CS 591: Formal Methods in Security and Privacy

Differential Privacy

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From the previous classes

Releasing the mean of Some Data

```
Mean(d : private data) : public real
   i:=0;
   s:=0;
   while (i<size(d))
      s:=s + d[i]
      i:=i+1;
   return (s/i)
```

Releasing the mean of Some Data

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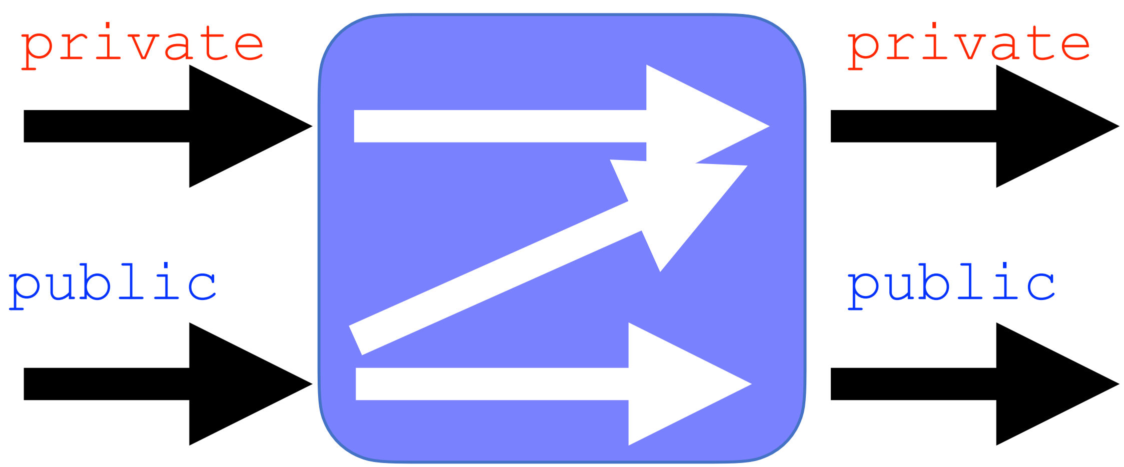
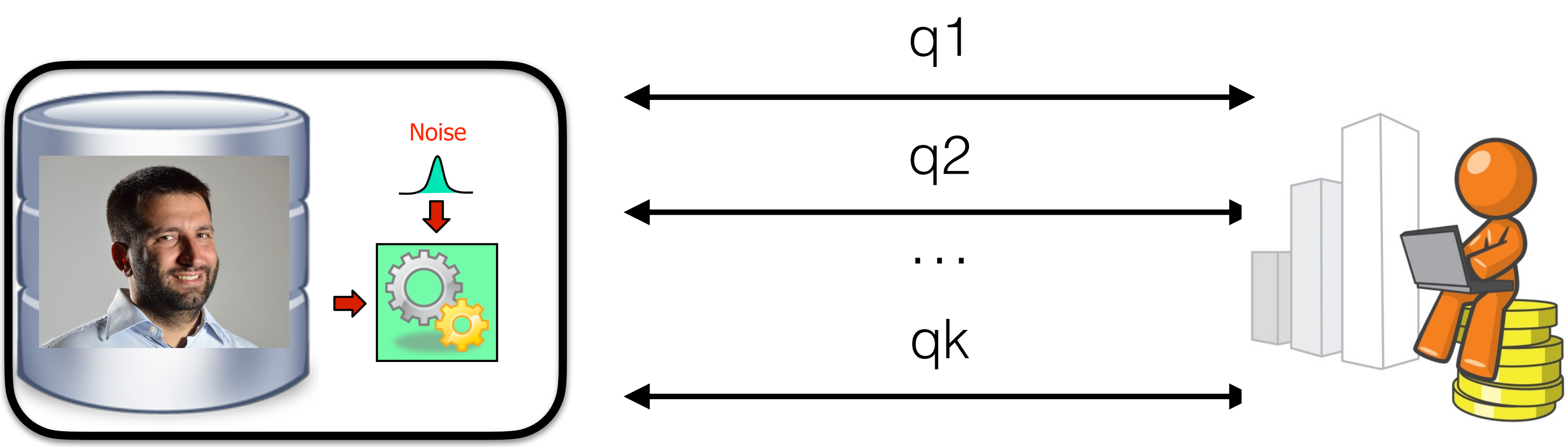


Privacy-preserving data analysis?

- The analyst learn almost the same about me after the analysis as what she would have learnt if I didn't contribute my data.



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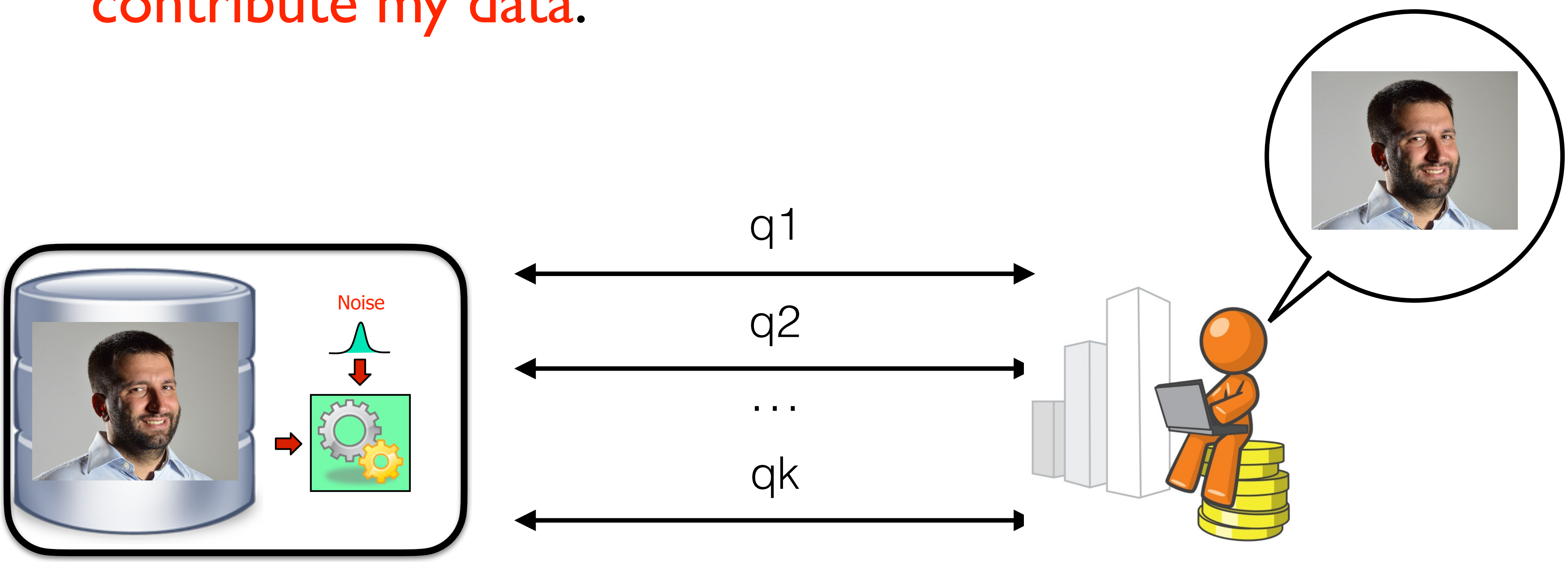

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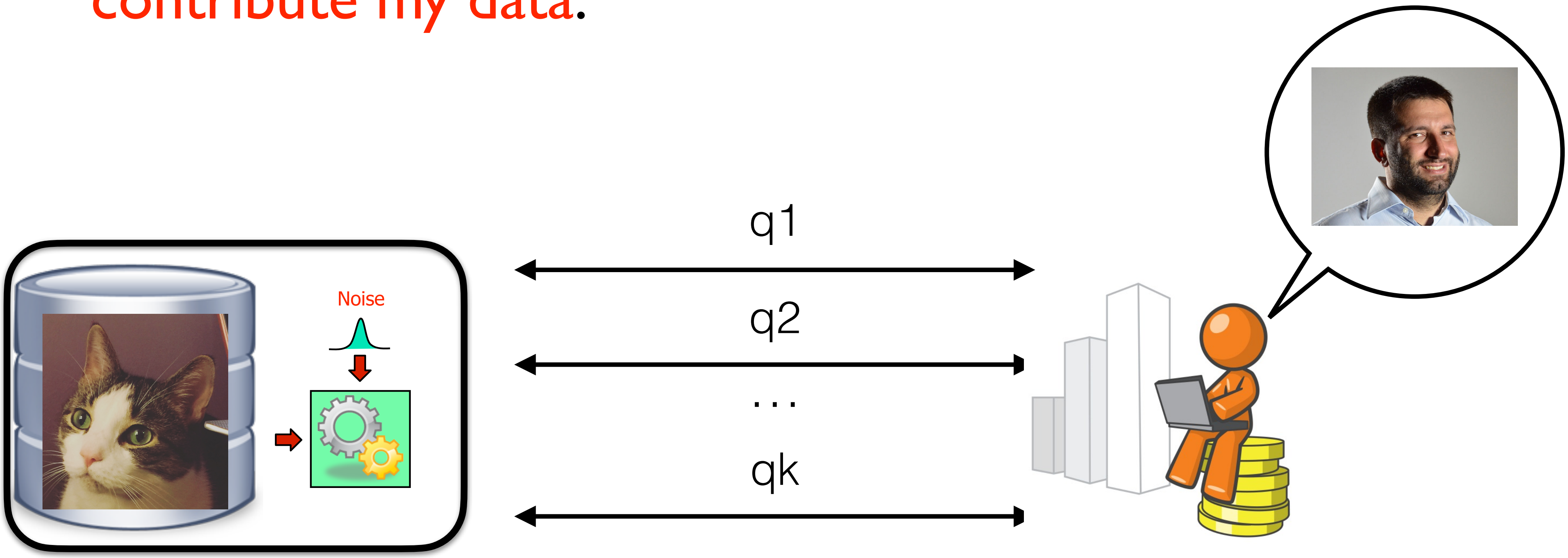


(ε,δ)-Differential Privacy

Definition

Given $\varepsilon,\delta \geq 0$, a probabilistic query $Q: X^n \rightarrow R$ is (ε,δ)-differentially private iff

for all adjacent database b_1, b_2 and for every $S \subseteq R$:

$$\Pr[Q(b_1) \in S] \leq \exp(\varepsilon)\Pr[Q(b_2) \in S] + \delta$$

(ε,δ)-indistinguishability

We can define a ε-skewed version of statistical distance. We call this notion ε-distance.

$$\Delta_\varepsilon(\mu 1,\mu 2)=\sup_{E\subseteq A} \max(\mu_1(E)-e^{\varepsilon}\mu_2(E),\ \mu_2(E)-e^{\varepsilon}\mu_1(E),0)$$

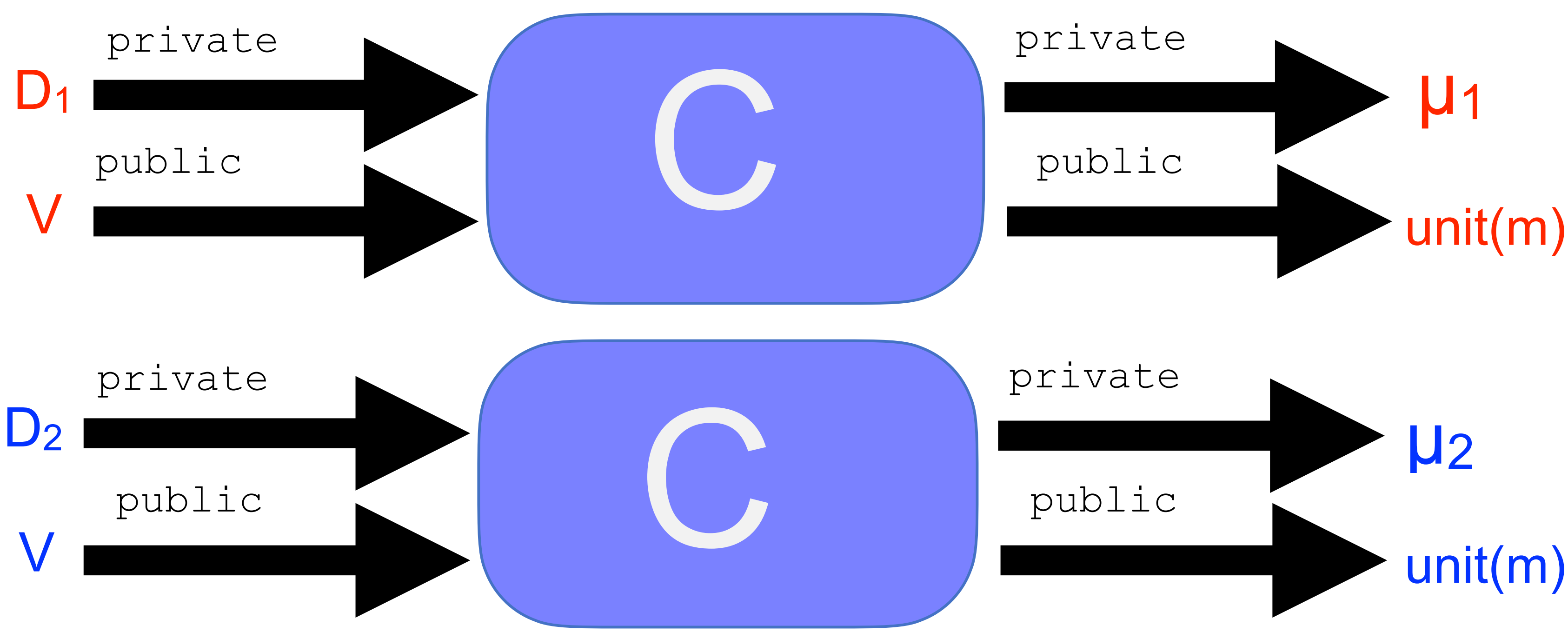
We say that two distributions $\mu_1, \mu_2 \in D(A)$, are at (ε,δ)-indistinguishable if:

$$\Delta_\varepsilon(\mu 1,\mu 2) \leq \delta$$

Differential Privacy as a Relational Property

c is differentially private if and only if for every $m_1 \sim m_2$ (extending the notion of adjacency to memories):

$\{c\}_{m1}=\mu_1$ and $\{c\}_{m2}=\mu_2$ implies $\Delta_\varepsilon(\mu_1,\mu_2) \leq \delta$



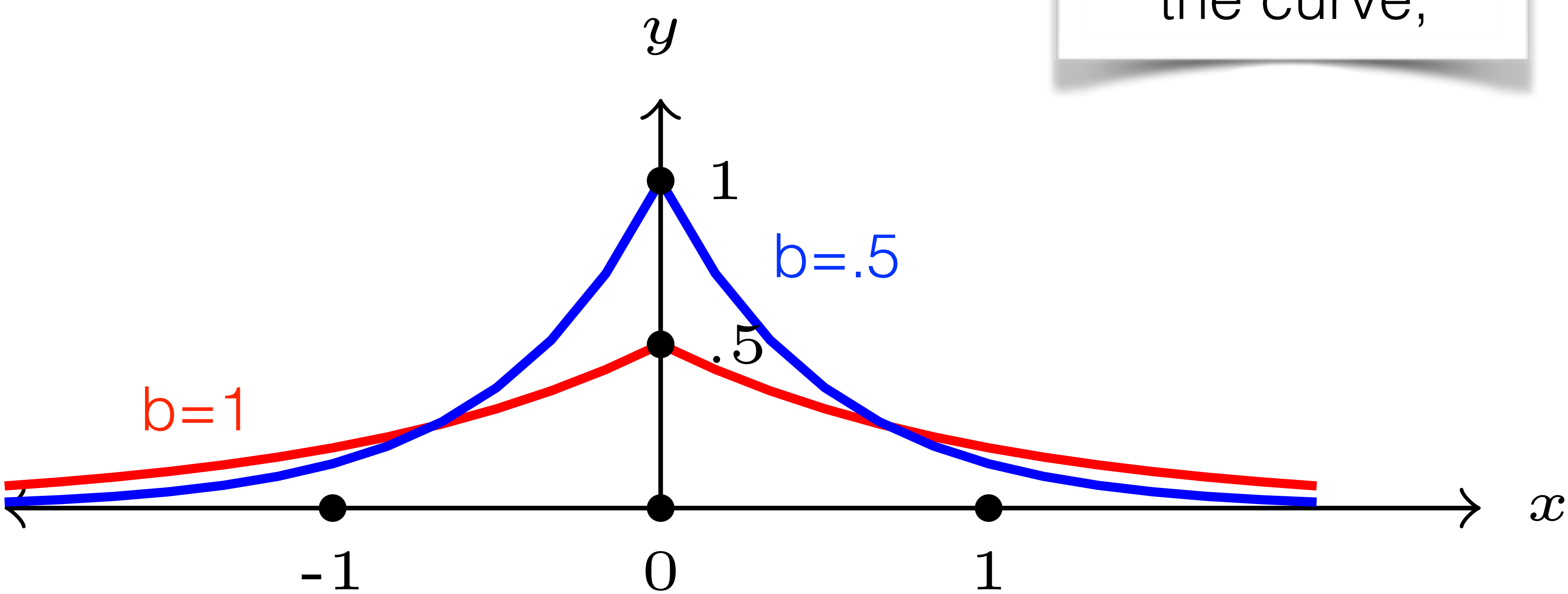
Releasing privately the mean of Some Data

```
Mean(d : private data) : public real
   i:=0;
   s:=0;
   while (i<size(d))
      s:=s + d[i]
      i:=i+1;
   z:=$ Laplace(sens/eps,0)
   z:= (s/i)+z
   return z
```

Laplace Distribution

$$\mathsf{Lap}(b,\mu)(X) = \frac{1}{2b}\exp\Big(-\frac{|\mu - X|}{b}\Big)$$

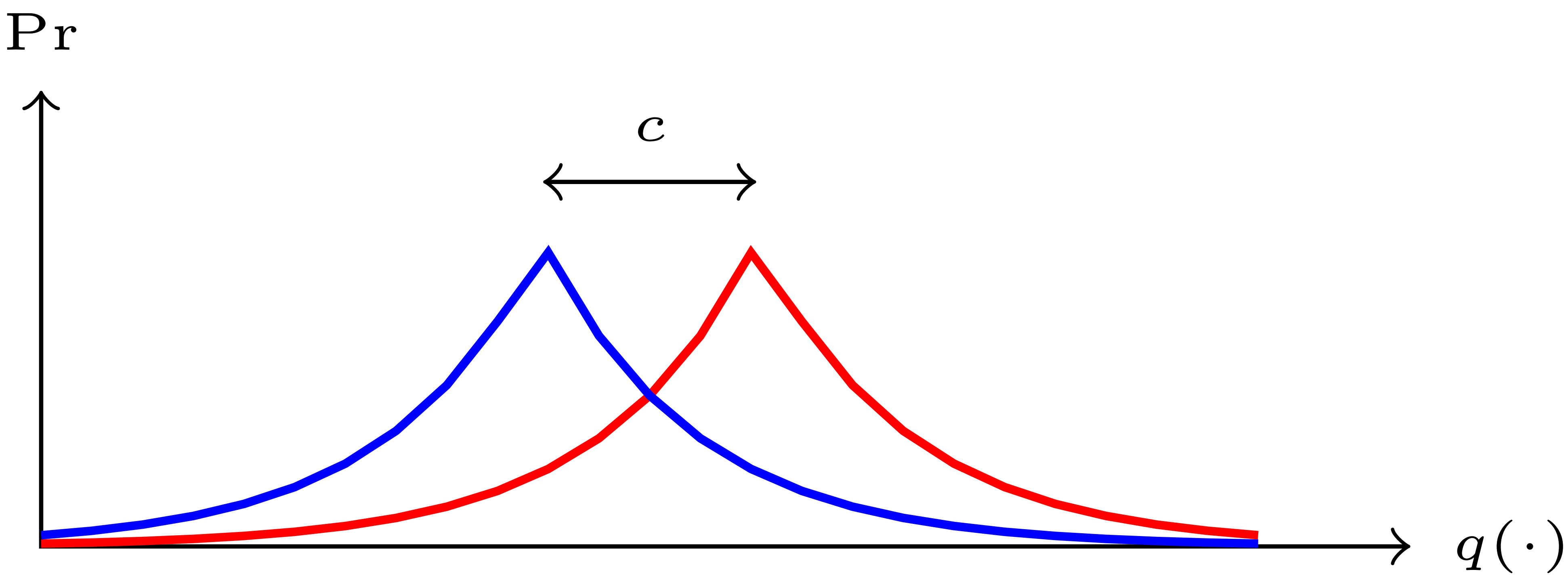
b regulates the skewness of the curve,



Laplace Mechanism

Theorem (Privacy of the Laplace Mechanism)
The Laplace mechanism is (ε,0)-differentially private.

Proof: Intuitively



Today

Laplace Mechanism

Theorem (Privacy of the Laplace Mechanism)
The Laplace mechanism is $(\varepsilon,0)$-differentially private.

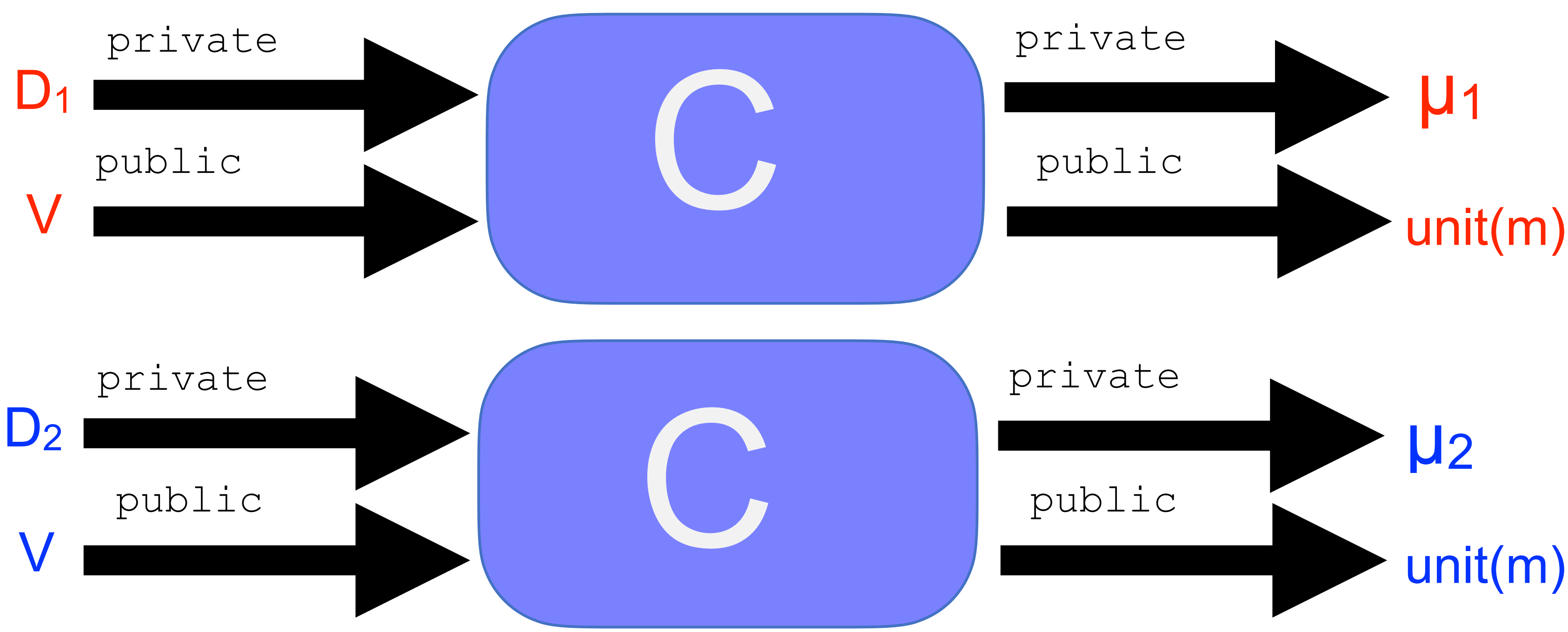
Laplace Mechanism

Question: How accurate is the answer that we get from the Laplace Mechanism?

Differential Privacy as a Relational Property

c is differentially private if and only if for every $m_1 \sim m_2$ (extending the notion of adjacency to memories):

$\{c\}_{m1}=\mu_1$ and $\{c\}_{m2}=\mu_2$ implies $\Delta_\varepsilon(\mu_1,\mu_2) \leq \delta$



apRHL

Indistinguishability parameter

Precondition (a logical formula)

$$\vdash_{\epsilon,\delta} c_1 \sim c_2 : P \Rightarrow Q$$

Probabilistic Program

Probabilistic Program

Postcondition (a logical formula)

Validity of apRHL judgments

We say that the quadruple $\vdash_{\varepsilon,\delta} \mathtt{c}_1 \sim \mathtt{c}_2 : P \Rightarrow Q$ is valid if and only if for every pair of memories $\mathtt{m}_1, \mathtt{m}_2$ such that $P(\mathtt{m}_1, \mathtt{m}_2)$ we have:

$\{\mathtt{c}_1\}_{\mathtt{m}1} = \mu_1$ and $\{\mathtt{c}_2\}_{\mathtt{m}2} = \mu_2$ implies $Q_{\varepsilon,\delta}{}^{*}(\mu_1, \mu_2)$.

R-(ε,δ)-Coupling

Given two distributions $\mu_1 \in D(A)$, and $\mu_2 \in D(B)$, we have an R-(ε,δ)-coupling between them, for $R \subseteq AxB$ and $0 \leq \delta \leq 1$, $\varepsilon \geq 0$, if there are two joint distributions $\mu_L, \mu_R \in D(AxB)$ such that:

1) $\pi_1(\mu_L)=\mu_1$ and $\pi_2(\mu_R)=\mu_2$,
2) the support of μ_L and μ_R is contained in R. That is, if $\mu_L(a,b)>0$, then $(a,b) \in R$, and if $\mu_R(a,b)>0$, then $(a,b) \in R$.
3) $\Delta_\varepsilon(\mu_L, \mu_R) \leq \delta$

(ε,δ)-indistinguishability revisited

For discrete distributions we can rewrite the notion of ε-distance as follows:

$\Delta\varepsilon(\mu 1,\mu 2)=1/2 * \Sigma_{a \in A} \max(\mu 1(a)-e^{\varepsilon}\mu 2(a),\ \mu 2(E)-e^{\varepsilon}\mu 1(E),0)$

Example of R-(ε,δ)-Coupling

μ_1

00	0.25
01	0.25
10	0.25
11	0.25

μ_2

00	0.20
01	0.25
10	0.25
11	0.30

$$R(a,b) = \{a=b\}$$

μ_L	00	01	10	11
00	0.25			
01		0.25		
10			0.25	
11				0.25

μ_R	00	01	10	11
00	0.20			
01		0.25		
10			0.25	
11				0.30

$$\Delta_0(\mu_L, \mu_R) = 0.05$$

Example of R-(ε,δ)-Coupling

μ_1

00	0.25
01	0.25
10	0.25
11	0.25

μ_2

00	0.20
01	0.25
10	0.25
11	0.30

$$R(a,b) = \{a=b\}$$

μ_L	00	01	10	11
00	0.25			
01		0.25		
10			0.25	
11				0.25

μ_R	00	01	10	11
00	0.20			
01		0.25		
10			0.25	
11				0.30

$$\Delta_1(\mu_L, \mu_R) = 0$$

Example of R-(ε,δ)-Coupling

μ_1

00	0.25
01	0.25
10	0.25
11	0.25

μ_2

00	0.20
01	0.25
10	0.25
11	0.30

$$R(a,b) = \{a=b\}$$

μ_L	00	01	10	11
00	0.25			
01		0.25		
10			0.25	
11				0.25

μ_R	00	01	10	11
00	0.20			
01		0.25		
10			0.25	
11				0.30

$$\Delta_{0.3}(\mu_L, \mu_R) = 0$$

Example of R-(ε,δ)-Coupling

$e^{0.3} \sim 1.3$

μ_L	00	01	10	11
00	0.25			
01		0.25		
10			0.25	
11				0.25

μ_R	00	01	10	11
00	0.20			
01		0.25		
10			0.25	
11				0.30

$\max(\mu_L(00,00)-e^{0.3}\mu R(00,00),\mu R(00,00)-e^{0.3}\mu_L(00,00),0)=0$

$+\ \max(\mu_L(01,01)-e^{0.3}\mu R(01,01),\mu R(01,01)-e^{0.3}\mu_L(01,01),0)=0$

$+\ \max(\mu_L(10,10)-e^{0.3}\mu R(10,10),\mu R(10,10)-e^{0.3}\mu_L(10,10),0)=0$

$+\ \max(\mu_L(11,11)-e^{0.3}\mu R(11,11),\mu R(11,11)-e^{0.3}\mu_L(11,11),0)=0$

$$\Delta_{0.3}(\mu_L,\mu_R)=0$$

Example of R-(ε,δ)-Coupling

μ_1

00	0.2
01	0.25
10	0.25
11	0.3

μ_2

00	0
01	0.40
10	0
11	0.6

$$R(a,b) = \{a \leq b\}$$

μ_L	00	01	10	11
00		0.20		
01		0.25		
10				0.25
11				0.30

μ_R	00	01	10	11
00		0.20		
01		0.20		
10				0.3
11				0.3

$$\Delta_0(\mu_L, \mu_R) = 0.05$$

Example of R-(ε,δ)-Coupling

μ_1

00	0.25
01	0.25
10	0.25
11	0.25

μ_2

00	0
01	0
10	0.5
11	0.5

Example of R-(ε,δ)-Coupling

μ_1

OO	0.25
O1	0.25
1O	0.25
11	0.25

$$R(a,b) = \{a=b\}$$

μ_2

OO	0
O1	0
1O	0.5
11	0.5

Example of R-(ε,δ)-Coupling

μ_1

00	0.25
01	0.25
10	0.25
11	0.25

$$R(a,b) = \{a=b\}$$

μ_2

00	0
01	0
10	0.5
11	0.5

μ_L	00	01	10	11
00	0.25			
01		0.25		
10			0.25	
11				0.25

μ_R	00	01	10	11
00	0			
01		0		
10			0.5	
11				0.5

Example of R-(ε,δ)-Coupling

μ_1

00	0.25
01	0.25
10	0.25
11	0.25

μ_2

00	0
01	0
10	0.5
11	0.5

$$R(a,b) = \{a=b\}$$

μ_L	00	01	10	11
00	0.25			
01		0.25		
10			0.25	
11				0.25

μ_R	00	01	10	11
00	0			
01		0		
10			0.5	
11				0.5

$$\Delta_0(\mu_L, \mu_R) = 0.5$$

Example of R-(ε,δ)-Coupling

μ_1

00	0.25
01	0.25
10	0.25
11	0.25

μ_2

00	0
01	0
10	0.5
11	0.5

Example of R-(ε , δ)-Coupling

μ_1

00	0.25
01	0.25
10	0.25
11	0.25

$$R(a,b) = \{a=b\}$$

μ_2

00	0
01	0
10	0.5
11	0.5

Example of R-(ε,δ)-Coupling

μ_1

00	0.25
01	0.25
10	0.25
11	0.25

$$R(a,b) = \{a=b\}$$

μ_2

00	0
01	0
10	0.5
11	0.5

μ_L	00	01	10	11
00	0.25			
01		0.25		
10			0.25	
11				0.25

μ_R	00	01	10	11
00	0			
01		0		
10			0.5	
11				0.5

Example of R-(ε, δ)-Coupling

μ_1

00	0.25
01	0.25
10	0.25
11	0.25

μ_2

00	0
01	0
10	0.5
11	0.5

$$R(a,b) = \{a=b\}$$

μ_L	00	01	10	11
00	0.25			
01		0.25		
10			0.25	
11				0.25

μ_R	00	01	10	11
00	0			
01		0		
10			0.5	
11				0.5

$$\Delta_1(\mu_L, \mu_R) = 0.25$$

$\mathtt{R}$-(ε,δ)-Coupling and Indistinguishability

Given two distributions $\mu_1 \in D(A)$, and $\mu_2 \in D(A)$, if we have a $=$-(ε,δ)-coupling between them, then they are (ε,δ)-indistinguishable.

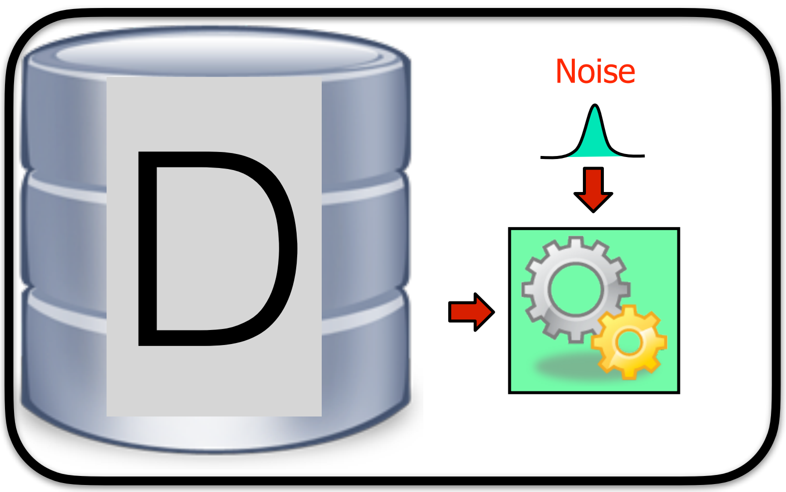
Probabilistic Relational Hoare Logic
Skip

$$\dfrac{}{\vdash_{0,0} \texttt{skip} \sim \texttt{skip} : P \Rightarrow P}$$

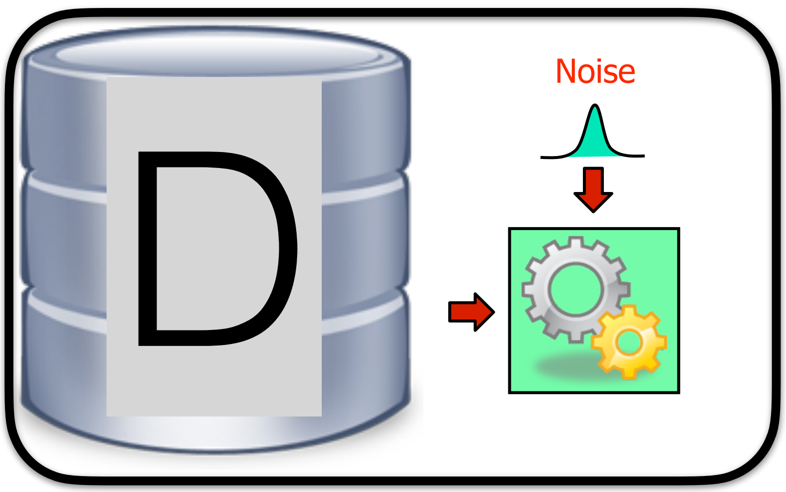
Probabilistic Relational Hoare Logic
Skip

$$\frac{}{\vdash_{\varepsilon,0} \begin{array}{l} x_1 := \$ \ \mathrm{Lap}(\varepsilon, y_1) \\ \sim \\ x_2 := \$ \ \mathrm{Lap}(\varepsilon, y_2) \\ : \ |y_1 - y_2| \leq 1 \Rightarrow = \end{array}}$$

Composition

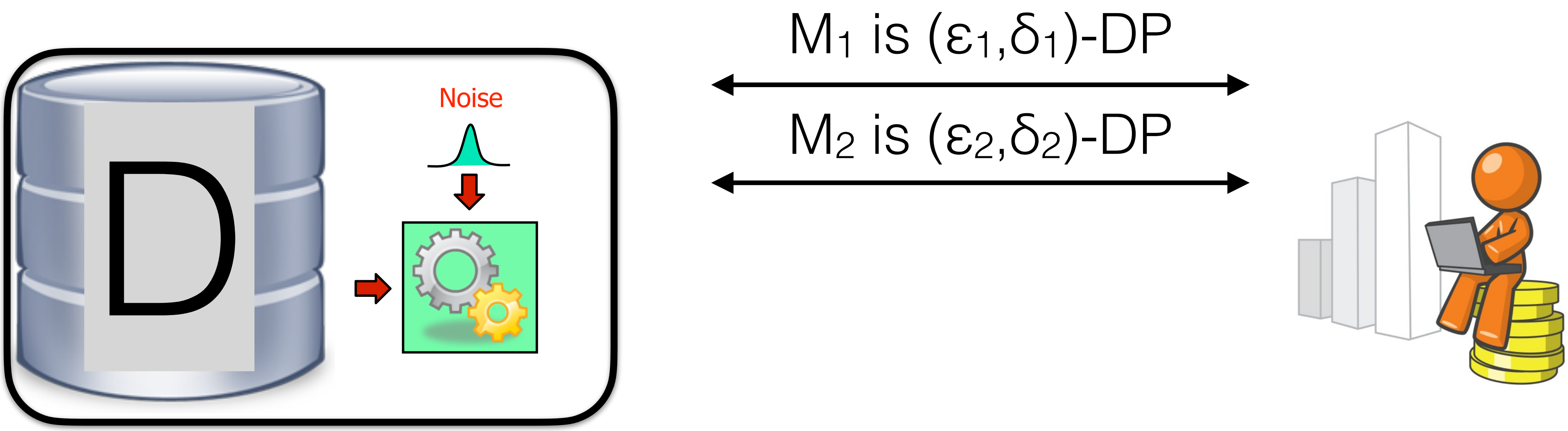


Composition



M_1 is (ε_1,δ_1)-DP

Composition



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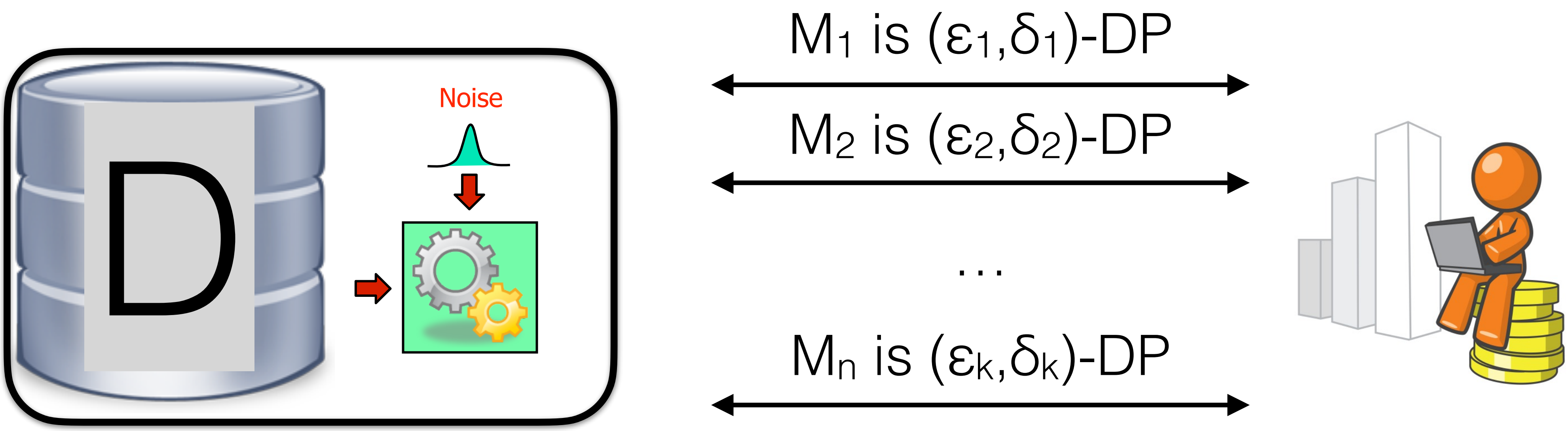
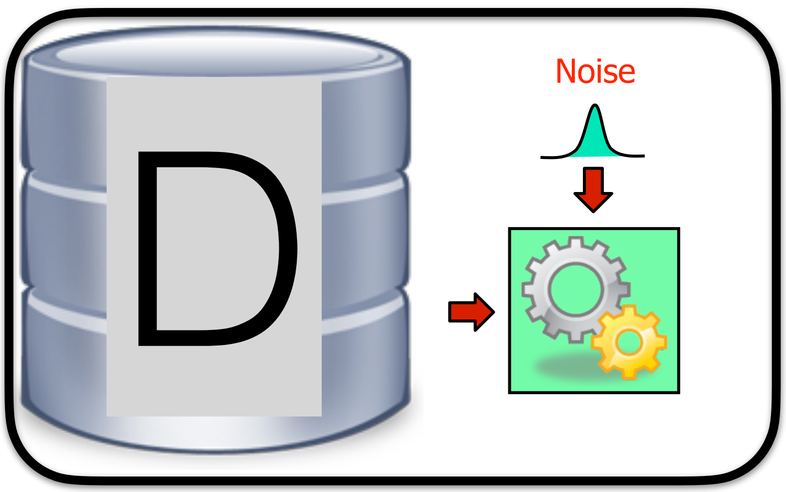


Composition



M_1 is (ε_1,δ_1)-DP

M_2 is (ε_2,δ_2)-DP

…

M_n is (ε_k,δ_k)-DP

The overall process is $(\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_k,\delta_1+\delta_2+\ldots+\delta_k)$-DP

Composition

Let M_1:DB $\rightarrow R_1$ be a (ε_1,δ_1)-differentially private program and M_2:DB $\rightarrow R_2$ be a (ε_2,δ_1)-differentially private program. Then, their composition $M_{1,2}$:DB$\rightarrow R_1$xR_2 defined as

$$M_{1,2}(D)=(M_1(D),M_2(D))$$

is $(\varepsilon_1+\varepsilon_2,\delta_1+\delta_2)$-differentially private.

Composition

Question: Why composition is important?

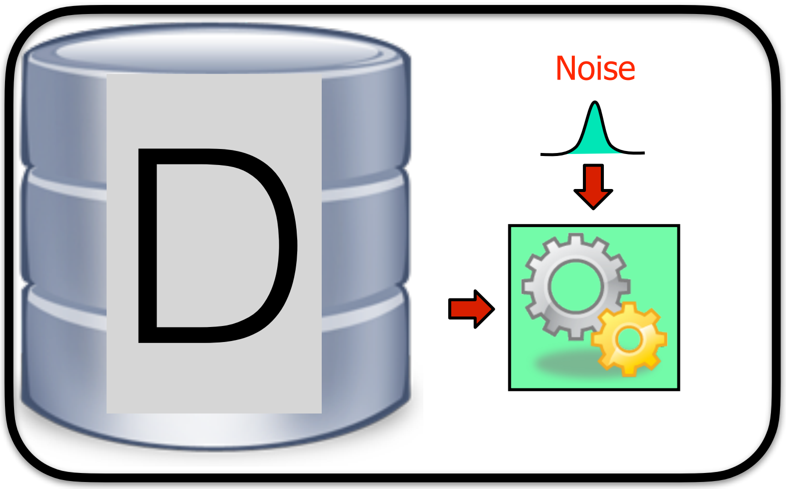
Composition

Question: Why composition is important?

Answer: Because it allows to reason about privacy as a budget!

Composition

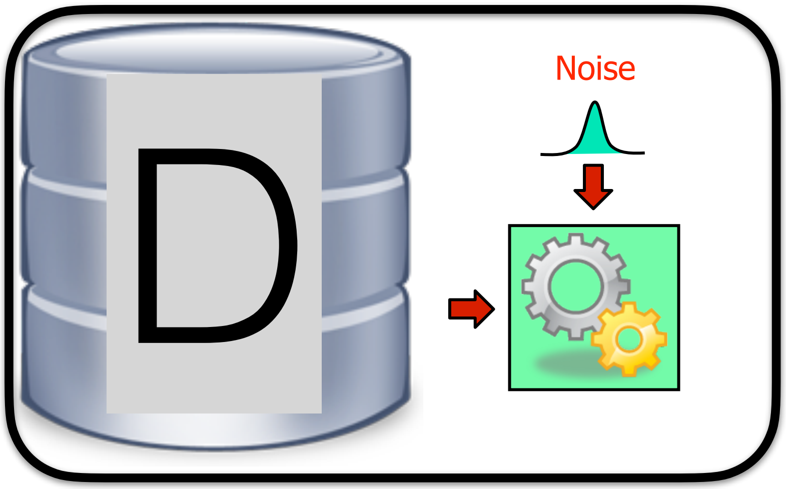
Budget=ε_{global}



Composition

Budget=ε_{global}

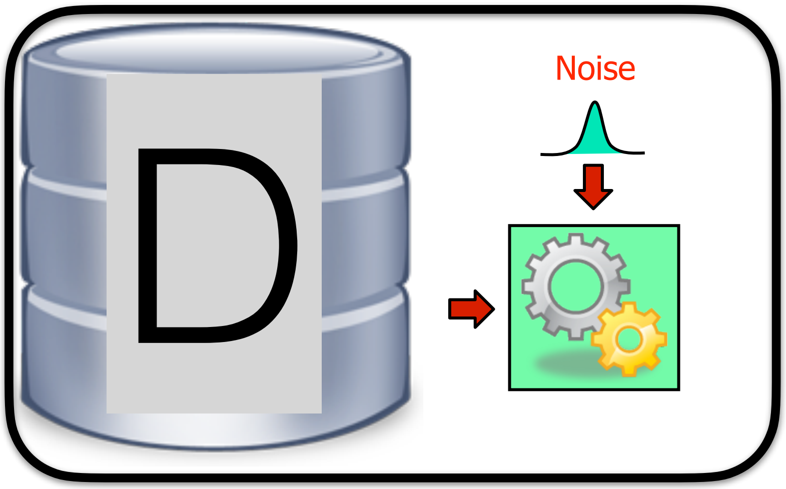
M_1 is ε_1-DP



Composition

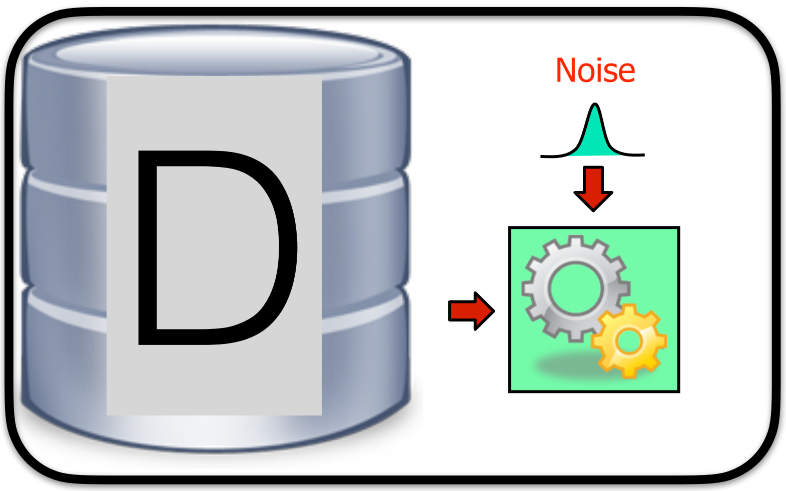
Budget$=\varepsilon_{global} - \varepsilon_1$

M_1 is ε_1-DP



Composition

Budget=ε_{global} - ε_1

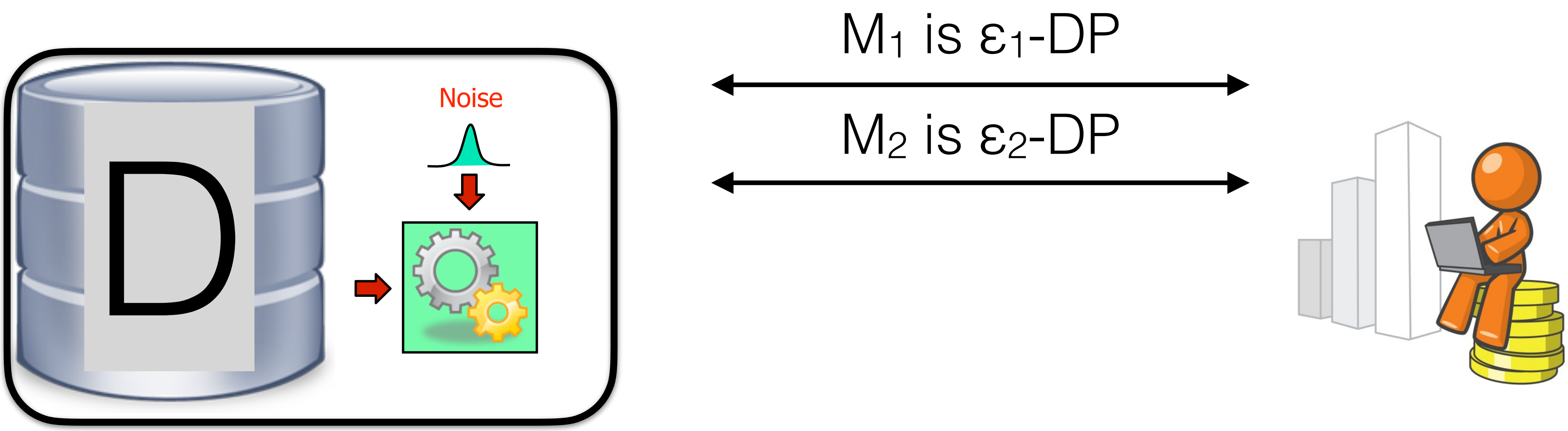


M_1 is ε_1-DP

M_2 is ε_2-DP

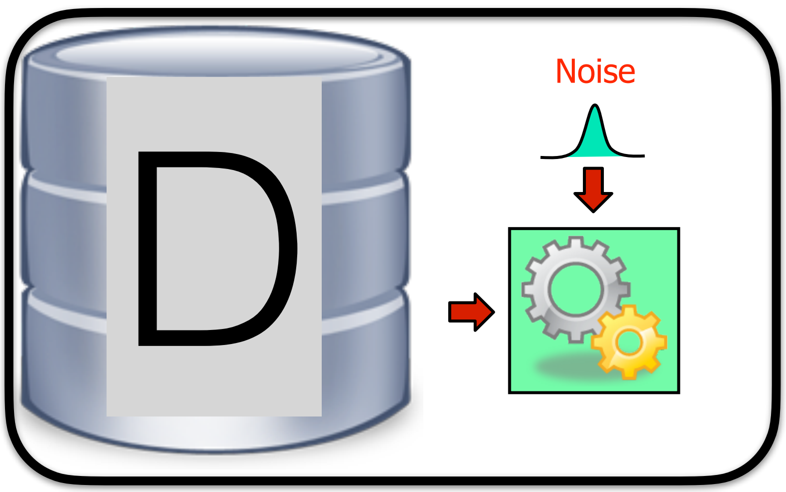
Composition

Budget=ε_{global} - ε_1 - ε_2



Composition

Budget=ε_{global} - ε_1 - ε_2 ...



M_1 is ε_1-DP

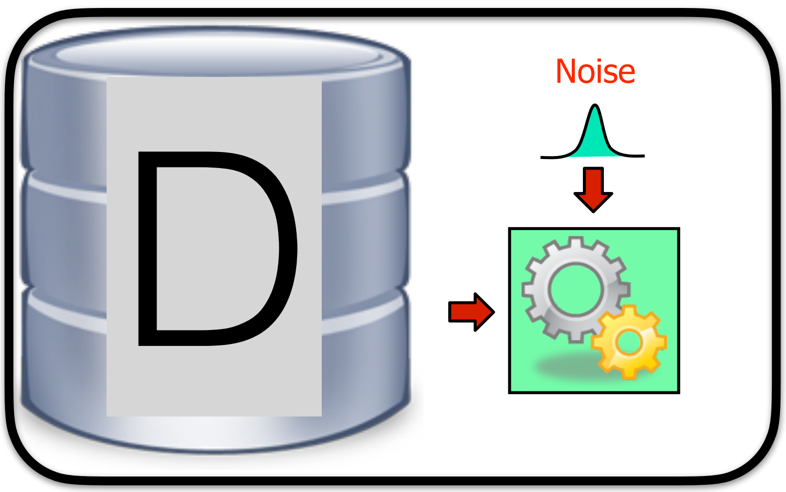
M_2 is ε_2-DP

...

M_n is ε_n-DP

Composition

Budget=ε_{global} - ε_1 - ε_2 … - ε_n



M_1 is ε_1-DP

M_2 is ε_2-DP

…

M_n is ε_n-DP

CDF

$$\text{Budget}=\varepsilon_{global} - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 - \varepsilon_6 - \varepsilon_7 - \varepsilon_8$$

$X=\{0,1\}^3$ ordered wrt binary encoding.

$D \in X^{10} =$

	D1	D2	D3
I1	0	0	0
I2	1	0	1
I3	0	1	0
I4	1	0	1
I5	0	0	0
I6	0	0	1
I7	1	1	0
I8	0	0	0
I9	0	1	0
I10	1	0	1

$q^*_{000}(D) = .3+L(1/\varepsilon_1)$

$q^*_{001}(D) = .4+L(1/\varepsilon_2)$

$q^*_{010}(D) = .6+L(1/\varepsilon_3)$

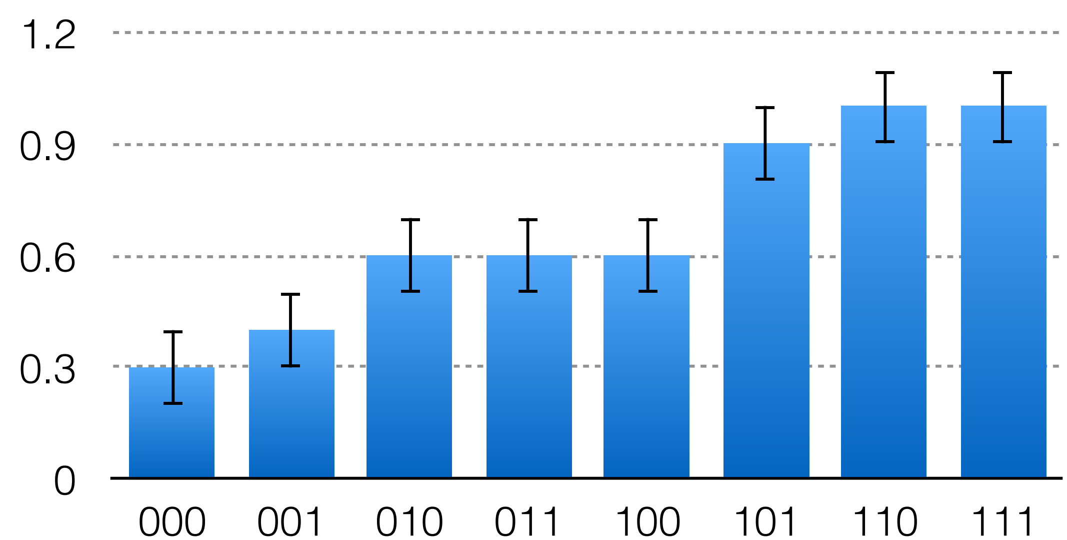
$q^*_{011}(D) = .6+L(1/\varepsilon_4)$

$q^*_{100}(D) = .6+L(1/\varepsilon_5)$

$q^*_{101}(D) = .9+L(1/\varepsilon_6)$

$q^*_{110}(D) = 1+L(1/\varepsilon_7)$

$q^*_{111}(D) = 1+L(1/\varepsilon_8)$



Marginals

Budget=ε_{global} - ε_1 - ε_2 - ε_3

$D \in X^{10} =$

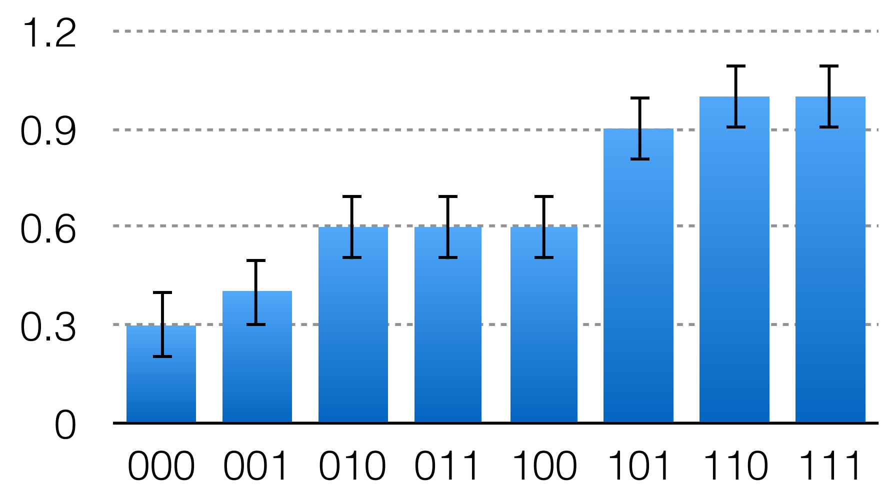
	D1	D2	D3
I1	0	0	0
I2	1	0	1
I3	0	1	0
I4	1	0	1
I5	0	0	0
I6	0	0	1
I7	1	1	0
I8	0	0	0
I9	0	1	0
I10	1	0	1
margin	$.4+Y_1$	$.3+Y_2$	$.4+Y_3$

$q^*_1(D) = .4+L(1/(10^*\varepsilon_1))$

$q^*_2(D) = .3+L(1/(10^*\varepsilon_2))$

$q^*_3(D) = .4+L(1/(10^*\varepsilon_3))$

$$\text{Budget} = \varepsilon_{\text{global}} - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 - \varepsilon_6 - \varepsilon_7 - \varepsilon_8$$



$$\text{Budget} = \varepsilon_{\text{global}} - \varepsilon_1 - \varepsilon_2 - \varepsilon_3$$

	D1	D2	D3
I1	0	0	0
I2	1	0	1
I3	0	1	0
I4	1	0	1
I5	0	0	0
I6	0	0	1
I7	1	1	0
I8	0	0	0
I9	0	1	0
I10	1	0	1
margin	$.4+Y_1$	$.3+Y_2$	$.4+Y_3$

Releasing partial sums

```
DummySum(d : {0,1} list) : real list
   i:= 0;
   s:= 0;
   r:= [];
   while (i<size d)
      s:= s + d[i]
      z:=$ Lap(eps,s)
      r:= r ++ [z];
      i:= i+1;
   return r
```

I am using the easycrypt notation here where `Lap(eps,a)` corresponds to adding to the value `a` noise from the Laplace distribution with `b=1/eps` and mean `mu=0`.

Probabilistic Relational Hoare Logic

Composition

$$\frac{\vdash_{\varepsilon 1,\delta 1} c_1 \sim c_2 : P \Rightarrow R \quad \vdash_{\varepsilon 2,\delta 2} c_1{}' \sim c_2{}' : R \Rightarrow S}{\vdash_{\varepsilon 1+\varepsilon 2,\delta 1+\delta 2} c_1 ; c_1{}' \sim c_2 ; c_2{}' : P \Rightarrow S}$$

Releasing partial sums

```
DummySum(d : {0,1} list) : real list
    i:=0;
    s:=0;
    r:=[];
    while (i<size d)
        z:=$ Lap(eps,d[i])
        s:= s + z
        r:= r ++ [s];
        i:= i+1;
    return r
```

Parallel Composition

Let M_1:DB $\rightarrow$R be a (ε_1,δ_1)-differentially private program and M_2:DB $\rightarrow$R be a (ε_2,δ_2)-differentially private program. Suppose that we partition D in a data-independent way into two datasets D_1 and D_2. Then, the composition $M_{1,2}$:DB$\rightarrow$R defined as

$$MP_{1,2}(D)=(M_1(D_1),M_2(D_2))$$

is $(\max(\varepsilon_1,\varepsilon_2),\max(\delta_1,\delta_2))$-differentially private.

Properties of Differential Privacy

Some important properties

- Resilience to post-processing
- Group privacy
- Composition

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- Resilience to post-processing
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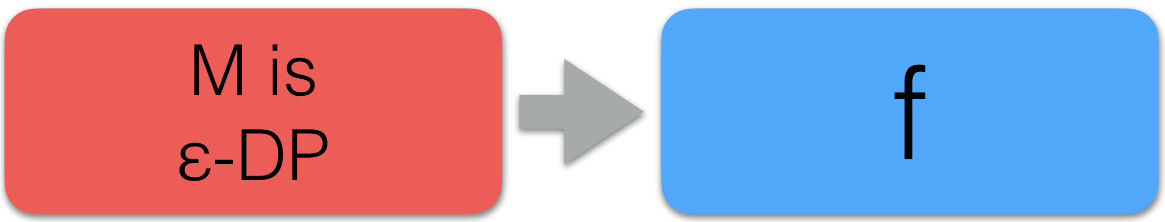
We will look at them in the context of $(\varepsilon,0)$-differential privacy.

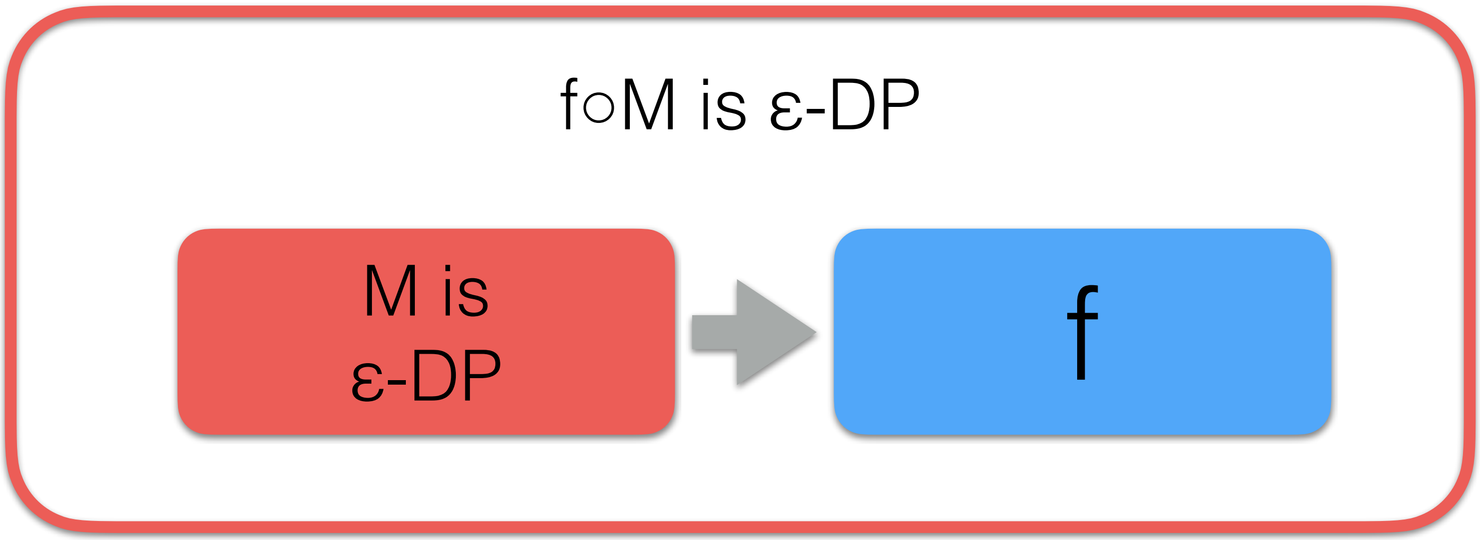
Resilience to Post-processing

M is
ε-DP

Resilience to Post-processing



Resilience to Post-processing



Resilience to Post-processing

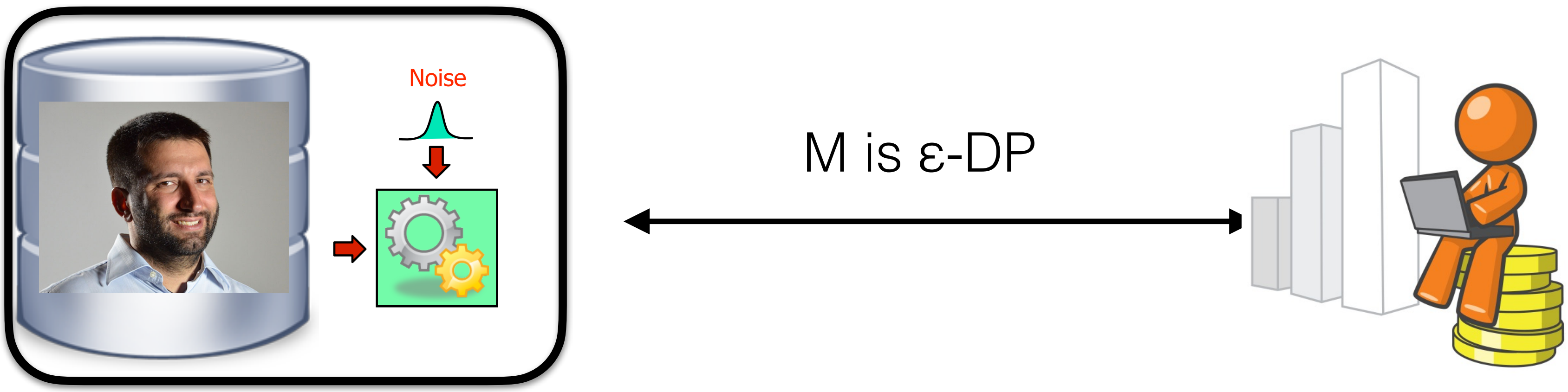
Question: Why is resilience to post-processing important?

Resilience to Post-processing

Question: Why is resilience to post-processing important?

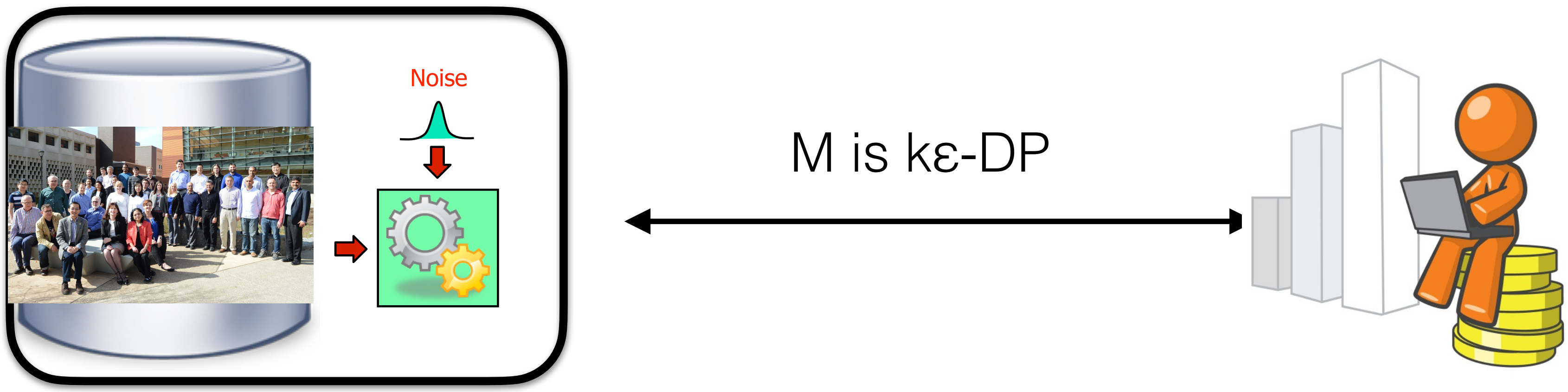
Answer: Because it is what allows us to publicly release the result of a differentially private analysis!

Group Privacy



$$\Pr[\mathcal{M}(D) = r] \leq e^{\epsilon} \Pr[\mathcal{M}(D') = r]$$

Group Privacy



$$\Pr[\mathcal{M}(D) \in S] \leq \exp(k\epsilon) \Pr[\mathcal{M}(D') \in S]$$

Group Privacy

Question: Why is group privacy important?

Group Privacy

Question: Why is group privacy important?

Answer: Because it allows to reason about privacy at different level of granularities!

Privacy Budget vs Epsilon

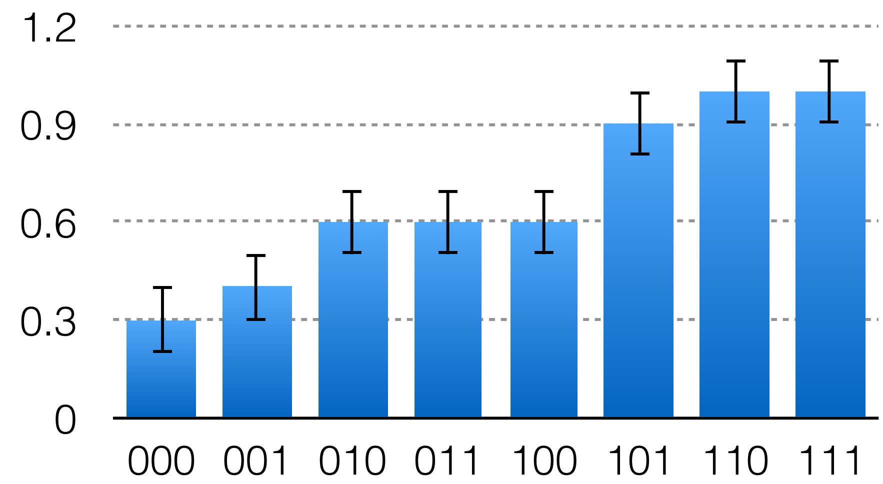
Sometimes is more convenient to think in terms of Privacy Budget: $\text{Budget} = \varepsilon_{global} - \sum \varepsilon_{local}$

Sometimes is more convenient to think in terms of epsilon: $\varepsilon_{global} = \sum \varepsilon_{local}$

Also making them uniforms is sometimes more informative.

$$\text{Budget} = \varepsilon_{global} - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 - \varepsilon_6 - \varepsilon_7 - \varepsilon_8$$

$$\varepsilon_{global} = \varepsilon+\varepsilon+\varepsilon+\varepsilon+\varepsilon+\varepsilon+\varepsilon+\varepsilon=8\varepsilon$$



$$\text{Budget} = \varepsilon_{global} - \varepsilon_1 - \varepsilon_2 - \varepsilon_3$$

$$\varepsilon_{global} = \varepsilon+\varepsilon+\varepsilon=3\varepsilon$$

	D1	D2	D3
I1	0	0	0
I2	1	0	1
I3	0	1	0
I4	1	0	1
I5	0	0	0
I6	0	0	1
I7	1	1	0
I8	0	0	0
I9	0	1	0
I10	1	0	1
margin	$.4+Y_1$	$.3+Y_2$	$.4+Y_3$