being at node i is proportional to the (weighted) degree of the vertex

- Random walks on undirected graphs are not "interesting"

Research on PageRank

- Specialized PageRank
  - personalization [BP98]
    - instead of picking a node uniformly at random favor specific nodes that are related to the user
  - topic sensitive PageRank [H02]
    - compute many PageRank vectors, one for each topic
    - estimate relevance of query with each topic
    - produce final PageRank as a weighted combination
- Updating PageRank [Chien et al 2002]
- Fast computation of PageRank
  - numerical analysis tricks
  - node aggregation techniques
  - dealing with the "Web frontier"

Topic-sensitive pagerank

- HITS-based scores are very inefficient to compute
- PageRank scores are independent of the queries
- Can we bias PageRank rankings to take into account query keywords?

Topic-sensitive PageRank

Topic-sensitive PageRank

- Conventional PageRank computation:
- $r^{(t+1)}(v)=\Sigma_{u\in N(v)} r^{(t)}(u)/d(v)$
- $N(v)$: neighbors of v
- $d(v)$: degree of v
- $r = Mxr$
- $M' = (1-\alpha)P + \alpha\left[1/n\right]_{nxn}$
- $r = (1-\alpha)Pr + \alpha\left[1/n\right]_{nxn}r = (1-\alpha)Mr + \alpha p$
- $p = [1/n]_{nx1}$

Topic-sensitive PageRank

- $\mathbf{r} = (1-\alpha)\mathbf{Pr} + \alpha\mathbf{p}$
- **Conventional PageRank:** **p** is a uniform vector with values **1/n**

- Topic-sensitive PageRank uses a **non-uniform** personalization vector **p**

- Not simply a post-processing step of the PageRank computation

- Personalization vector **p** introduces bias in all iterations of the iterative computation of the PageRank vector

Personalization vector

- In the random-walk model, the personalization vector represents the addition of a set of transition edges, where the probability of an artificial edge **(u,v)** is αp_v

- Given a graph the result of the PageRank computation only depends on **α** and **p :** **PR(α,p)**

Topic-sensitive PageRank: Overall approach

- Preprocessing
  - Fix a set of **k** topics
  - For each topic c_j compute the PageRank scores of page **u** wrt to the **j**-th topic: $r(u,j)$

- Query-time processing:
  - For query q compute the total score of page **u** wrt **q** as $score(u,q) = \Sigma_{j=1\ldots k} Pr(c_j|q)\ r(u,j)$

Topic-sensitive PageRank: Preprocessing

- Create **k** different biased PageRank vectors using some pre-defined set of k categories **($c_1,\ldots,c_k$)**
- **T_j**: set of URLs in the **j-**th category
- Use non-uniform personalization vector **$p=w_j$** such that:

$$w_j(v) = \begin{cases} \dfrac{1}{T_j}, v \in T_j \\ 0, \text{o/w} \end{cases}$$

Topic-sensitive PageRank: Query-time processing

- **D_j**: class term vectors consisting of all the terms appearing in the **k** pre-selected categories

$$\Pr(c_j \mid q) = \frac{\Pr(c_j)\Pr(q \mid c_j)}{\Pr(q)} \propto \Pr(c_j)\prod_i \Pr(q_i \mid c_j)$$

- How can we compute **$P(c_j)$**?
- How can we compute **$Pr(q_i \mid c_j)$**?

- Comparing results of Link Analysis Ranking algorithms
- Comparing and aggregating rankings

Comparing LAR vectors

$$w_1 = [\ 1 \quad 0.8 \quad 0.5 \quad 0.3 \quad 0\ ]$$

$$w_2 = [\ 0.9 \quad 1 \quad 0.7 \quad 0.6 \quad 0.8\ ]$$

- How close are the LAR vectors w_1, w_2?

Distance between LAR vectors

- Geometric distance: how close are the numerical weights of vectors w_1, w_2?

$$d_1(w_1, w_2) = \sum \left| w_1[i] - w_2[i] \right|$$

$$w_1 = [\ 1.0 \quad 0.8 \quad 0.5 \quad 0.3 \quad 0.0\ ]$$

$$w_2 = [\ 0.9 \quad 1.0 \quad 0.7 \quad 0.6 \quad 0.8\ ]$$

$$d_1(w_1, w_2) = \quad 0.1+0.2+0.2+0.3+0.8 = 1.6$$

Distance between LAR vectors

- Rank distance: how close are the ordinal rankings induced by the vectors w_1, w_2?
  - Kendal's τ distance

$$d_r(w_1, w_2) = \frac{\text{pairs ranked in a different order}}{\text{total number of distinct pairs}}$$

Outline

- Rank Aggregation
  - Computing aggregate scores
  - Computing aggregate rankings - voting

Rank Aggregation

- Given a set of rankings $R_1,R_2,\ldots,R_m$ of a set of objects $X_1,X_2,\ldots,X_n$ produce a single ranking R that is in agreement with the existing rankings

Examples

- Voting
  - rankings $R_1,R_2,\ldots,R_m$ are the voters, the objects $X_1,X_2,\ldots,X_n$ are the candidates.

Examples

- Combining multiple scoring functions
  - rankings $R_1, R_2, \ldots, R_m$ are the scoring functions, the objects $X_1, X_2, \ldots, X_n$ are data items.
    - Combine the PageRank scores with term-weighting scores
    - Combine scores for multimedia items
      - color, shape, texture
    - Combine scores for database tuples
      - find the best hotel according to price and location

Examples

- Combining multiple sources
  - rankings $R_1, R_2, \ldots, R_m$ are the sources, the objects $X_1, X_2, \ldots, X_n$ are data items.
    - meta-search engines for the Web
    - distributed databases
    - P2P sources

Variants of the problem

- Combining scores
  - we know the scores assigned to objects by each ranking, and we want to compute a single score
- Combining ordinal rankings
  - the scores are not known, only the ordering is known
  - the scores are known but we do not know how, or do not want to combine them
    - e.g. price and star rating

Combining scores

- Each object X_i has m scores $(r_{i1}, r_{i2}, \ldots, r_{im})$
- The score of object X_i is computed using an aggregate scoring function $f(r_{i1}, r_{i2}, \ldots, r_{im})$

	R_1	R_2	R_3
X_1	1	0.3	0.2
X_2	0.8	0.8	0
X_3	0.5	0.7	0.6
X_4	0.3	0.2	0.8
X_5	0.1	0.1	0.1

Combining scores

- Each object X_i has m scores $(r_{i1}, r_{i2}, \ldots, r_{im})$
- The score of object X_i is computed using an aggregate scoring function $f(r_{i1}, r_{i2}, \ldots, r_{im})$
  - $f(r_{i1}, r_{i2}, \ldots, r_{im}) = \min\{r_{i1}, r_{i2}, \ldots, r_{im}\}$

	R_1	R_2	R_3	R
X_1	1	0.3	0.2	0.2
X_2	0.8	0.8	0	0
X_3	0.5	0.7	0.6	0.5
X_4	0.3	0.2	0.8	0.2
X_5	0.1	0.1	0.1	0.1

Combining scores

- Each object X_i has m scores $(r_{i1}, r_{i2}, \ldots, r_{im})$
- The score of object X_i is computed using an aggregate scoring function $f(r_{i1}, r_{i2}, \ldots, r_{im})$
  - $f(r_{i1}, r_{i2}, \ldots, r_{im}) = \max\{r_{i1}, r_{i2}, \ldots, r_{im}\}$

	R_1	R_2	R_3	R
X_1	1	0.3	0.2	1
X_2	0.8	0.8	0	0.8
X_3	0.5	0.7	0.6	0.7
X_4	0.3	0.2	0.8	0.8
X_5	0.1	0.1	0.1	0.1

Combining scores

- Each object X_i has m scores $(r_{i1},r_{i2},\dots,r_{im})$
- The score of object X_i is computed using an aggregate scoring function $f(r_{i1},r_{i2},\dots,r_{im})$
  - $f(r_{i1},r_{i2},\dots,r_{im}) = r_{i1} + r_{i2} + \dots + r_{im}$

	R_1	R_2	R_3	R
X_1	1	0.3	0.2	1.5
X_2	0.8	0.8	0	1.6
X_3	0.5	0.7	0.6	1.8
X_4	0.3	0.2	0.8	1.3
X_5	0.1	0.1	0.1	0.3

Top-k

- Given a set of n objects and m scoring lists sorted in decreasing order, find the top-k objects according to a scoring function f
- top-k: a set T of k objects such that $f(r_{j1},\dots,r_{jm}) \leq f(r_{i1},\dots,r_{im})$ for every object X_i in T and every object X_j not in T
- Assumption: The function f is monotone
  - $f(r_1,\dots,r_m) \leq f(r_1',\dots,r_m')$ if $r_i \leq r_i'$ for all i
- Objective: Compute top-k with the minimum cost

Cost function

- We want to minimize the number of accesses to the scoring lists
- Sorted accesses: sequentially access the objects in the order in which they appear in a list
  - cost C_s
- Random accesses: obtain the cost value for a specific object in a list
  - cost C_r
- If s sorted accesses and r random accesses minimize $s\,C_s + r\,C_r$

Example

R_1	
X_1	1
X_2	0.8
X_3	0.5
X_4	0.3
X_5	0.1

R_2	
X_2	0.8
X_3	0.7
X_1	0.3
X_4	0.2
X_5	0.1

R_3	
X_4	0.8
X_3	0.6
X_1	0.2
X_5	0.1
X_2	0

- Compute top-2 for the sum aggregate function

Fagin's Algorithm

1. Access sequentially all lists in parallel until there are k objects that have been seen in all lists

R_1	
X_1	1
X_2	0.8
X_3	0.5
X_4	0.3
X_5	0.1

R_2	
X_2	0.8
X_3	0.7
X_1	0.3
X_4	0.2
X_5	0.1

R_3	
X_4	0.8
X_3	0.6
X_1	0.2
X_5	0.1
X_2	0

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R_1	
X_1	1
X_2	0.8
X_3	0.5
X_4	0.3
X_5	0.1

R_2	
X_2	0.8
X_3	0.7
X_1	0.3
X_4	0.2
X_5	0.1

R_3	
X_4	0.8
X_3	0.6
X_1	0.2
X_5	0.1
X_2	0

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R_1	
X_1	1
X_2	0.8
X_3	0.5
X_4	0.3
X_5	0.1

R_2	
X_2	0.8
X_3	0.7
X_1	0.3
X_4	0.2
X_5	0.1

R_3	
X_4	0.8
X_3	0.6
X_1	0.2
X_5	0.1
X_2	0

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R_1	
X_1	1
X_2	0.8
X_3	0.5
X_4	0.3
X_5	0.1

R_2	
X_2	0.8
X_3	0.7
X_1	0.3
X_4	0.2
X_5	0.1

R_3	
X_4	0.8
X_3	0.6
X_1	0.2
X_5	0.1
X_2	0

Fagin's Algorithm

1. Access sequentially all lists in parallel until there are k objects that have been seen in all lists

R_1	
X_1	1
X_2	0.8
X_3	0.5
X_4	0.3
X_5	0.1

R_2	
X_2	0.8
X_3	0.7
X_1	0.3
X_4	0.2
X_5	0.1

R_3	
X_4	0.8
X_3	0.6
X_1	0.2
X_5	0.1
X_2	0

Fagin's Algorithm

2. Perform random accesses to obtain the scores of all seen objects

R_1	
X_1	1
X_2	0.8
X_3	0.5
X_4	0.3
X_5	0.1

R_2	
X_2	0.8
X_3	0.7
X_1	0.3
X_4	0.2
X_5	0.1

R_3	
X_4	0.8
X_3	0.6
X_1	0.2
X_5	0.1
X_2	0

Fagin's Algorithm

3. Compute score for all objects and find the top-k

R_1	
X_1	1
X_2	0.8
X_3	0.5
X_4	0.3
X_5	0.1

R_2	
X_2	0.8
X_3	0.7
X_1	0.3
X_4	0.2
X_5	0.1

R_3	
X_4	0.8
X_3	0.6
X_1	0.2
X_5	0.1
X_2	0

R	
X_3	1.8
X_2	1.6
X_1	1.5
X_4	1.3

Fagin's Algorithm

- X_5 cannot be in the top-2 because of the monotonicity property
  - $f(X_5) \leq f(X_1) \leq f(X_3)$

R_1	
X_1	1
X_2	0.8
X_3	0.5
X_4	0.3
X_5	0.1

R_2	
X_2	0.8
X_3	0.7
X_1	0.3
X_4	0.2
X_5	0.1

R_3	
X_4	0.8
X_3	0.6
X_1	0.2
X_5	0.1
X_2	0

R	
X_3	1.8
X_2	1.6
X_1	1.5
X_4	1.3

Fagin's Algorithm

- The algorithm is cost optimal under some probabilistic assumptions for a restricted class of aggregate functions

Threshold algorithm

1. Access the elements sequentially

R_1	
X_1	1
X_2	0.8
X_3	0.5
X_4	0.3
X_5	0.1

R_2	
X_2	0.8
X_3	0.7
X_1	0.3
X_4	0.2
X_5	0.1

R_3	
X_4	0.8
X_3	0.6
X_1	0.2
X_5	0.1
X_2	0

Threshold algorithm

1. At each sequential access
   a. Set the threshold t to be the aggregate of the scores seen in this access

R_1	
X_1	1
X_2	0.8
X_3	0.5
X_4	0.3
X_5	0.1

R_2	
X_2	0.8
X_3	0.7
X_1	0.3
X_4	0.2
X_5	0.1

R_3	
X_4	0.8
X_3	0.6
X_1	0.2
X_5	0.1
X_2	0

t = 2.6

Threshold algorithm

1. At each sequential access
   b. Do random accesses and compute the score of the objects seen

R_1	
X_1	1
X_2	0.8
X_3	0.5
X_4	0.3
X_5	0.1

R_2	
X_2	0.8
X_3	0.7
X_1	0.3
X_4	0.2
X_5	0.1

R_3	
X_4	0.8
X_3	0.6
X_1	0.2
X_5	0.1
X_2	0

$t = 2.6$

X_1	1.5
X_2	1.6
X_4	1.3

Threshold algorithm

1. At each sequential access
   c. Maintain a list of top-k objects seen so far

R_1	
X_1	1
X_2	0.8
X_3	0.5
X_4	0.3
X_5	0.1

R_2	
X_2	0.8
X_3	0.7
X_1	0.3
X_4	0.2
X_5	0.1

R_3	
X_4	0.8
X_3	0.6
X_1	0.2
X_5	0.1
X_2	0

t = 2.6

X_2	1.6
X_1	1.5

Threshold algorithm

1. At each sequential access
   d. When the scores of the top-k are greater or equal to the threshold, stop

R_1	
X_1	1
X_2	0.8
X_3	0.5
X_4	0.3
X_5	0.1

R_2	
X_2	0.8
X_3	0.7
X_1	0.3
X_4	0.2
X_5	0.1

R_3	
X_4	0.8
X_3	0.6
X_1	0.2
X_5	0.1
X_2	0

$t = 2.1$

X_3	1.8
X_2	1.6

Threshold algorithm

1. At each sequential access
   d. When the scores of the top-k are greater or equal to the threshold, stop

R_1	
X_1	1
X_2	0.8
X_3	0.5
X_4	0.3
X_5	0.1

R_2	
X_2	0.8
X_3	0.7
X_1	0.3
X_4	0.2
X_5	0.1

R_3	
X_4	0.8
X_3	0.6
X_1	0.2
X_5	0.1
X_2	0

$t = 1.0$

X_3	1.8
X_2	1.6

Threshold algorithm

2. Return the top-k seen so far

R_1	
X_1	1
X_2	0.8
X_3	0.5
X_4	0.3
X_5	0.1

R_2	
X_2	0.8
X_3	0.7
X_1	0.3
X_4	0.2
X_5	0.1

R_3	
X_4	0.8
X_3	0.6
X_1	0.2
X_5	0.1
X_2	0

$t = 1.0$

X_3	1.8
X_2	1.6

Threshold algorithm

- From the monotonicity property for any object not seen, the score of the object is less than the threshold
  - $f(X_5) \leq t \leq f(X_2)$

- The algorithm is instance cost-optimal
  - within a constant factor of the best algorithm on any database

Combining rankings

- In many cases the scores are not known
  - e.g. meta-search engines – scores are proprietary information
- … or we do not know how they were obtained
  - one search engine returns score 10, the other 100. What does this mean?
- … or the scores are incompatible
  - apples and oranges: does it make sense to combine price with distance?

- In this cases we can only work with the rankings

The problem

- Input: a set of rankings $R_1,R_2,\ldots,R_m$ of the objects $X_1,X_2,\ldots,X_n$. Each ranking R_i is a total ordering of the objects
  - for every pair X_i,X_j either X_i is ranked above X_j or X_j is ranked above X_i
- Output: A total ordering R that aggregates rankings $R_1,R_2,\ldots,R_m$

Voting theory

- A voting system is a rank aggregation mechanism
- Long history and literature
  - criteria and axioms for good voting systems

What is a good voting system?

- The Condorcet criterion
  – if object A defeats every other object in a pairwise majority vote, then A should be ranked first

- Extended Condorcet criterion
  – if the objects in a set X defeat in pairwise comparisons the objects in the set Y then the objects in X should be ranked above those in Y

- Not all voting systems satisfy the Condorcet criterion!

Pairwise majority comparisons

- Unfortunately the Condorcet winner does not always exist
  - irrational behavior of groups

	V_1	V_2	V_3
1	A	B	C
2	B	C	A
3	C	A	B

A > B B > C C > A

Pairwise majority comparisons

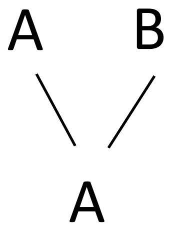
- Resolve cycles by imposing an agenda

	V_1	V_2	V_3
1	A	D	E
2	B	E	A
3	C	A	B
4	D	B	C
5	E	C	D

Pairwise majority comparisons

- Resolve cycles by imposing an agenda

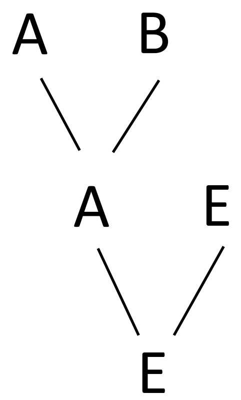
	V_1	V_2	V_3
1	A	D	E
2	B	E	A
3	C	A	B
4	D	B	C
5	E	C	D



Pairwise majority comparisons

- Resolve cycles by imposing an agenda

	V_1	V_2	V_3
1	A	D	E
2	B	E	A
3	C	A	B
4	D	B	C
5	E	C	D



Pairwise majority comparisons

- Resolve cycles by imposing an agenda

	V_1	V_2	V_3
1	A	D	E
2	B	E	A
3	C	A	B
4	D	B	C
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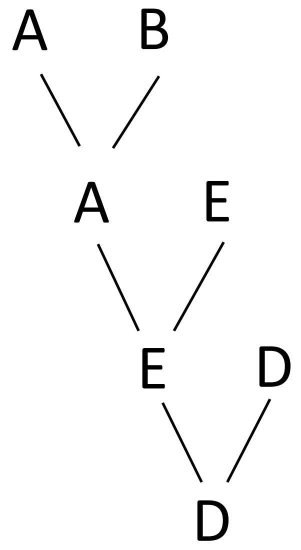
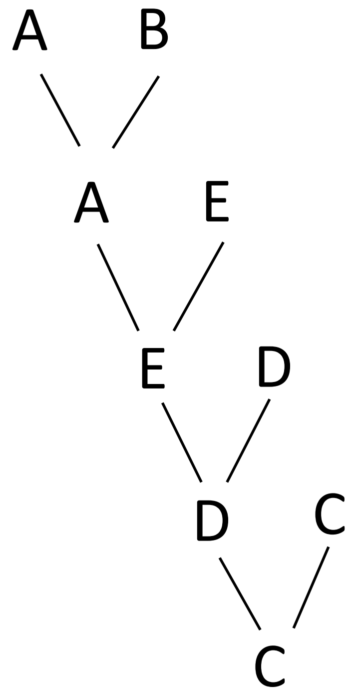


Pairwise majority comparisons

- Resolve cycles by imposing an agenda

	V_1	V_2	V_3
1	A	D	E
2	B	E	A
3	C	A	B
4	D	B	C
5	E	C	D

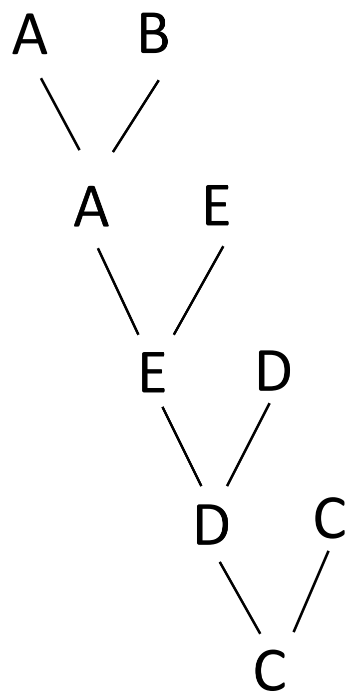


- C is the winner

Pairwise majority comparisons

- Resolve cycles by imposing an agenda

	V_1	V_2	V_3
1	A	D	E
2	B	E	A
3	C	A	B
4	D	B	C
5	E	C	D



- But everybody prefers A or B over C

Pairwise majority comparisons

- The voting system is not Pareto optimal
  - there exists another ordering that everybody prefers

- Also, it is sensitive to the order of voting

Plurality vote

- Elect first whoever has more 1st position votes

voters	10	8	7
1	A	C	B
2	B	A	C
3	C	B	A

- Does not find a Condorcet winner (C in this case)

Plurality with runoff

- If no-one gets more than 50% of the 1st position votes, take the majority winner of the first two

voters	10	8	7	2
1	A	C	B	B
2	B	A	C	A
3	C	B	A	C

first round: A 10, B 9, C 8
second round: A 18, B 9
winner: A

Plurality with runoff

- If no-one gets more than 50% of the 1st position votes, take the majority winner of the first two

voters	10	8	7	2
1	A	C	B	A
2	B	A	C	B
3	C	B	A	C

change the order of A and B in the last column

first round: A 12, B 7, C 8
second round: A 12, C 15
winner: C!

Positive Association axiom

- Plurality with runoff violates the positive association axiom

- Positive association axiom: positive changes in preferences for an object should not cause the ranking of the object to decrease

Borda Count

- For each ranking, assign to object X, number of points equal to the number of objects it defeats
  - first position gets n-1 points, second n-2, …, last 0 points
- The total weight of X is the number of points it accumulates from all rankings

Borda Count

voters	3	2	2
1 (3p)	A	B	C
2 (2p)	B	C	D
3 (1p)	C	D	A
4 (0p)	D	A	B

A: 3*3 + 2*0 + 2*1 = 11p
B: 3*2 + 2*3 + 2*0 = 12p
C: 3*1 + 2*2 + 2*3 = 13p
D: 3*0 + 2*1 + 2*2 = 6p

BC
C
B
A
D

- Does not always produce Condorcet winner

Borda Count

- Assume that D is removed from the vote

voters	3	2	2
1 (2p)	A	B	C
2 (1p)	B	C	A
3 (0p)	C	A	B

A: 3*2 + 2*0 + 2*1 = 7p
B: 3*1 + 2*2 + 2*0 = 7p
C: 3*0 + 2*1 + 2*2 = 6p

BC
B
A
C

- Changing the position of D changes the order of the other elements!

Independence of Irrelevant Alternatives

- The relative ranking of X and Y should not depend on a third object Z
  - heavily debated axiom

Borda Count

- The Borda Count of an an object X is the aggregate number of pairwise comparisons that the object X wins
  - follows from the fact that in one ranking X wins all the pairwise comparisons with objects that are under X in the ranking

Voting Theory

- Is there a voting system that does not suffer from the previous shortcomings?

Arrow's Impossibility Theorem

- There is no voting system that satisfies the following axioms
  - Universality
    - all inputs are possible
  - Completeness and Transitivity
    - for each input we produce an answer and it is meaningful
  - Positive Assosiation
  - Independence of Irrelevant Alternatives
  - Non-imposition
  - Non-dictatoriship

- **KENNETH J. ARROW** *Social Choice and Individual Values* (1951). Won Nobel Prize in 1972

Kemeny Optimal Aggregation

- Kemeny distance $K(R_1,R_2)$: The number of pairs of nodes that are ranked in a different order (Kendall-tau)
  - number of bubble-sort swaps required to transform one ranking into another
- Kemeny optimal aggregation minimizes

$$K(R, R_1, \ldots, R_m) = \sum_{i=1}^{m} K(R, R_i)$$

- Kemeny optimal aggregation satisfies the Condorcet criterion and the extended Condorcet criterion
  - maximum likelihood interpretation: produces the ranking that is most likely to have generated the observed rankings
- …but it is NP-hard to compute
  - easy 2-approximation by obtaining the best of the input rankings, but it is not “interesting”

Locally Kemeny optimal aggregation

- A ranking R is locally Kemeny optimal if there is no bubble-sort swap that produces a ranking R' such that $K(R',R_1,\ldots,R_m) \leq K(R',R_1,\ldots,R_m)$
- Locally Kemeny optimal is not necessarily Kemeny optimal
- Definitions apply for the case of partial lists also

Locally Kemeny optimal aggregation

- Locally Kemeny optimal aggregation can be computed in polynomial time
  - At the i-th iteration insert the i-th element x in the bottom of the list, and bubble it up until there is an element y such that the majority places y over x
- Locally Kemeny optimal aggregation satisfies the Condorcet and extended Condorcet criterion

Rank Aggregation algorithm [DKNS01]

- Start with an aggregated ranking and make it into a locally Kemeny optimal aggregation
- How do we select the initial aggregation?
  - Use another aggregation method
  - Create a Markov Chain where you move from an object X, to another object Y that is ranked higher by the majority

Spearman's footrule distance

- Spearman's footrule distance: The difference between the ranks R(i) and R'(i) assigned to object i

$$F(R,R') = \sum_{i=1}^{n} \left| R(i) - R'(i) \right|$$

- Relation between Spearman's footrule and Kemeny distance

$$K(R,R') \le F(R,R') \le 2K(R,R')$$

Spearman's footrule aggregation

- Find the ranking R, that minimizes

$$F(R, R_1, \ldots, R_m) = \sum_{i=1}^{m} F(R, R_i)$$

- The optimal Spearman's footrule aggregation can be computed in polynomial time
  - It also gives a 2-approximation to the Kemeny optimal aggregation

- If the median ranks of the objects are unique then this ordering is optimal

Example

R_1	
1	A
2	B
3	C
4	D

R_2	
1	B
2	A
3	D
4	C

R_3	
1	B
2	C
3	A
4	D

R	
1	B
2	A
3	C
4	D

A: (1 , 2 , 3)

B: (1 , 1 , 2)

C: (3 , 3 , 4)

D: (3 , 4 , 4)

The MedRank algorithm

- Access the rankings sequentially

R_1	
1	A
2	B
3	C
4	D

R_2	
1	B
2	A
3	D
4	C

R_3	
1	B
2	C
3	A
4	D

R	
1	
2	
3	
4	

The MedRank algorithm

- Access the rankings sequentially
  - when an element has appeared in more than half of the rankings, output it in the aggregated ranking

R_1	
1	A
2	B
3	C
4	D

R_2	
1	B
2	A
3	D
4	C

R_3	
1	B
2	C
3	A
4	D

R	
1	B
2	
3	
4	

The MedRank algorithm

- Access the rankings sequentially
  - when an element has appeared in more than half of the rankings, output it in the aggregated ranking

R_1	
1	A
2	B
3	C
4	D

R_2	
1	B
2	A
3	D
4	C

R_3	
1	B
2	C
3	A
4	D

R	
1	B
2	A
3	
4	

The MedRank algorithm

- Access the rankings sequentially
  - when an element has appeared in more than half of the rankings, output it in the aggregated ranking

R_1	
1	A
2	B
3	C
4	D

R_2	
1	B
2	A
3	D
4	C

R_3	
1	B
2	C
3	A
4	D

R	
1	B
2	A
3	C
4	

The MedRank algorithm

- Access the rankings sequentially
  - when an element has appeared in more than half of the rankings, output it in the aggregated ranking

R_1	
1	A
2	B
3	C
4	D

R_2	
1	B
2	A
3	D
4	C

R_3	
1	B
2	C
3	A
4	D

R	
1	B
2	A
3	C
4	D

The Spearman's rank correlation

- Spearman's rank correlation

$$S(R,R') = \sum_{i=1}^{n} \left(R(i) - R'(i)\right)^2$$

- Computing the optimal rank aggregation with respect to Spearman's rank correlation is the same as computing Borda Count
  - Computable in polynomial time

Extensions and Applications

- Rank distance measures between partial orderings and top-k lists
- Similarity search
- Ranked Join Indices
- Analysis of Link Analysis Ranking algorithms
- Connections with machine learning

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