Dynamics of Social Balance on Networks

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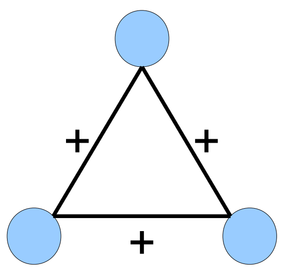
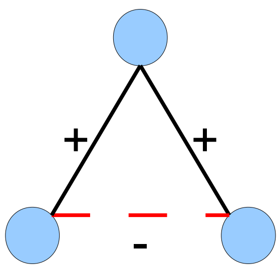
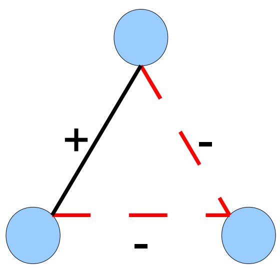
Definition: Social Balance



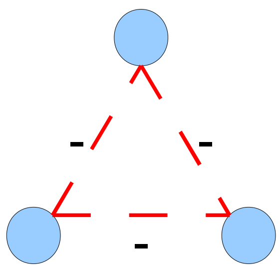
balanced



unbalanced

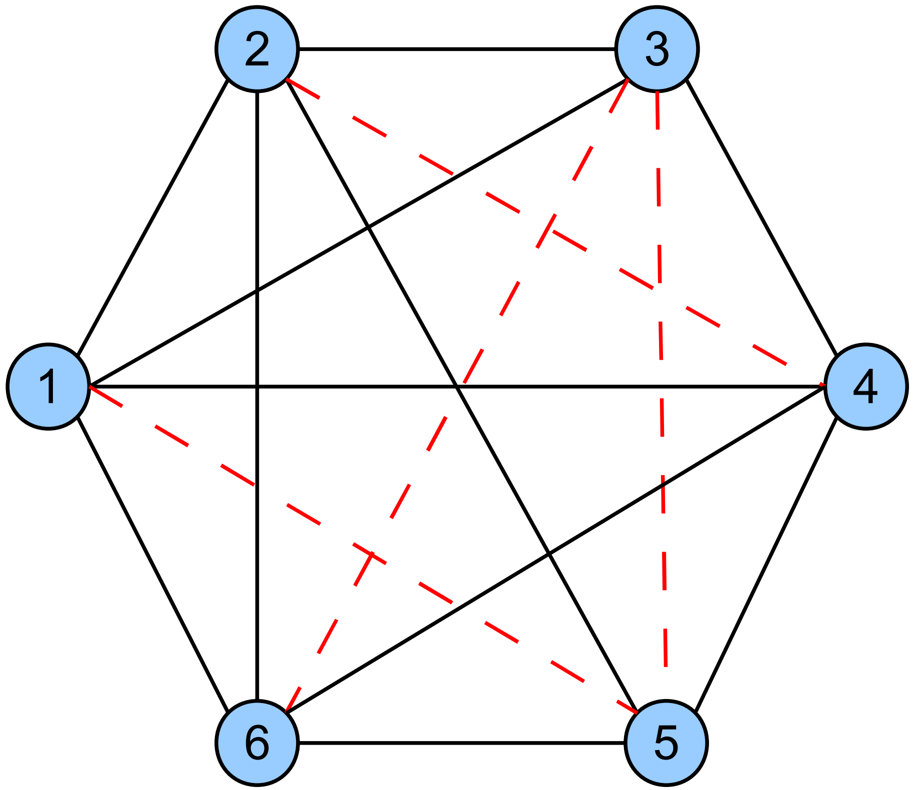


balanced



unbalanced

Example



Balanced?

Definition: Social Balance

A network is balanced, if ***all*** *of its consisting triangles are balanced.*

→ How can this be achieved?

<u>Theorem:</u> In a **completely connected** graph, the only balanced solution is: Two groups X and Y such that for all nodes x_1, $x_2 \in X$ and y_1, $y_2 \in Y$:

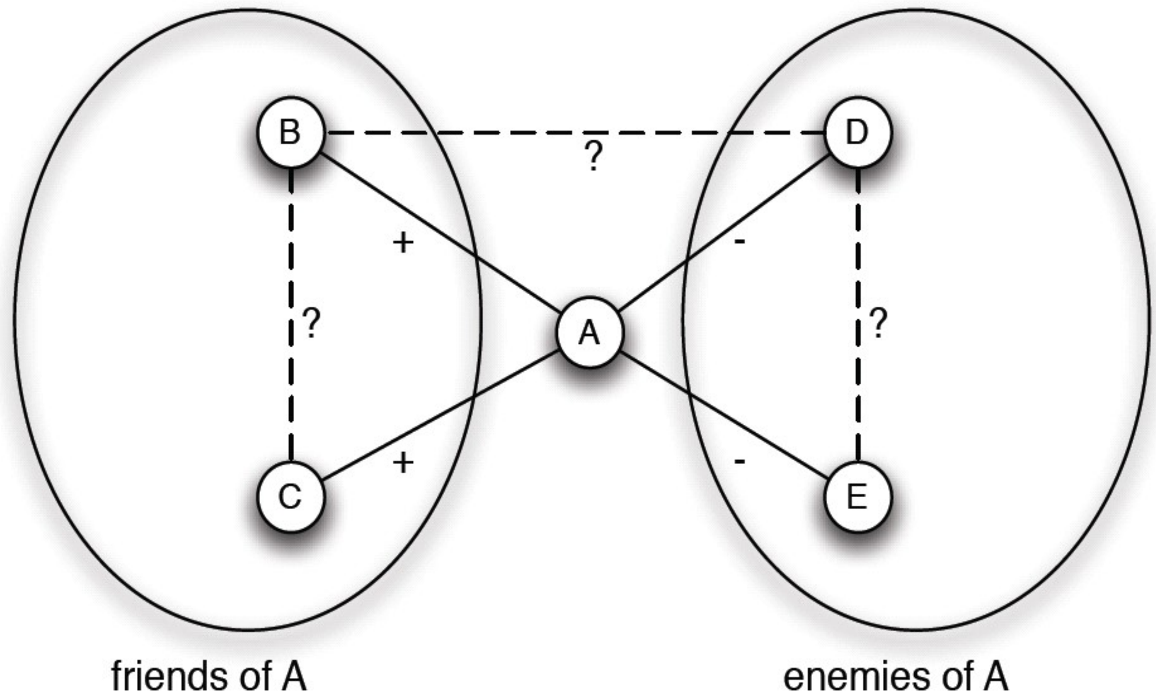
$$(x_1, x_2) = (y_1, y_2) = +1 , \quad (x_1, x_2) = -1$$

[4]

Definition: Social Balance

Proof:

- Pick an arbitrary node A
- Split the graph into the friends and the enemies of A

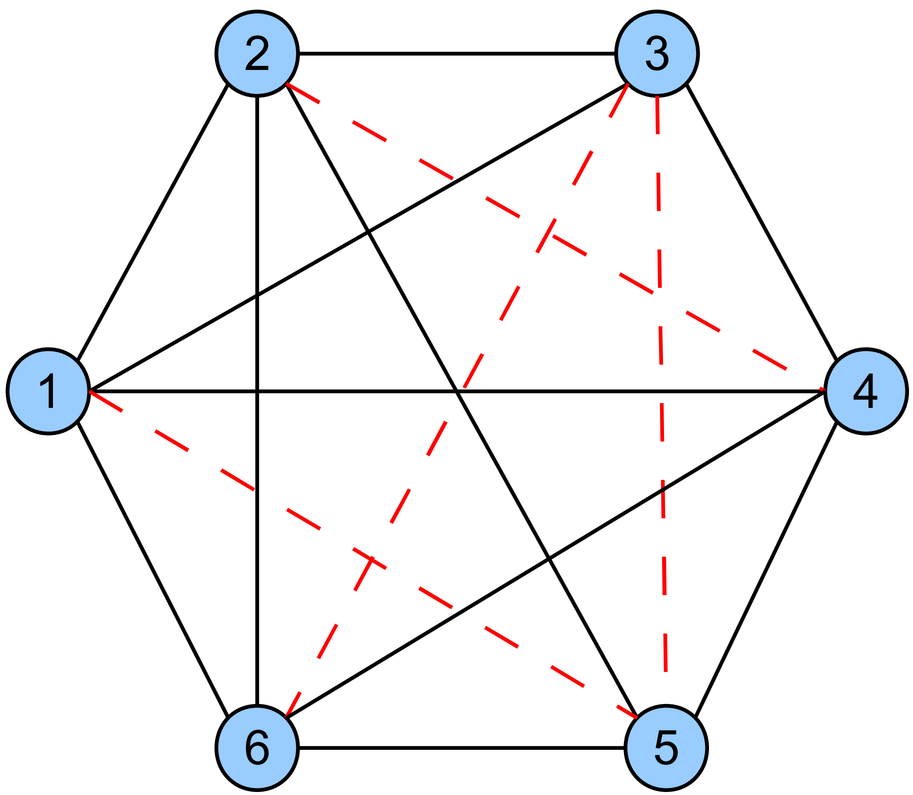


- Consider any of the open links

[3]

- From trivial conclusions follows, that there are exactly two groups

Example

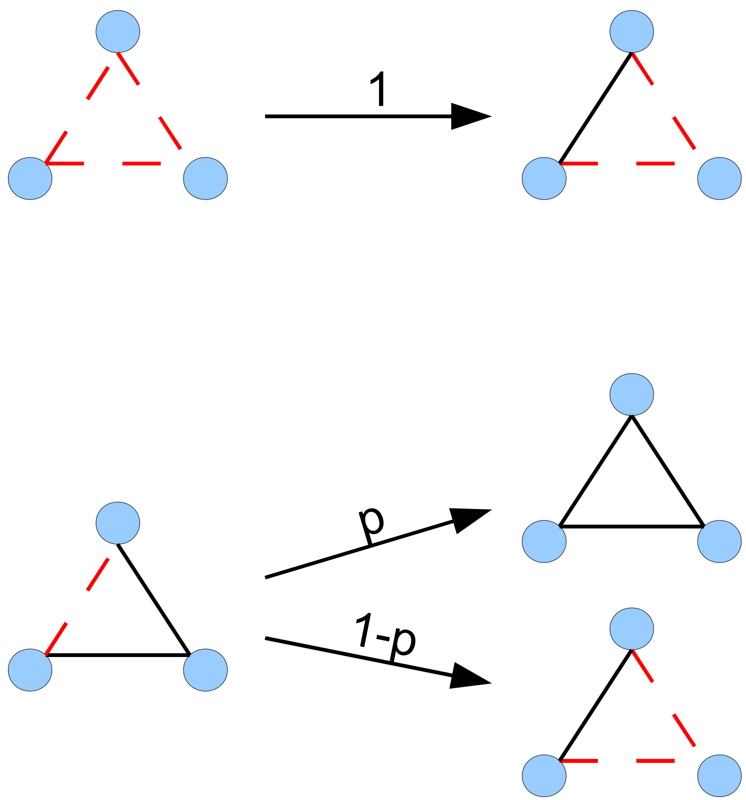


Balanced?

- 123, 126, 456 → yes
- 156, 125, 126 → no!
- ...

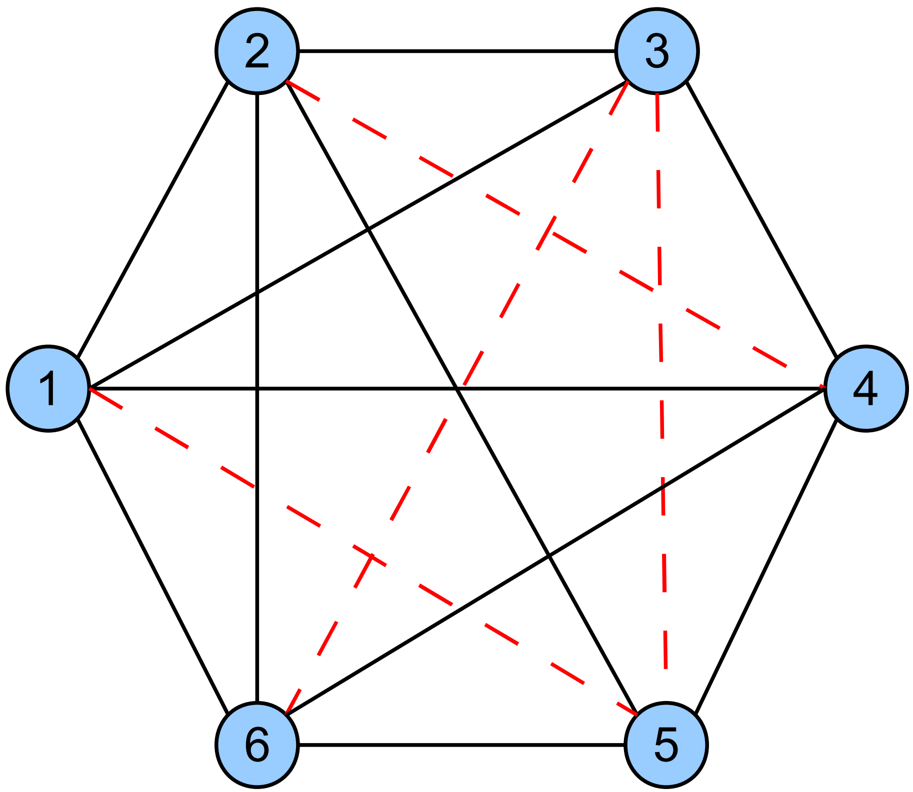
→ system will evolve

Local Triad Dynamics (LTD)

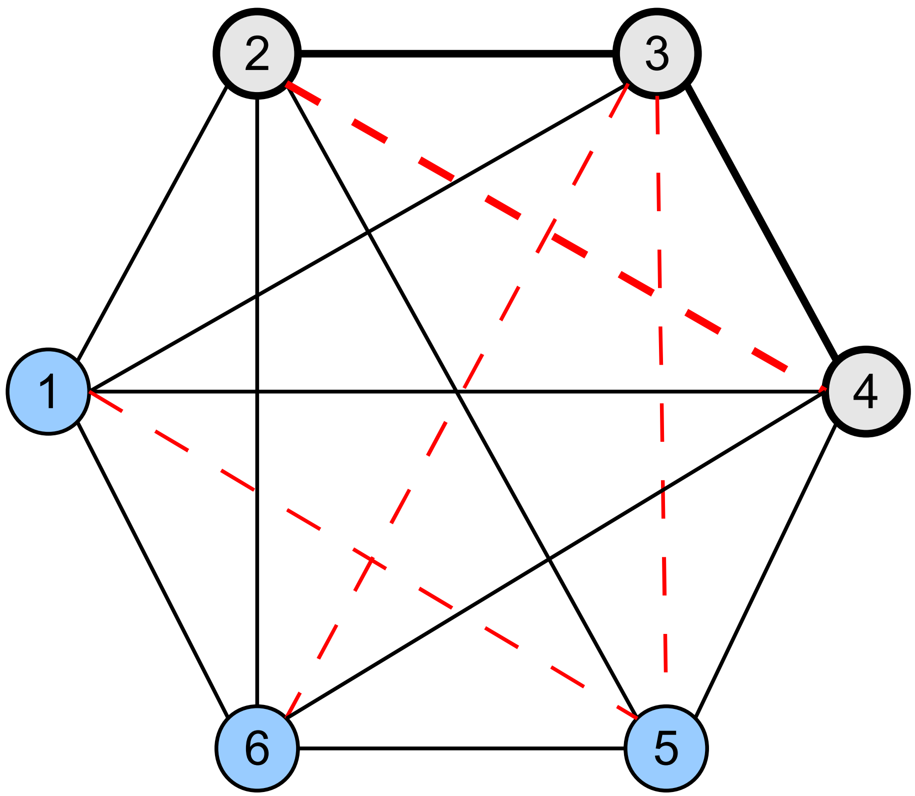


- Update rule:
  - Pick a triangle at random
  - Transforms the selected triad into a balanced one
  - repeat
- p is the control parameter of that model

Example: LTD-Update

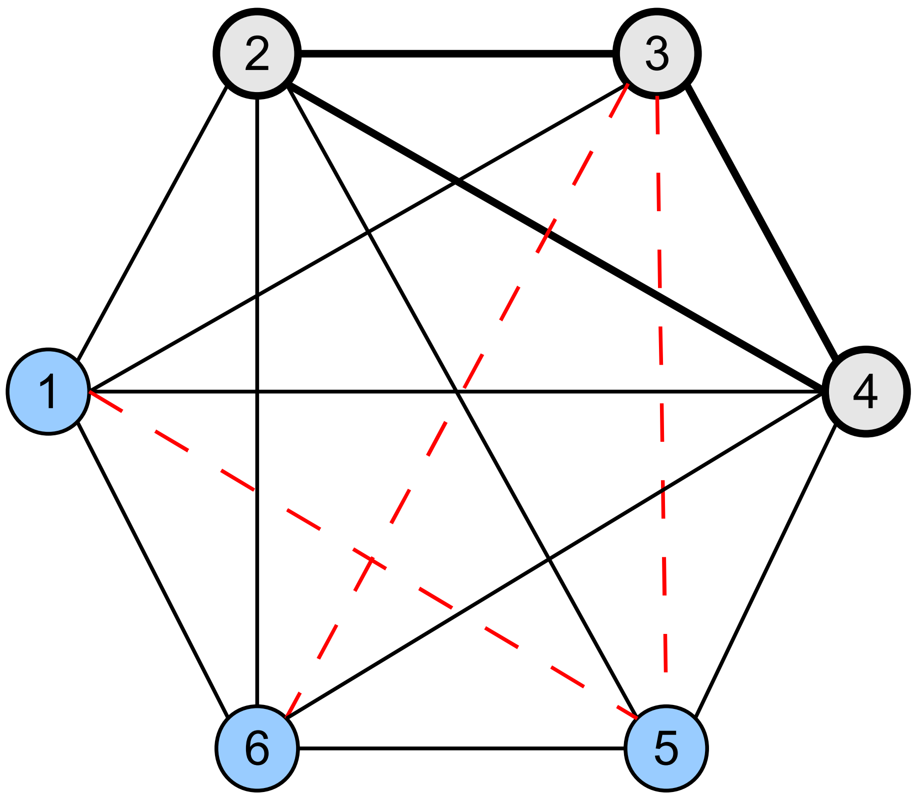


Example: LTD-Update



Switch (2,4) from unfriendly to friendly!

Example: LTD-Update



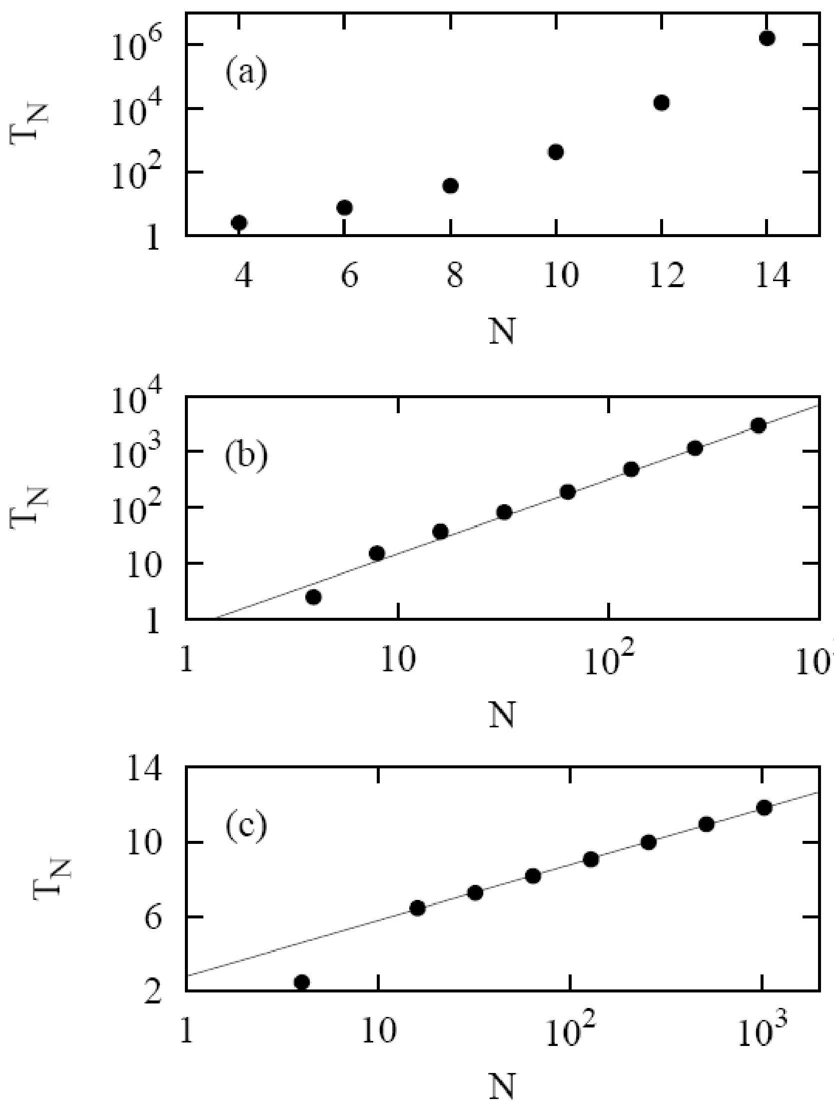
Switch (2,4) from unfriendly to friendly!

LTD: Final States

Some results:

- All evolutions end up in a balanced state
- The following measures only depend on p:
  - The time to reach a balanced state
  - The ratio of positive links in the final state (ρ_∞)
  - The densities of triangles with 0, 1, 2 and 3 negative links (n_0, n_1, n_2, n_3)

LTD: Final States



Average time to reach balance as a function of the network size N for an initially antagonistic society ($\rho_0 = 0$) for:

(a) p = 1/3

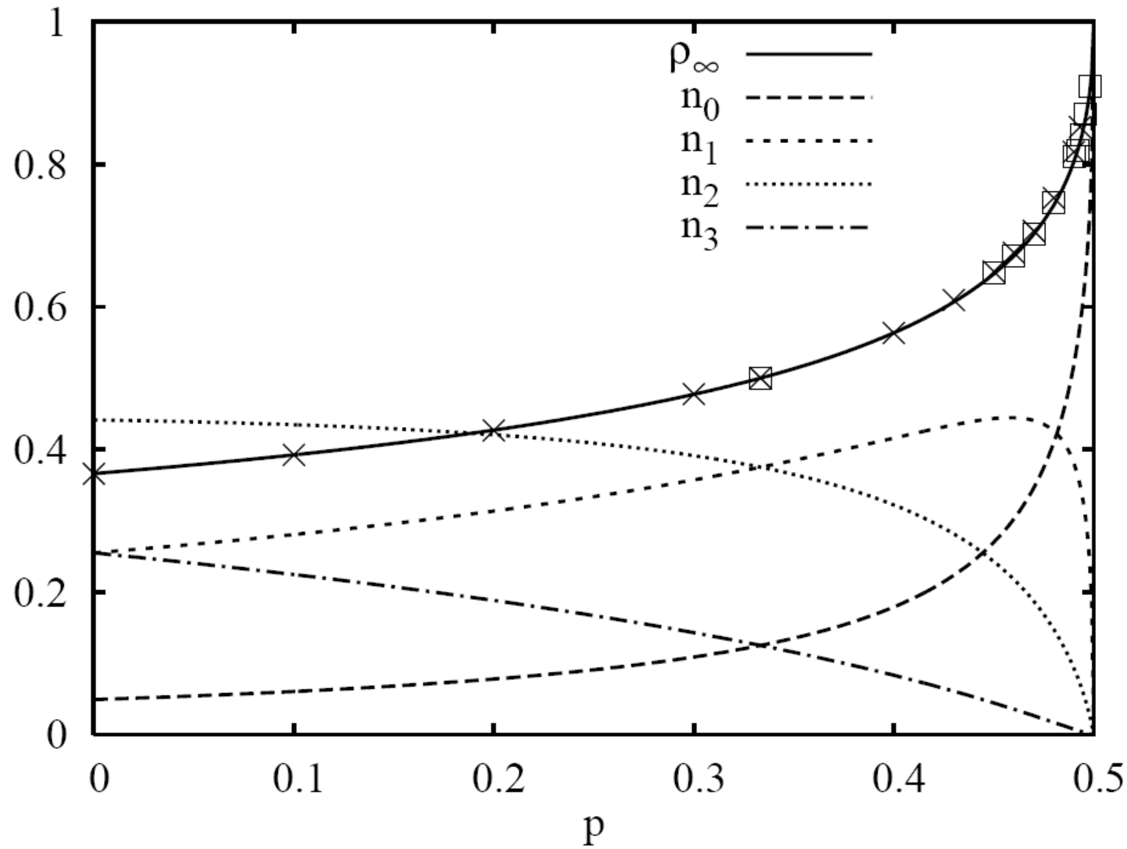
(b) p = 1/2

(c) p=3/4

$$T_N \propto \begin{cases} e^{N^2} & p < 1/2 \\ N^{4/3} & p = 1/2 \\ (2p-1)^{-1} \ln N & p > 1/2 \end{cases}$$

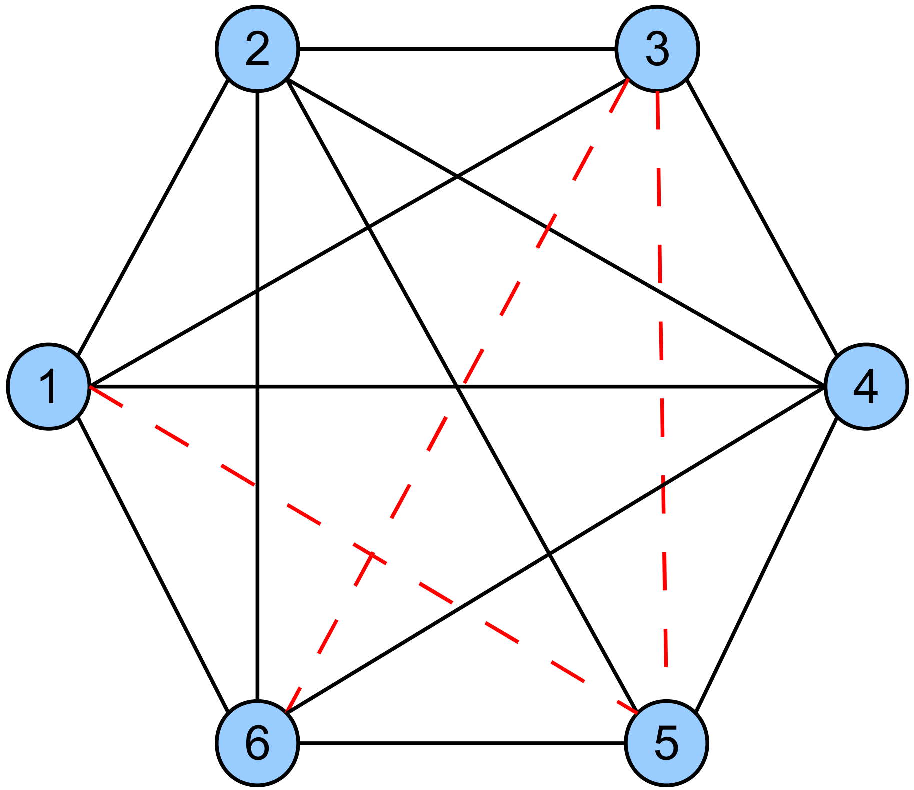
[1]

LTD: Final States

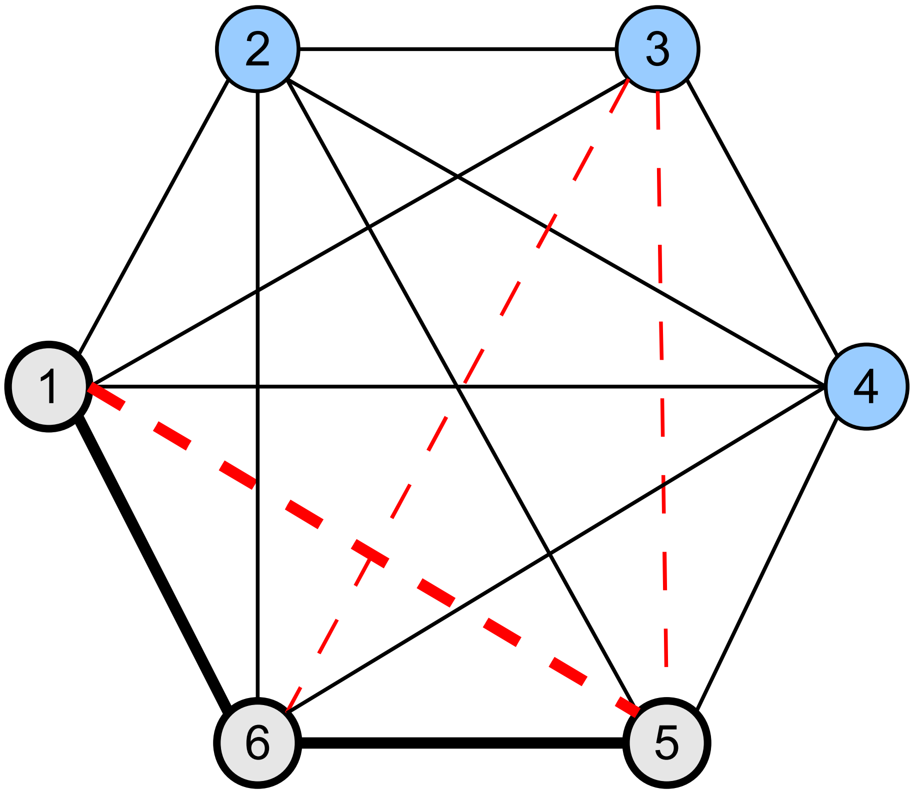


The stationary densities $n_k(p)$ of triads with k unfriendly links and the density of friendly links ρ_∞ as a function of p. Simulation results for ρ_∞ for N = 64 (crosses) and 256 (boxes) are also shown. [1]

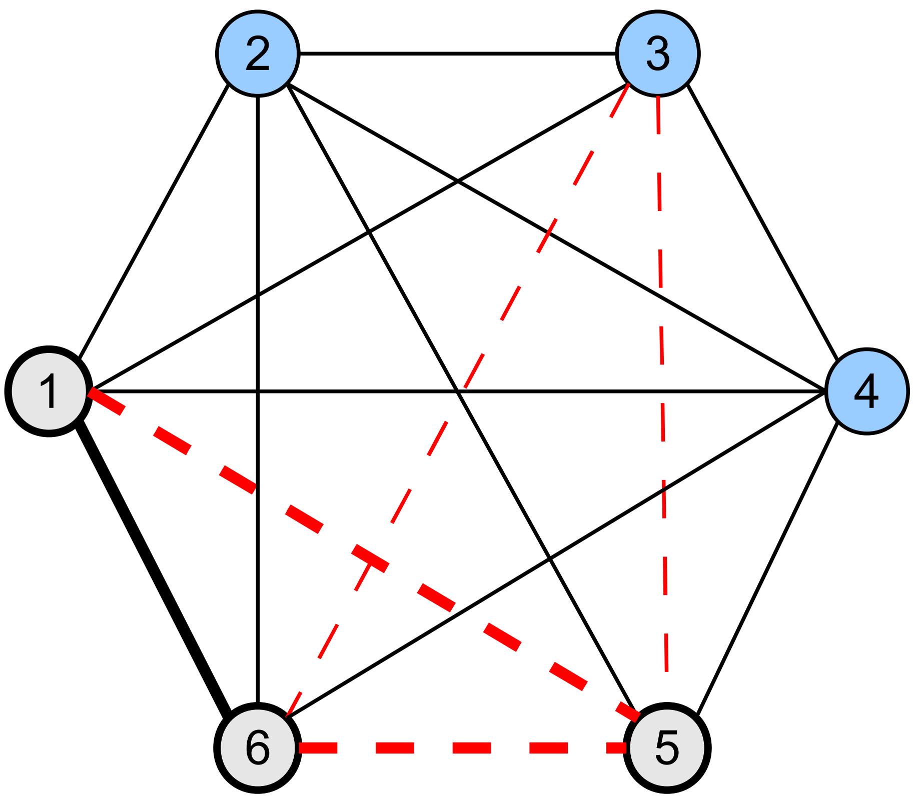
Example: LTD-Update



Example: LTD-Update



Example: LTD-Update



Effects of changing (5,6) from + to – :

- Unbalanced → bal.:

  (1,5,6)

- Balanced → unbal.:

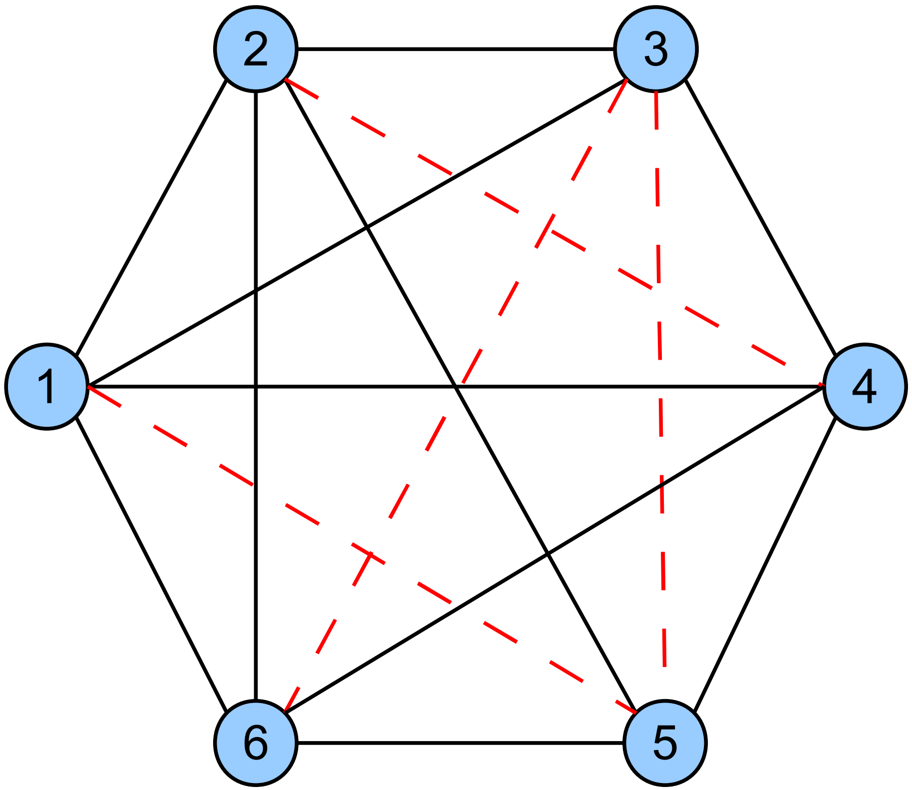
  (2,5,6), (3,5,6), (4,5,6)

Constrained Triad Dynamics (CTD)

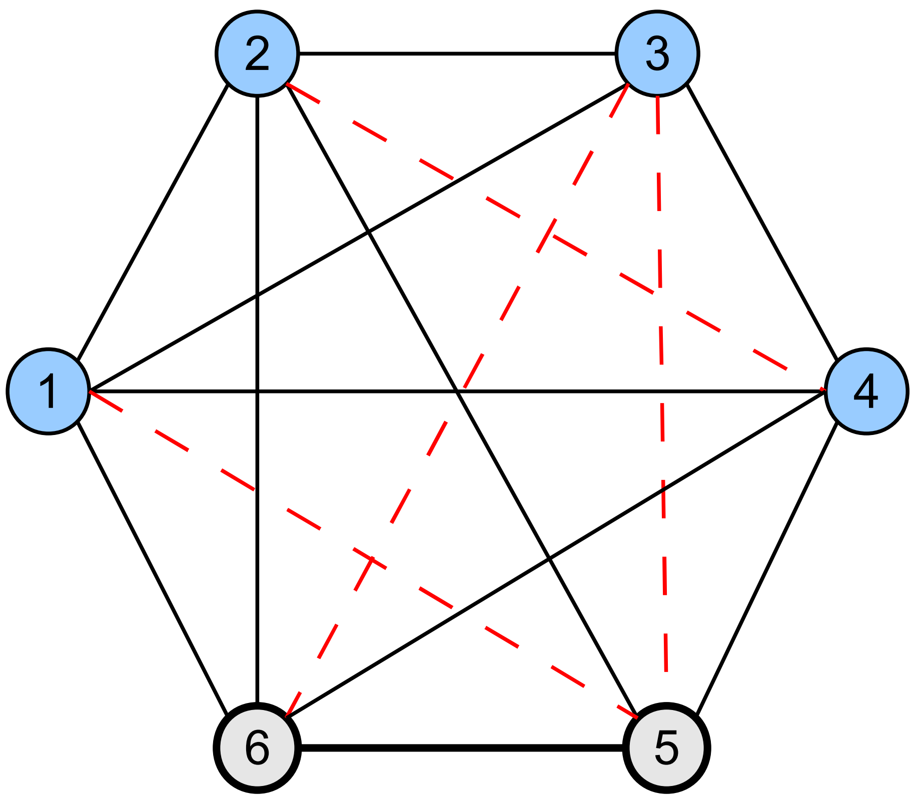
Update rule:

- Randomly pick an edge e_{ij}
- Define:
  u_{ij} = # of unbalanced triangles e_{ij} is part of
  b_{ij} analogously
  - Flip that edge, if $u_{ij} > b_{ij}$
  - Flip with probability 50%, if $u_{ij} = b_{ij}$
  - Else, don't flip
- Repeat

Example: CTD-Update



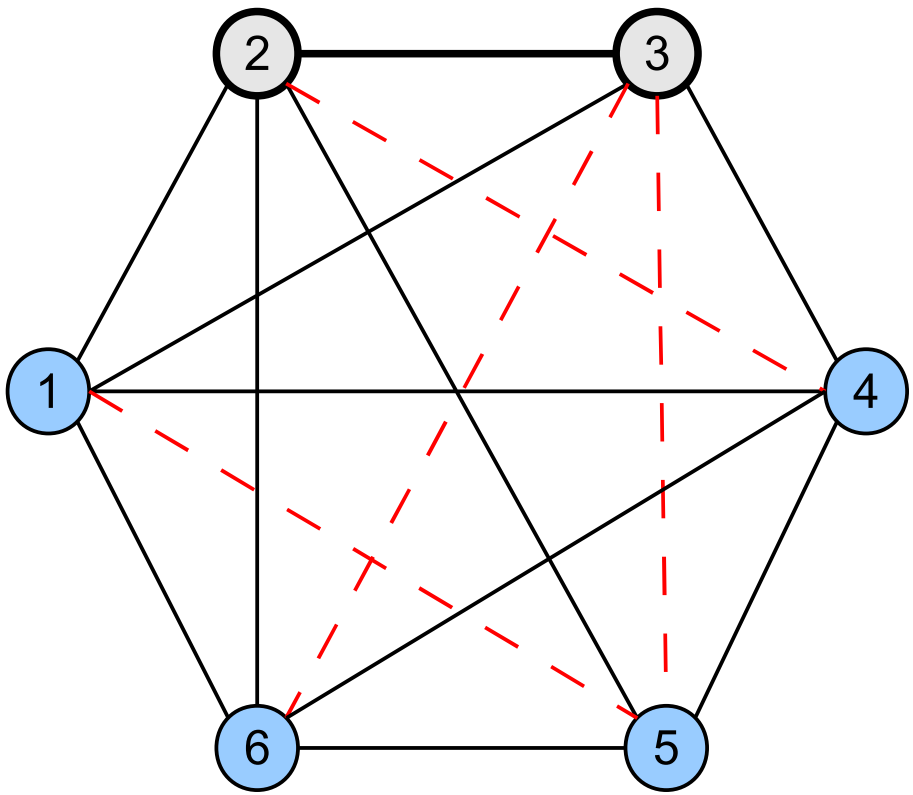
Example: CTD-Update



- Pick (5,6)
- $u_{56} = 1$, $b_{56} = 3$

  → don't flip!

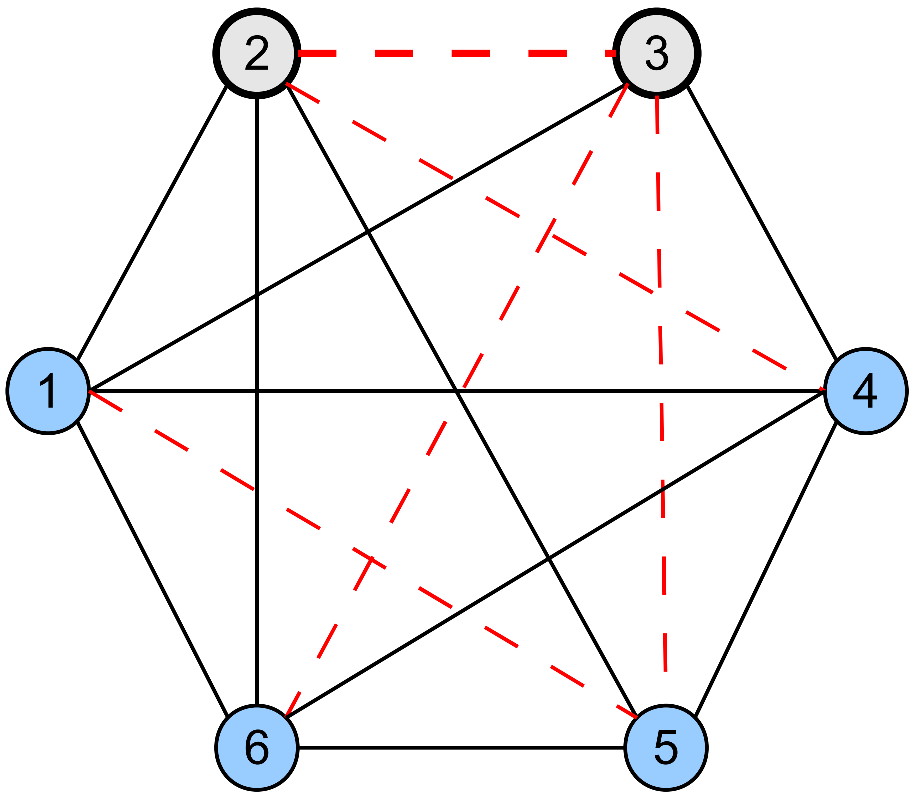
Example: CTD-Update



- Pick (2,3)
- $u_{23} = 3$, $b_{56} = 1$

  $\rightarrow$ flip!

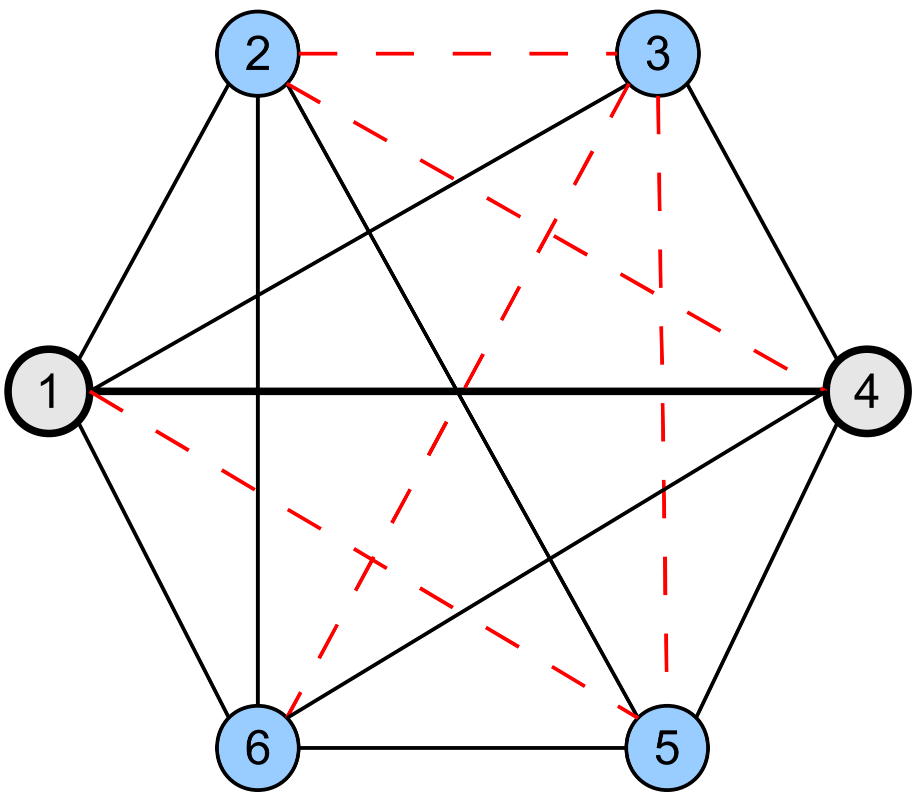
Example: CTD-Update



- Pick (2,3)
- $u_{23} = 3$, $b_{56} = 1$

  → flip!

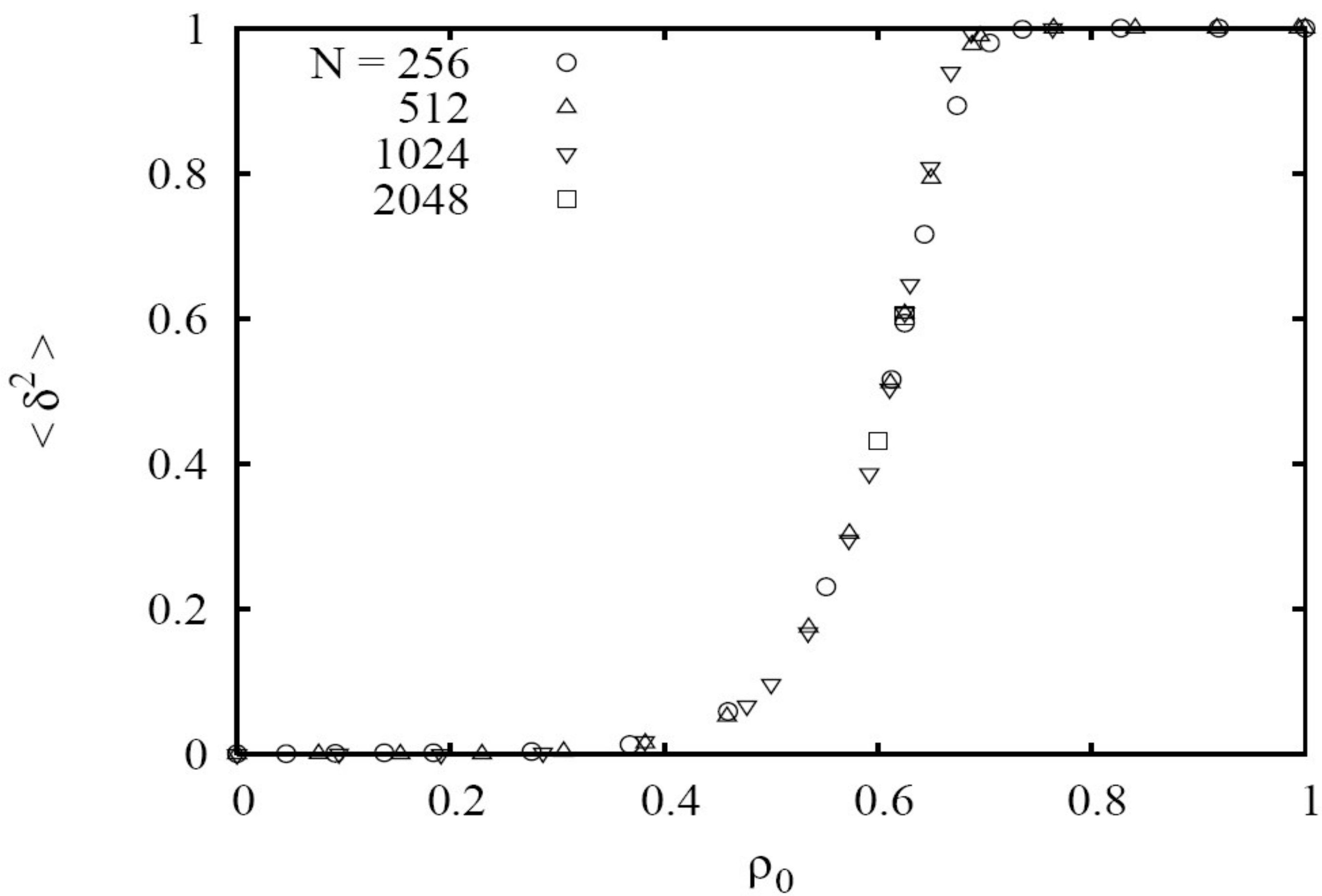
Example: CTD-Update



- Pick (1,4)
- $u_{14} = 2$, $b_{14} = 2$

  → toss a coin!

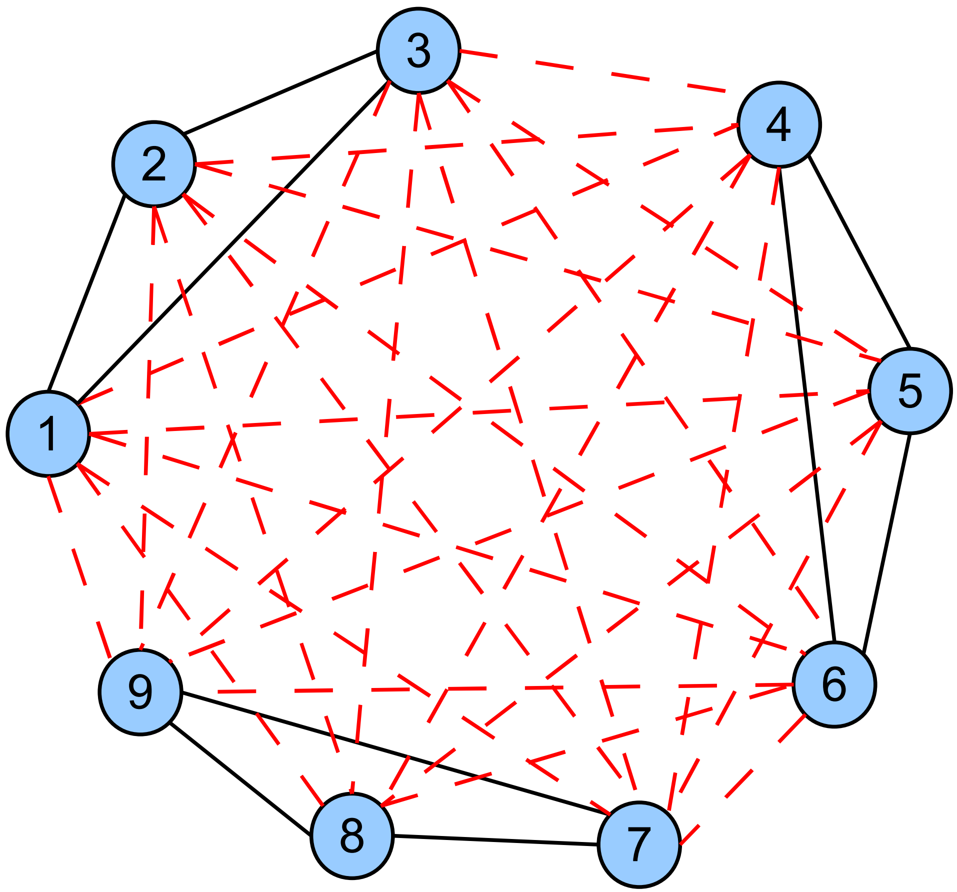
CTD: Final States



Asymmetry of the final state as a function of the initial friendship density ρ_0 for several network sizes.

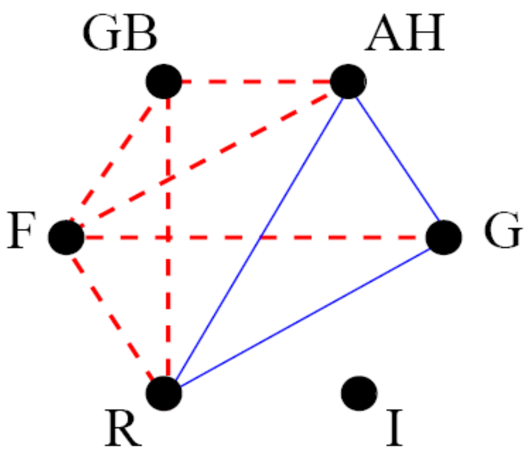
δ: size difference between the two groups in the final state

Constrained Triad Dynamics (CTD)



- CTD-dynamics may be trapped forever in 'jammed states'
- No local move can be made according to the update rules

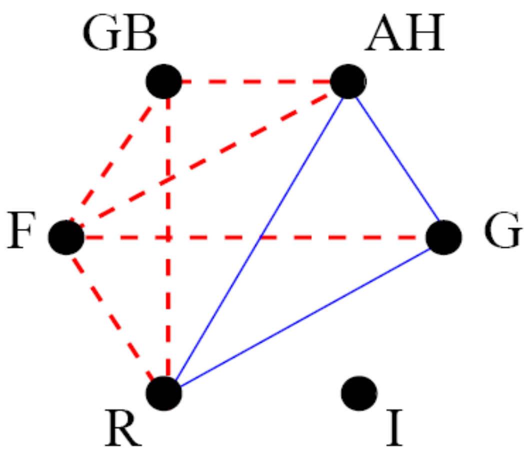
Real World Example



3 Emperor's league 1872–81

[2]

Real World Example



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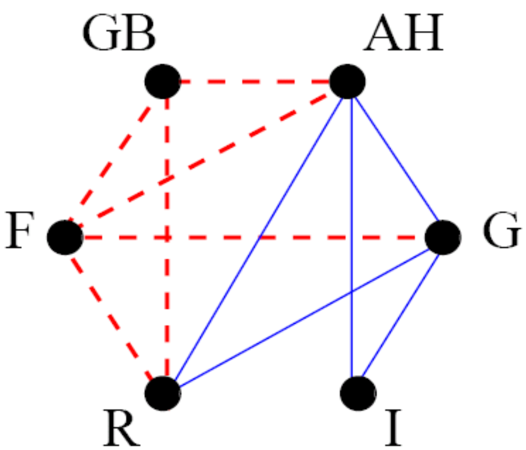
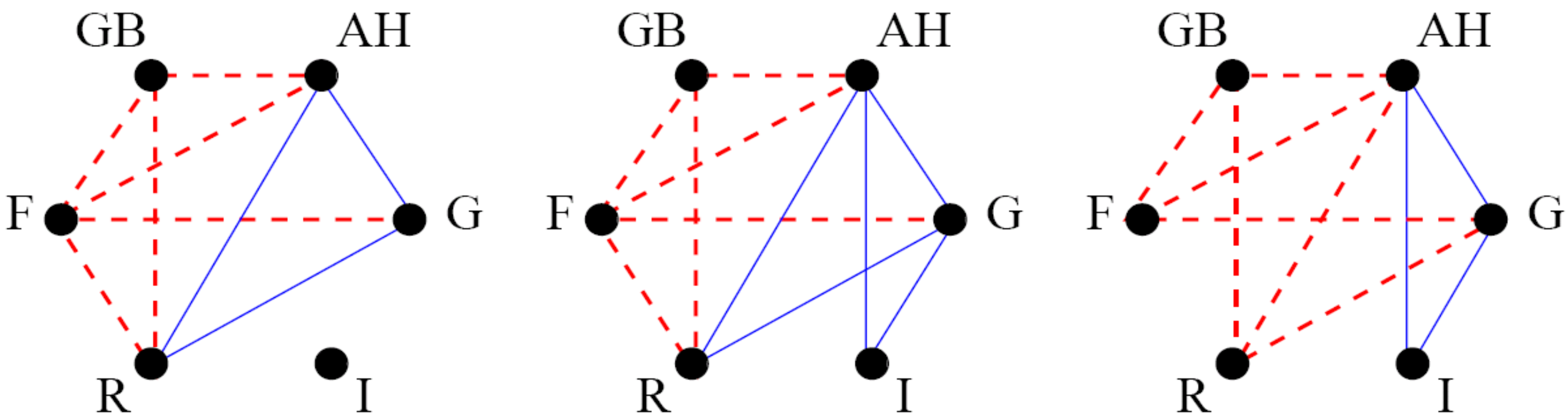


Triple Alliance 1882

[2]

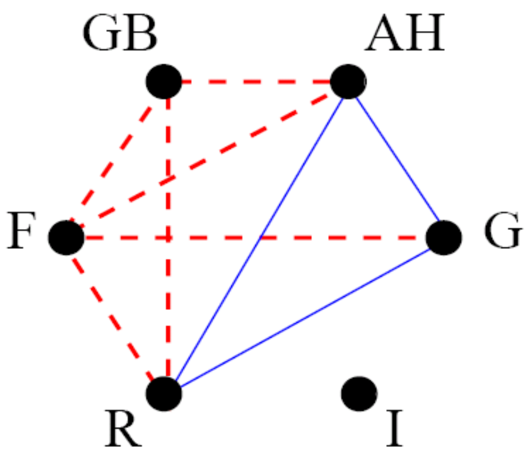
Real World Example



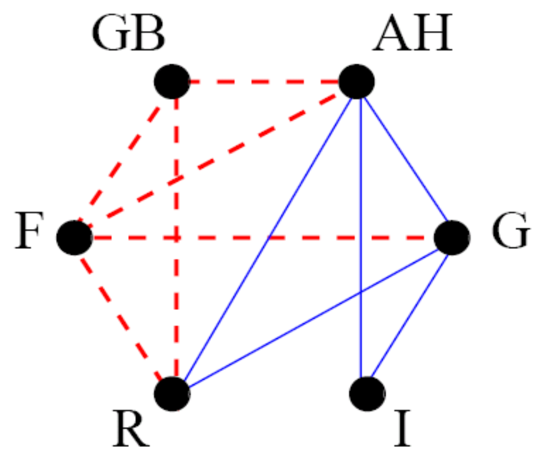
3 Emperor's league 1872−81 | Triple Alliance 1882 | German−Russian Lapse 1890

[2]

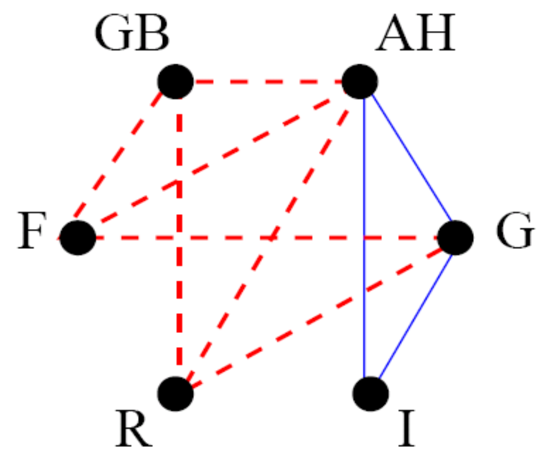
Real World Example



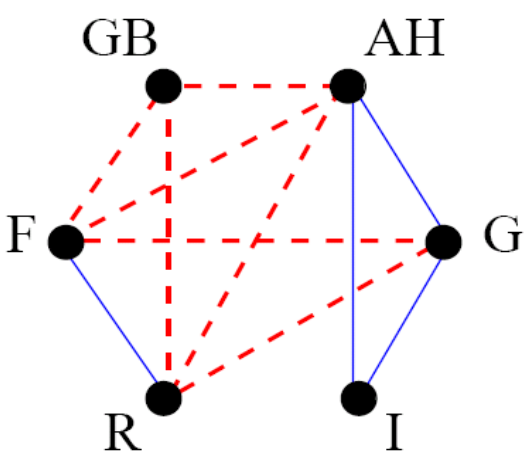
3 Emperor's league 1872−81



Triple Alliance 1882



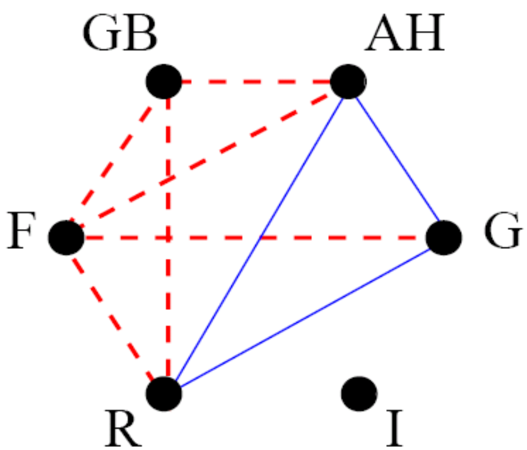
German−Russian Lapse 1890



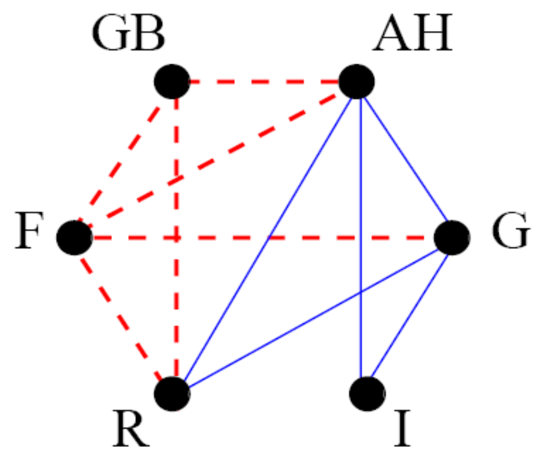
French−Russian Alliance 1891−94

[2]

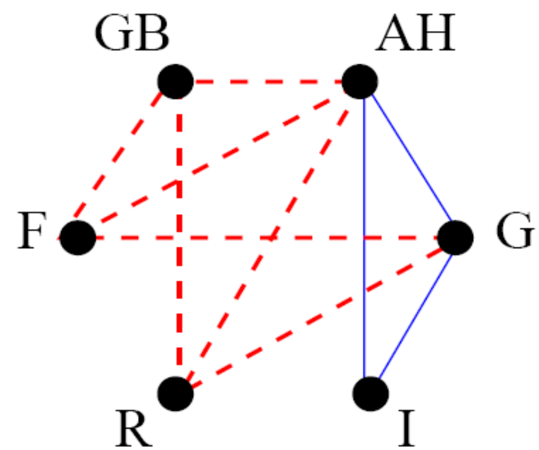
Real World Example



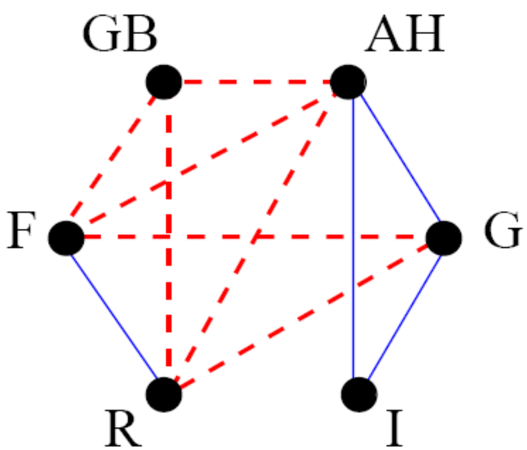
3 Emperor's league 1872–81



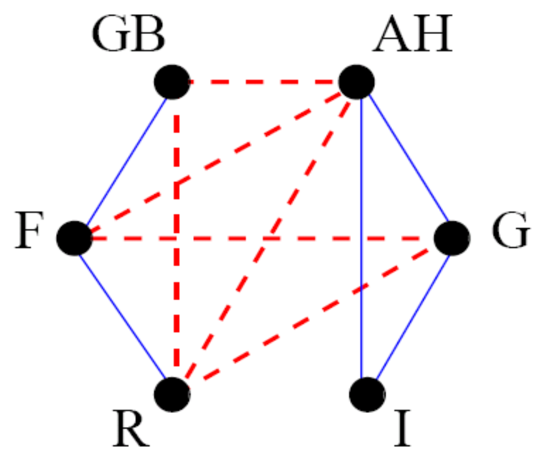
Triple Alliance 1882



German–Russian Lapse 1890



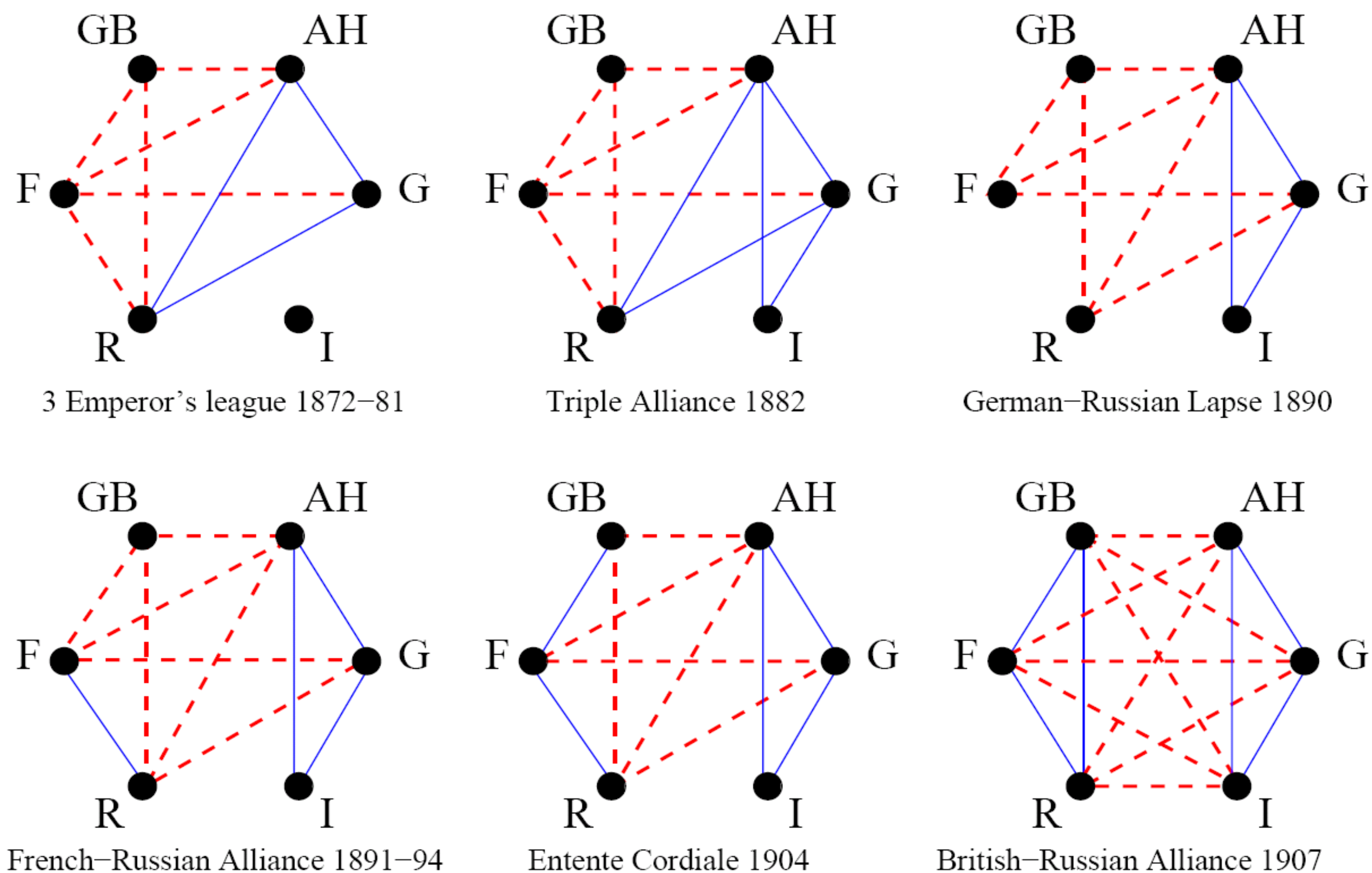
French–Russian Alliance 1891–94



Entente Cordiale 1904

[2]

Real World Example

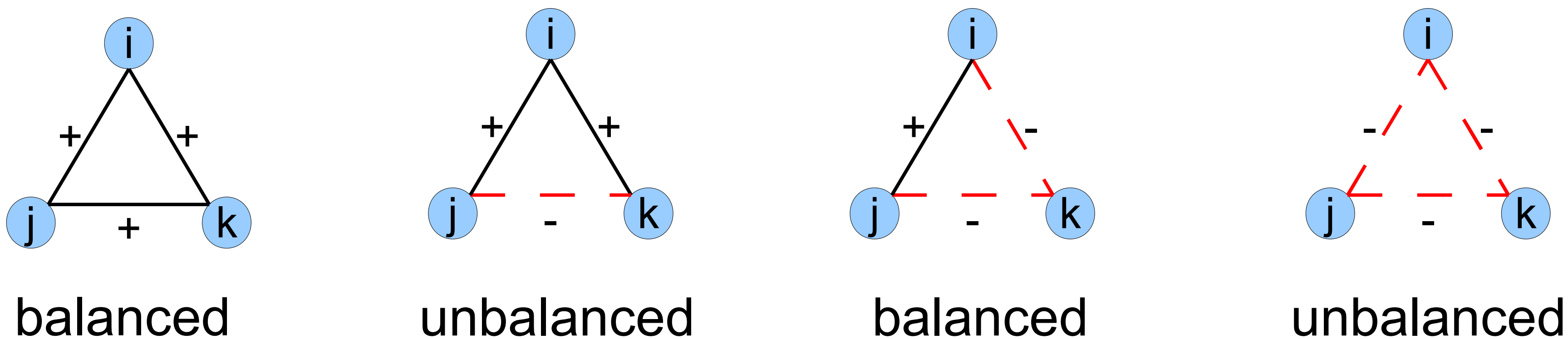


[2]

Project

- Generalization to neutral edges
- Introduction of an 'energy' U and a 'temperature' T on the network
- Definition of a dynamic that flips between positive, negative and neutral edges such that every state of the system is taken on with a probability corresponding to the energy of that state

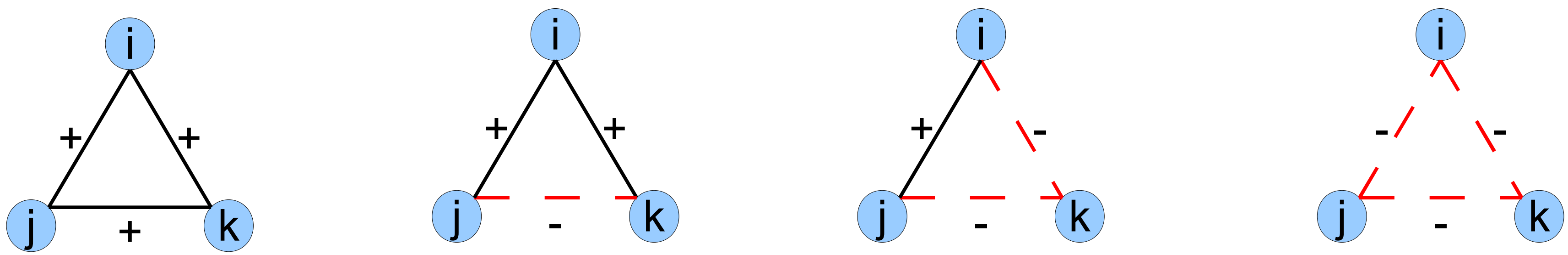
Definition: Social Balance



Note:

- For balanced triangles, it is: $e_{ij}\, e_{jk}\, e_{ki} = +1$

Definition: Social Balance



balanced unbalanced balanced unbalanced

Note:

- For balanced triangles, it is: $e_{ij}\ e_{jk}\ e_{ki} = +1$
- For unbalanced triangles, it is: $e_{ij}\ e_{jk}\ e_{ki} = -1$

Project

- Energy of edge e_{ij}:

$$U_{ij} = -\frac{1}{N-2}\sum_{k=1(k\neq i,j)}^{N} e_{jk}\cdot e_{ki}$$

- Probabilities:

$$P\left(e_{ij} = +1\right) = \frac{e^{\frac{-U_{ij}}{T}}}{e^{\frac{-U_{ij}}{T}} + e^{\frac{0}{T}} + e^{\frac{U_{ij}}{T}}}$$

$$P\left(e_{ij} = 0\right) = \frac{1}{e^{\frac{-U_{ij}}{T}} + 1 + e^{\frac{U_{ij}}{T}}}$$

$$P\left(e_{ij} = -1\right) = \frac{e^{\frac{U_{ij}}{T}}}{e^{\frac{-U_{ij}}{T}} + 1 + e^{\frac{U_{ij}}{T}}}$$

Project

Update rule:

- Pick one edge e_{ij} at a time
- Evaluate $U_{ij} = -\frac{1}{N-2} \sum_{k=1 (k \neq i,j)}^{N} e_{jk} \cdot e_{ki}$
- Set e_{ij} according to the probabilities
- repeat

References

[1] T. Antal, P.L. Krapivsky, S. Redner, *Dynamics of Social Balance on Networks*, Phys. Rev. E 72, 036121 (2005)

[2] T. Antal, P.L. Krapivsky, S. Redner, *Social Balance of Networks: The Dynamics of Friendship and Enmity*, Physica D 224,pp. 130–136 (2006)

[3] D. Easley, J. Kleinberg, *Networks, Crowds, and Markets - Reasoning About a Highly Connected World*, Cambridge University Press (2010)

[4] F. Harary, R.Z. Norman, D. Cartwright, Structural Models: *An Introduction to the Theory of Directed Graphs*, Wiley & Sons (1965)

[5] F. Heider, *Social Perception and Phenomenal Causality*, Psychological Rev., vol. 51, no. 6, pp. 358–374 (1944)