CS 591: Formal Methods in Security and Privacy

Differential Privacy

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From the previous classes

(ε,δ)-Differential Privacy

Definition

Given $\varepsilon,\delta \geq 0$, a probabilistic query $Q: X^n \rightarrow R$ is (ε,δ)-differentially private iff for all adjacent database b_1, b_2 and for every $S \subseteq R$:

$$\Pr[Q(b_1) \in S] \leq \exp(\varepsilon)\Pr[Q(b_2) \in S] + \delta$$

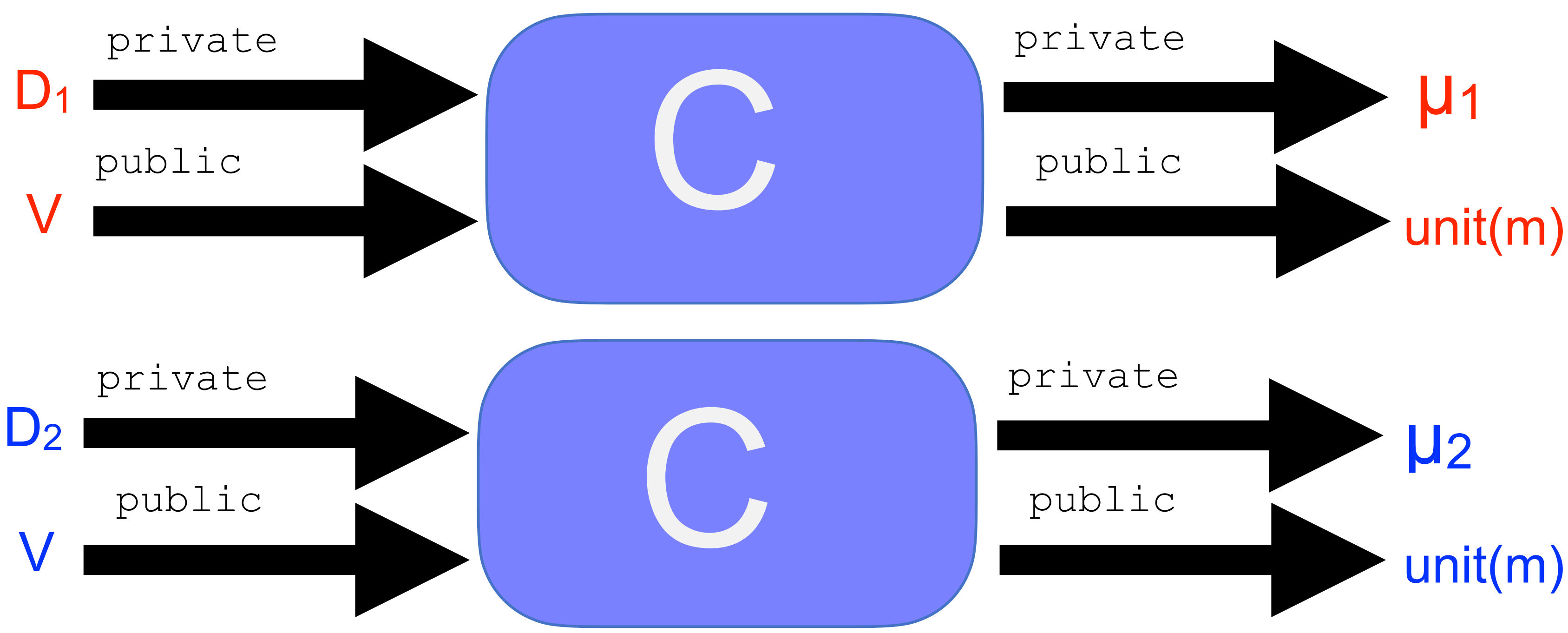
Releasing privately the mean of Some Data

```
Mean(d : private data) : public real
   i:=0;
   s:=0;
   while (i<size(d))
       s:=s + d[i]
       i:=i+1;
   z:=$ Laplace(sens/eps,0)
   z:= (s/i)+z
   return z
```

Differential Privacy as a Relational Property

c is differentially private if and only if for every $m_1 \sim m_2$ (extending the notion of adjacency to memories):

$\{c\}_{m1}=\mu_1$ and $\{c\}_{m2}=\mu_2$ implies $\Delta_\varepsilon(\mu_1,\mu_2) \leq \delta$



apRHL

Indistinguishability parameter

Precondition (a logical formula)

$$\vdash_{\epsilon,\delta} c_1 \sim c_2 : P \Rightarrow Q$$

Probabilistic Program

Probabilistic Program

Postcondition (a logical formula)

Validity of apRHL judgments

We say that the quadruple $\vdash_{\varepsilon,\delta} c_1 \sim c_2 : P \Rightarrow Q$ is valid if and only if for every pair of memories m_1, m_2 such that $P(m_1, m_2)$ we have:

$\{c_1\}_{m1} = \mu_1$ and $\{c_2\}_{m2} = \mu_2$ implies $Q_{\varepsilon,\delta}{}^*(\mu_1, \mu_2)$.

R-(ε,δ)-Coupling

Given two distributions $\mu_1 \in D(A)$, and $\mu_2 \in D(B)$, we have an R-(ε,δ)-coupling between them, for $R \subseteq AxB$ and $0 \leq \delta \leq 1$, $\varepsilon \geq 0$, if there are two joint distributions $\mu_L, \mu_R \in D(AxB)$ such that:

1) $\pi_1(\mu_L) = \mu_1$ and $\pi_2(\mu_R) = \mu_2$,
2) the support of μ_L and μ_R is contained in R. That is, if $\mu_L(a,b) > 0$, then $(a,b) \in R$, and if $\mu_R(a,b) > 0$, then $(a,b) \in R$.
3) $\Delta_\varepsilon(\mu_L, \mu_R) \leq \delta$

Example of R-(ε, δ)-Coupling

μ_1

00	0.25
01	0.25
10	0.25
11	0.25

μ_2

00	0.20
01	0.25
10	0.25
11	0.30

$$R(a,b) = \{a=b\}$$

μ_L	00	01	10	11
00	0.25			
01		0.25		
10			0.25	
11				0.25

μ_R	00	01	10	11
00	0.20			
01		0.25		
10			0.25	
11				0.30

$$\Delta_{0.3}(\mu_L, \mu_R) = 0$$

Example of R-(ε , δ)-Coupling

μ_1

00	0.2
01	0.25
10	0.25
11	0.3

$$R(a,b) = \{a \leq b\}$$

μ_2

00	0
01	0.40
10	0
11	0.6

μ_L	00	01	10	11
00		0.20		
01		0.25		
10				0.25
11				0.30

μ_R	00	01	10	11
00		0.20		
01		0.20		
10				0.3
11				0.3

$$\Delta_0(\mu_L, \mu_R) = 0.05$$

Probabilistic Relational Hoare Logic
Skip

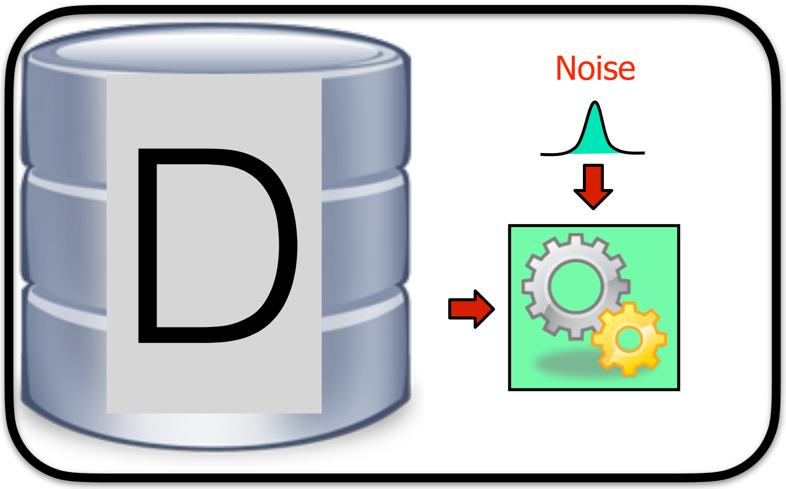
$$\frac{}{\vdash_{0,0} \texttt{skip} \sim \texttt{skip} : P \Rightarrow P}$$

Probabilistic Relational Hoare Logic
Skip

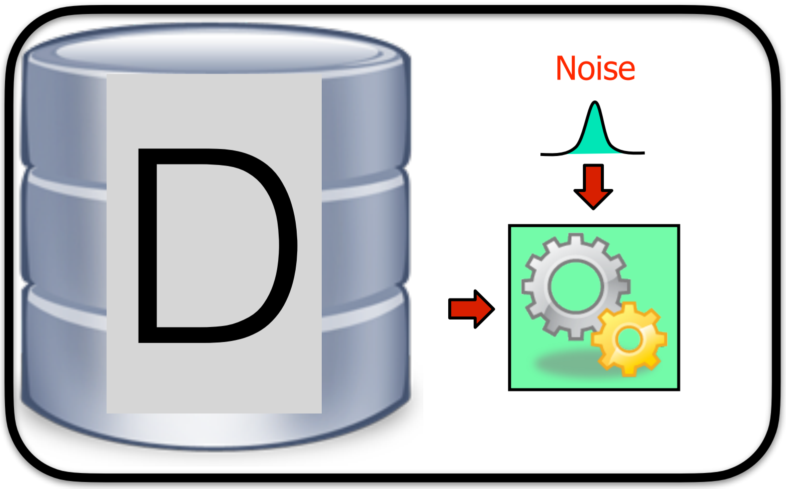
$$\dfrac{}{\vdash_{\varepsilon,0} \begin{array}{l} x_1 :=\$ \; \mathrm{Lap}(\varepsilon, y_1) \\ \sim \\ x_2 :=\$ \; \mathrm{Lap}(\varepsilon, y_2) \\ : \; |y_1 - y_2| \leq 1 \Rightarrow = \end{array}}$$

Today

Composition

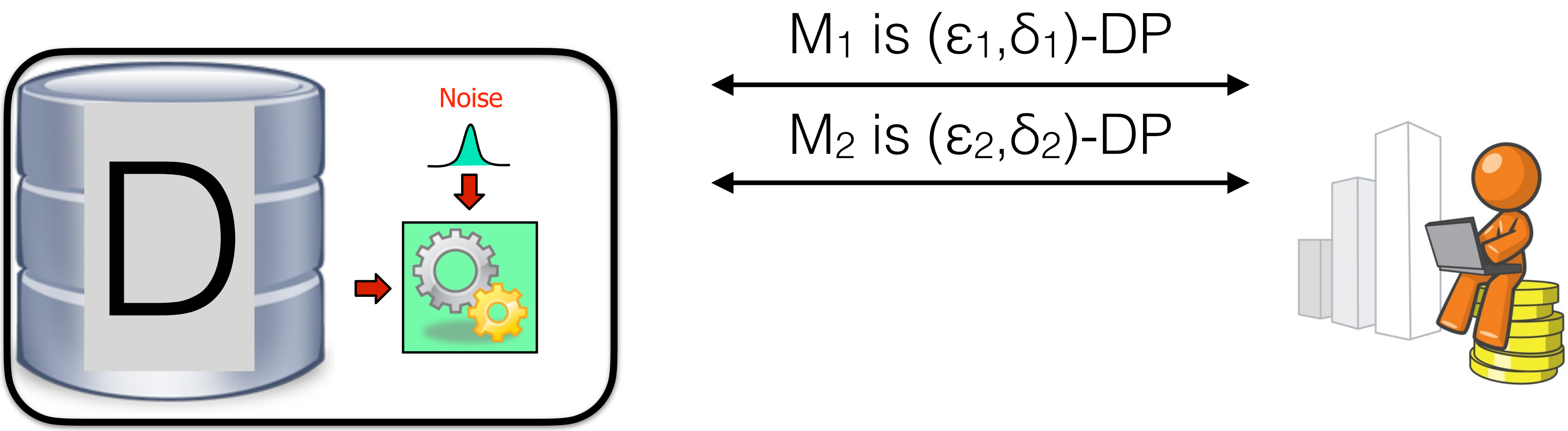


Composition

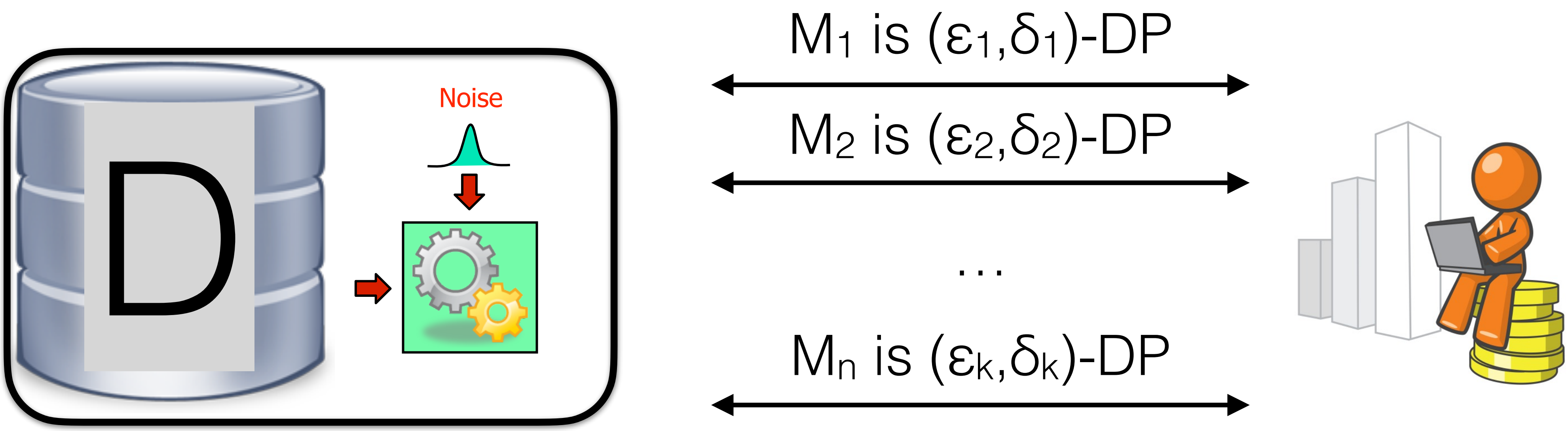


M_1 is (ε_1,δ_1)-DP

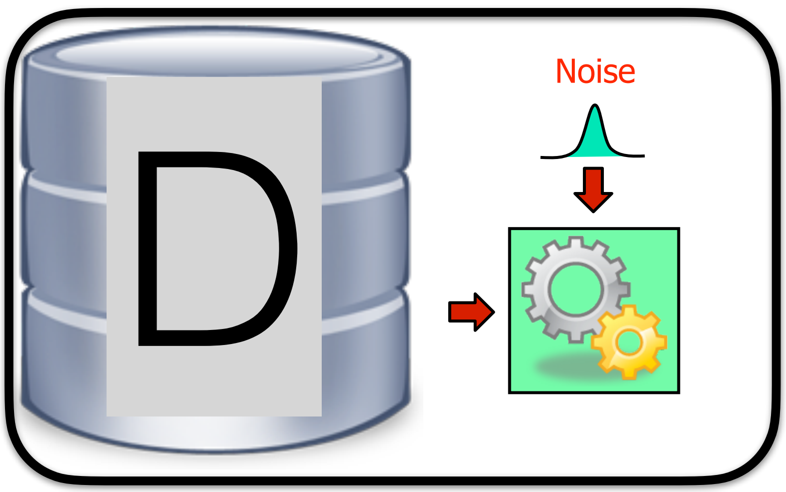
Composition



Composition



Composition



M_1 is (ε_1,δ_1)-DP

M_2 is (ε_2,δ_2)-DP

...

M_n is (ε_k,δ_k)-DP

The overall process is $(\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_k,\delta_1+\delta_2+\ldots+\delta_k)$-DP

Composition

Let M_1:DB $\rightarrow R_1$ be a (ε_1,δ_1)-differentially private program and M_2:DB $\rightarrow R_2$ be a (ε_2,δ_1)-differentially private program. Then, their composition $M_{1,2}$:DB$\rightarrow R_1 x R_2$ defined as

$$M_{1,2}(D)=(M_1(D),M_2(D))$$

is $(\varepsilon_1+\varepsilon_2,\delta_1+\delta_2)$-differentially private.

Composition

Question: Why composition is important?

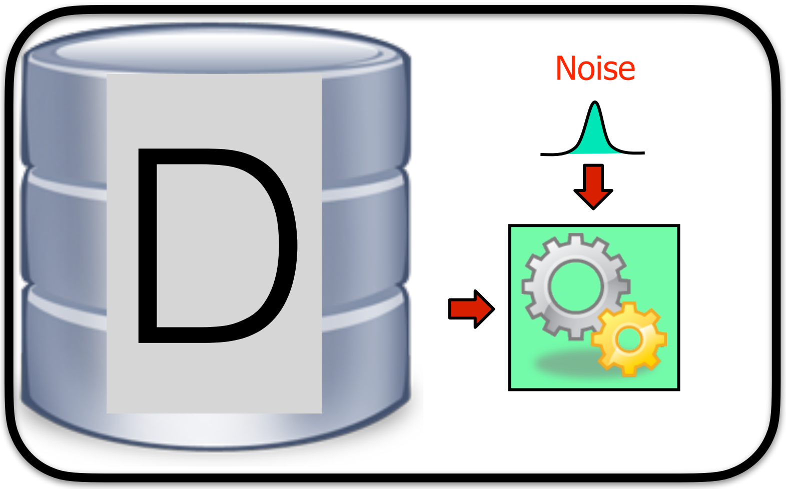
Composition

Question: Why composition is important?

Answer: Because it allows to reason about privacy as a budget!

Composition

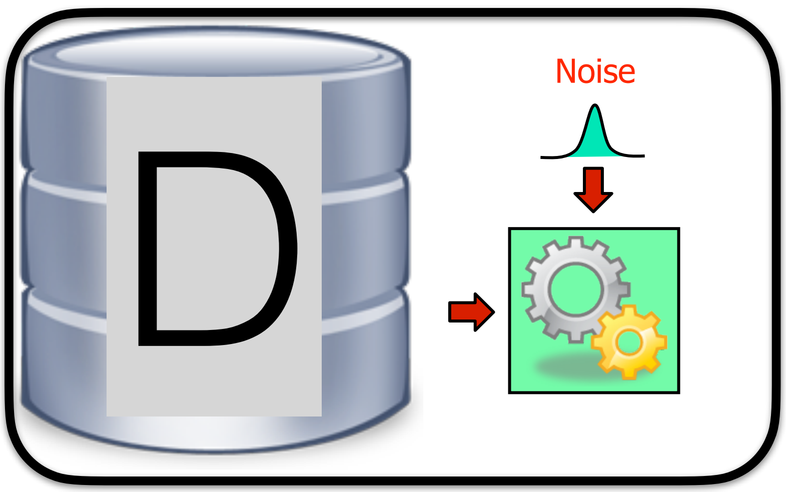
Budget=ε_{global}



Composition

Budget=ε_{global}

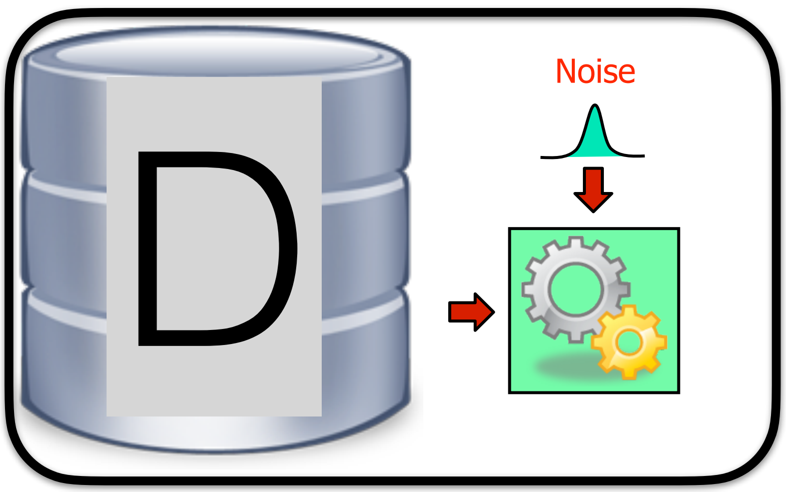
M_1 is ε_1-DP



Composition

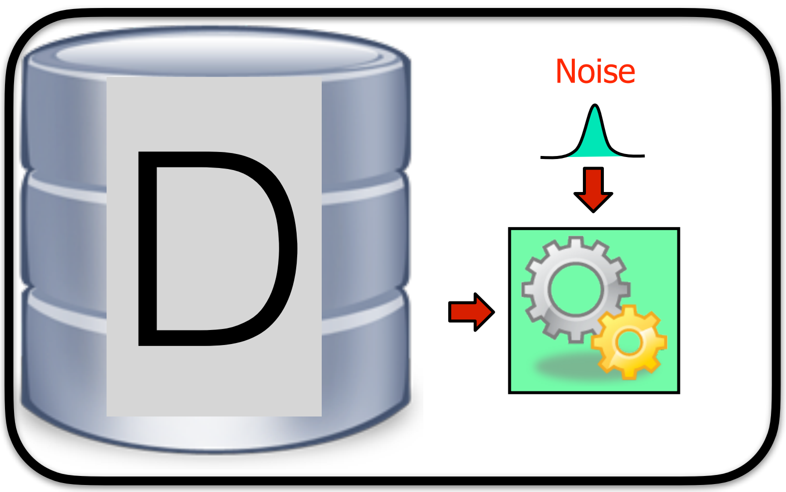
Budget=ε_{global} - ε_1

M_1 is ε_1-DP



Composition

Budget=ε_{global} - ε_1

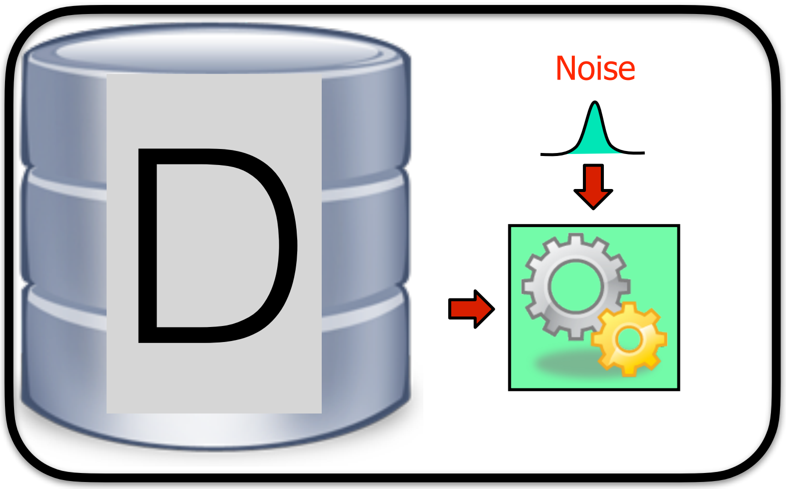


M_1 is ε_1-DP

M_2 is ε_2-DP

Composition

Budget=ε_{global} - ε_1 - ε_2

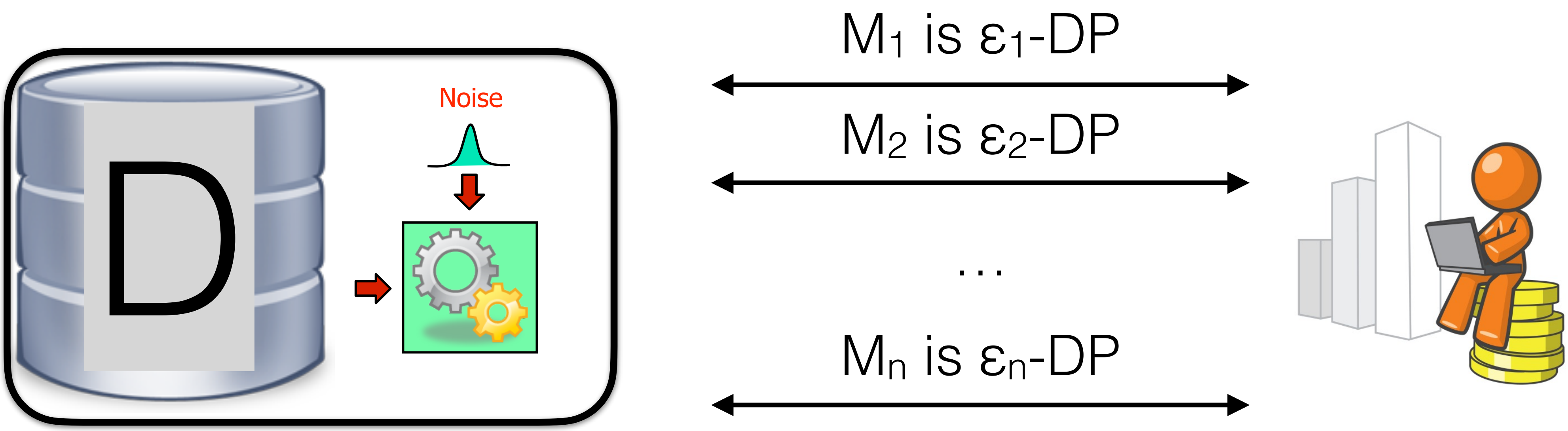


M_1 is ε_1-DP

M_2 is ε_2-DP

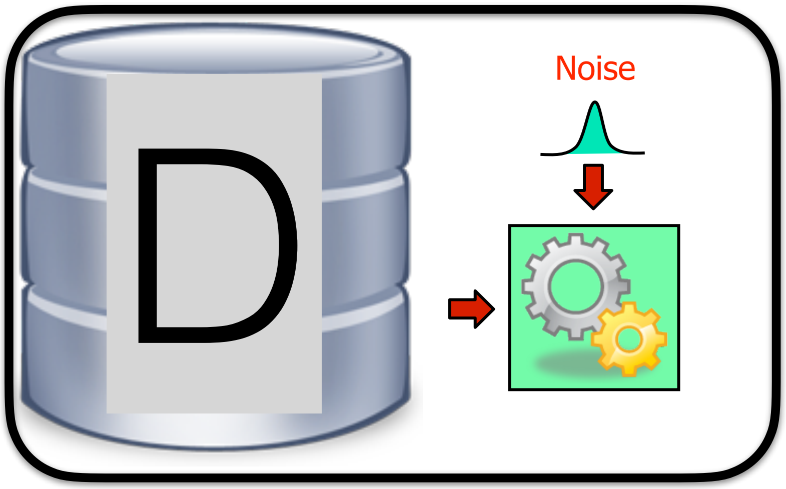
Composition

Budget=ε_{global} - ε_1 - ε_2 ...



Composition

Budget=ε_{global} - ε_1 - ε_2 … - ε_n



M_1 is ε_1-DP

M_2 is ε_2-DP

…

M_n is ε_n-DP

CDF

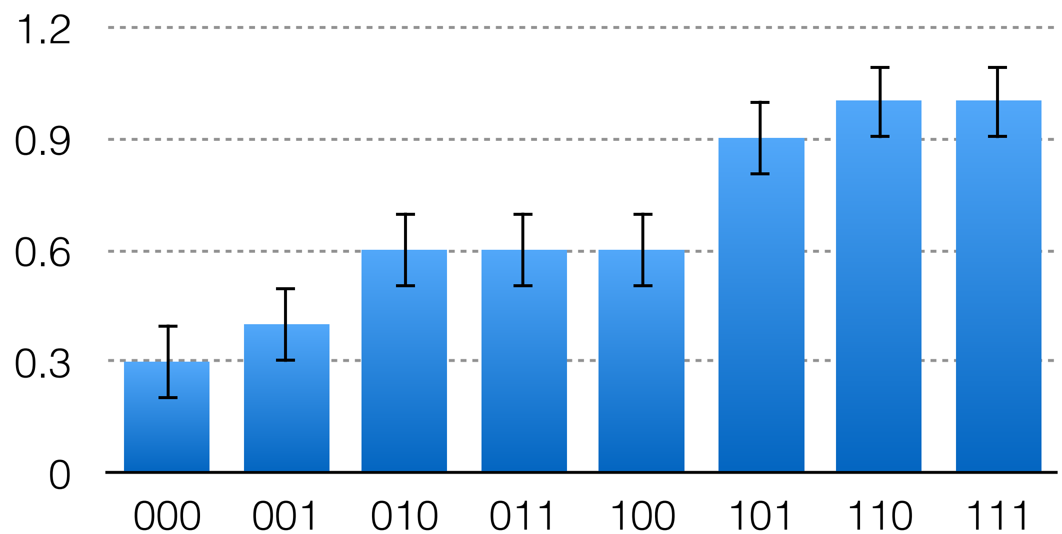
$$\text{Budget}=\varepsilon_{global} - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 - \varepsilon_6 - \varepsilon_7 - \varepsilon_8$$

$X=\{0,1\}^3$ ordered wrt binary encoding.

$$q^*_{000}(D) = .3+L(1/\varepsilon_1)$$
$$q^*_{001}(D) = .4+L(1/\varepsilon_2)$$
$$q^*_{010}(D) = .6+L(1/\varepsilon_3)$$
$$q^*_{011}(D) = .6+L(1/\varepsilon_4)$$
$$q^*_{100}(D) = .6+L(1/\varepsilon_5)$$
$$q^*_{101}(D) = .9+L(1/\varepsilon_6)$$
$$q^*_{110}(D) = 1+L(1/\varepsilon_7)$$
$$q^*_{111}(D) = 1+L(1/\varepsilon_8)$$

$D \in X^{10} =$

	D1	D2	D3
I1	0	0	0
I2	1	0	1
I3	0	1	0
I4	1	0	1
I5	0	0	0
I6	0	0	1
I7	1	1	0
I8	0	0	0
I9	0	1	0
I10	1	0	1



Marginals

Budget=ε_{global} - ε_1 - ε_2 - ε_3

$D \in X^{10}$ =

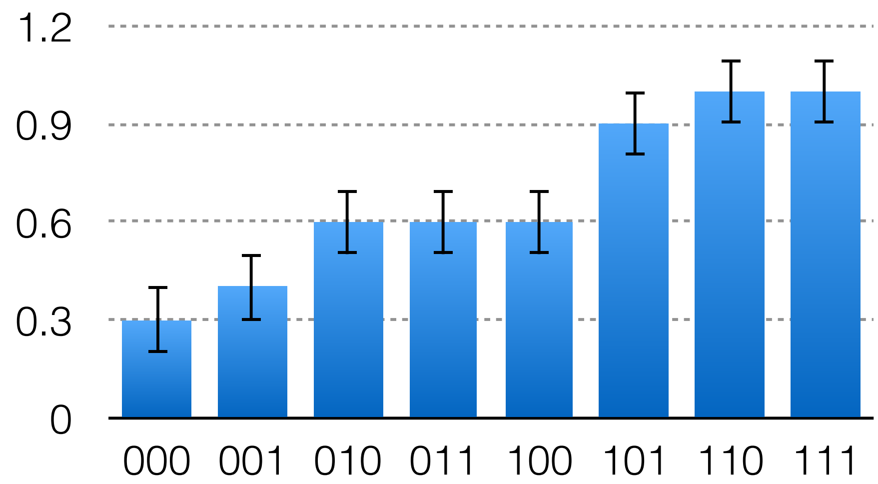
	D1	D2	D3
I1	0	0	0
I2	1	0	1
I3	0	1	0
I4	1	0	1
I5	0	0	0
I6	0	0	1
I7	1	1	0
I8	0	0	0
I9	0	1	0
I10	1	0	1
margin	$.4+Y_1$	$.3+Y_2$	$.4+Y_3$

$q^*_1(D) = .4+L(1/(10^*\varepsilon_1))$

$q^*_2(D) = .3+L(1/(10^*\varepsilon_2))$

$q^*_3(D) = .4+L(1/(10^*\varepsilon_3))$

$$\text{Budget} = \varepsilon_{global} - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 - \varepsilon_6 - \varepsilon_7 - \varepsilon_8$$



$$\text{Budget} = \varepsilon_{global} - \varepsilon_1 - \varepsilon_2 - \varepsilon_3$$

	D1	D2	D3
I1	0	0	0
I2	1	0	1
I3	0	1	0
I4	1	0	1
I5	0	0	0
I6	0	0	1
I7	1	1	0
I8	0	0	0
I9	0	1	0
I10	1	0	1
margin	$.4+Y_1$	$.3+Y_2$	$.4+Y_3$

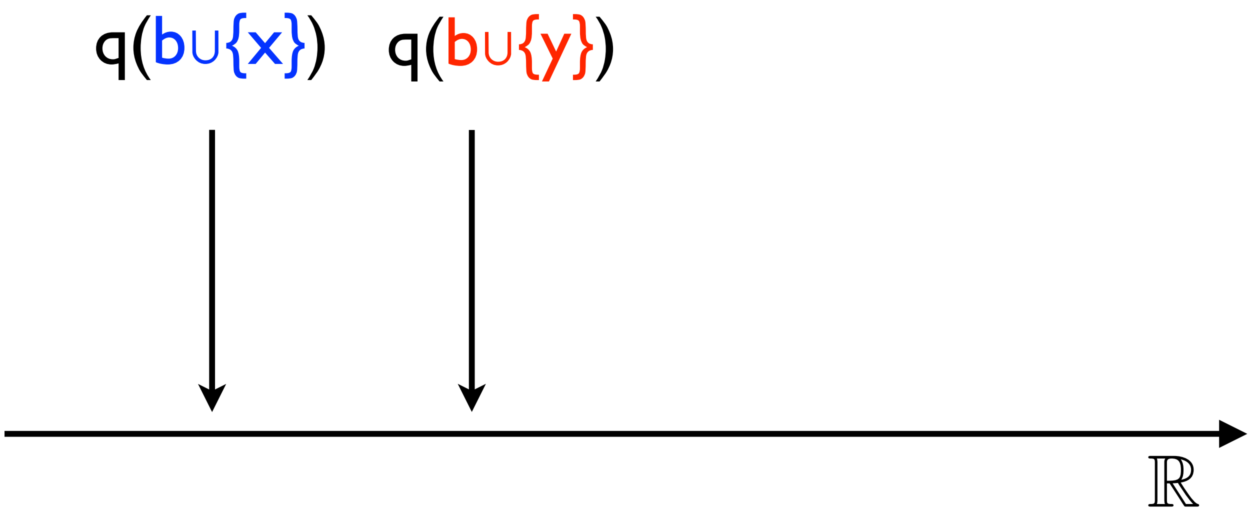
Releasing partial sums

```
DummySum(d : {0,1} list) : real list
   i:= 0;
   s:= 0;
   r:= [];
   while (i<size d)
      s:= s + d[i]
      z:=$ Lap(eps,s)
      r:= r ++ [z];
      i:= i+1;
   return r
```

I am using the easycrypt notation here where `Lap(eps,a)` corresponds to adding to the value `a` noise from the Laplace distribution with `b=1/eps` and mean `mu=0`.

Global Sensitivity

$$GS_q = \max\{\,|q(D) - q(D')|\ \text{s.t.}\ D \sim D'\}$$



Probabilistic Relational Hoare Logic
Composition

$$\frac{\vdash_{\varepsilon 1,\delta 1} c_1 \sim c_2 : P \Rightarrow R \quad \vdash_{\varepsilon 2,\delta 2} c_1' \sim c_2' : R \Rightarrow S}{\vdash_{\varepsilon 1+\varepsilon 2,\delta 1+\delta 2} c_1 ; c_1' \sim c_2 ; c_2' : P \Rightarrow S}$$

Releasing partial sums

```
DummySum(d : {0,1} list) : real list
   i:=0;
   s:=0;
   r:=[];
   while (i<size d)
       z:=$ Lap(eps,d[i])
       s:= s + z
       r:= r ++ [s];
       i:= i+1;
   return r
```

Parallel Composition

Let M_1:DB $\rightarrow$R be a (ε_1,δ_1)-differentially private program and M_2:DB $\rightarrow$R be a (ε_2,δ_2)-differentially private program. Suppose that we partition D in a data-independent way into two datasets D_1 and D_2. Then, the composition $M_{1,2}$:DB$\rightarrow$R defined as

$$MP_{1,2}(D)=(M_1(D_1),M_2(D_2))$$

is $(\max(\varepsilon_1,\varepsilon_2),\max(\delta_1,\delta_2))$-differentially private.

Probabilistic Relational Hoare Logic

Composition

$$\frac{\vdash_{\varepsilon 1, \delta 1} c_1 \sim c_2 : P \Rightarrow R \quad \vdash_{\varepsilon 2, \delta 2} c_1' \sim c_2' : R \Rightarrow S}{\vdash_{\varepsilon 1 + \varepsilon 2, \delta 1 + \delta 2} c_1 ; c_1' \sim c_2 ; c_2' : P \Rightarrow S}$$

apRHL

awhile

$$\frac{\begin{array}{c} P \land e<1> \leq 0 \Rightarrow \neg b1<1> \\[1em] \vdash_{\varepsilon_k,\delta_k} c1 \sim c2 : P \land b1<1> \land b2<2> \land k=e<1> \land e<1> \leq n \\ \Longrightarrow P \land b1<1>=b2<2> \land k < e<1> \end{array}}{\begin{array}{c} \text{while } b1 \text{ do } c1 \sim \text{while } b2 \text{ do } c2 \\ \vdash_{\sum\varepsilon_k,\sum\delta_k} : P \land b1<1>=b2<2> \land e<1> \leq n \\ \Longrightarrow P \land \neg b1<1> \land \neg b2<2> \end{array}}$$

Properties of Differential Privacy

Some important properties

- Resilience to post-processing
- Group privacy
- Composition

Some important properties

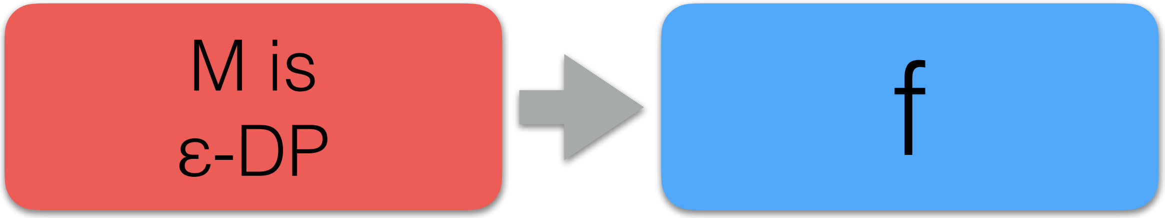
- Resilience to post-processing
- Group privacy
- Composition

We will look at them in the context of $(\varepsilon,0)$-differential privacy.

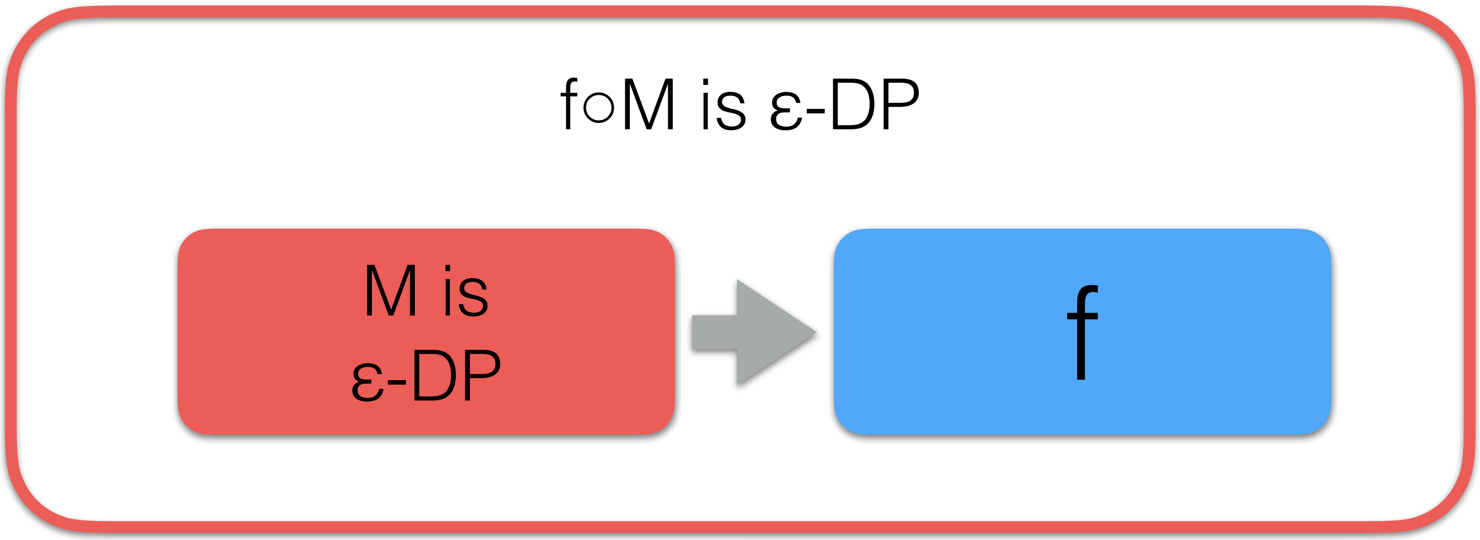
Resilience to Post-processing

M is
ε-DP

Resilience to Post-processing



Resilience to Post-processing



Resilience to Post-processing

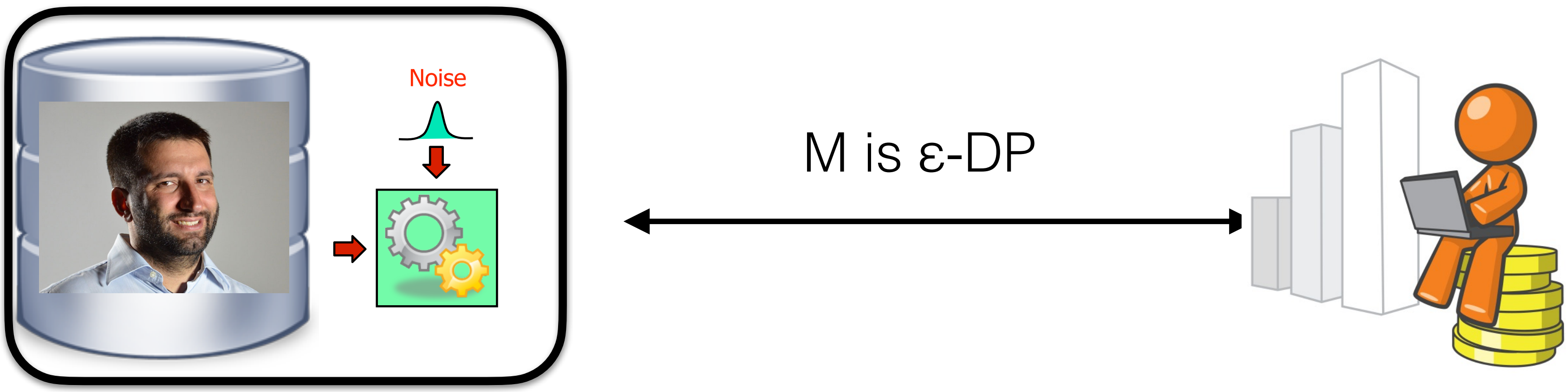
Question: Why is resilience to post-processing important?

Resilience to Post-processing

Question: Why is resilience to post-processing important?

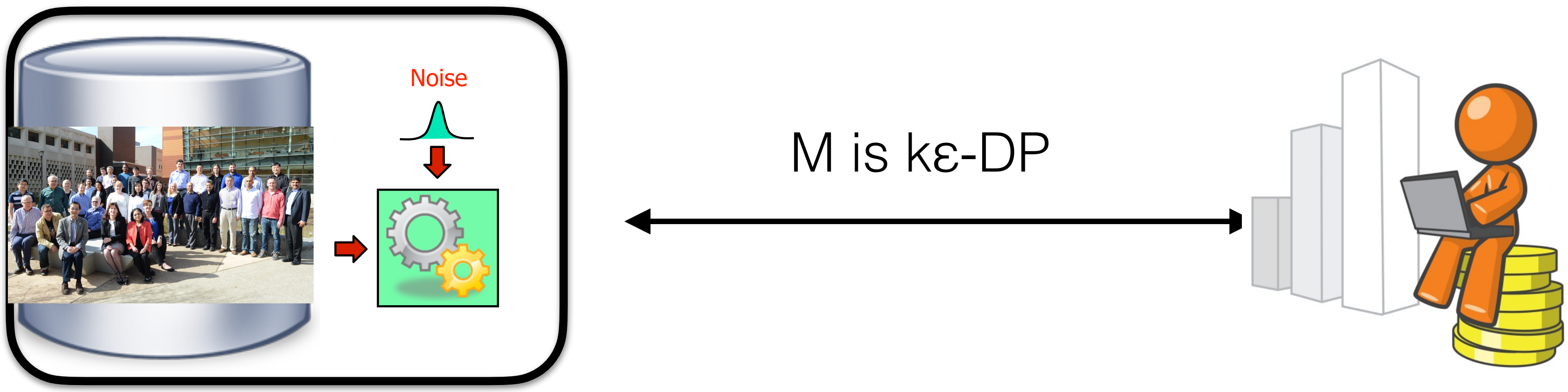
Answer: Because it is what allows us to publicly release the result of a differentially private analysis!

Group Privacy



$$\Pr[\mathcal{M}(D) = r] \leq e^{\epsilon} \Pr[\mathcal{M}(D') = r]$$

Group Privacy



$$\Pr[\mathcal{M}(D) \in S] \leq \exp(k\epsilon)\Pr[\mathcal{M}(D') \in S]$$

Group Privacy

Question: Why is group privacy important?

Group Privacy

Question: Why is group privacy important?

Answer: Because it allows to reason about privacy at different level of granularities!

Privacy Budget vs Epsilon

Sometimes is more convenient to think in terms of Privacy Budget: Budget=$\varepsilon_{global} - \sum \varepsilon_{local}$

Sometimes is more convenient to think in terms of epsilon: $\varepsilon_{global} = \sum \varepsilon_{local}$

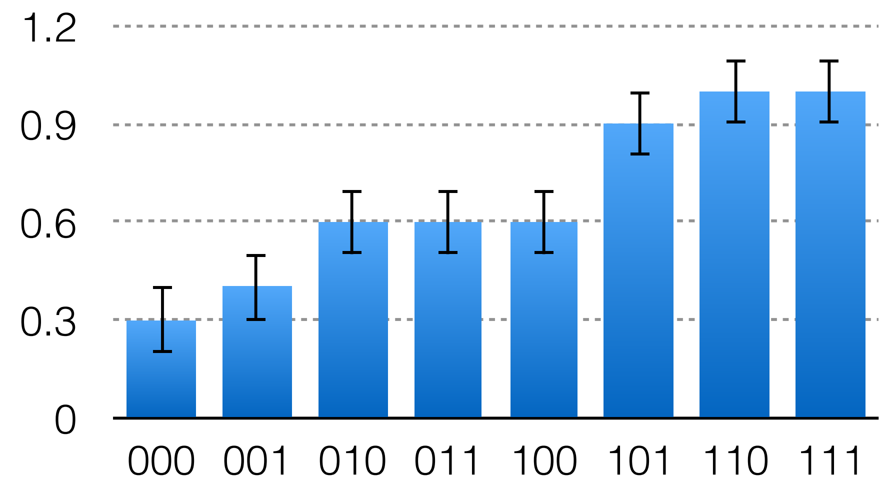
Also making them uniforms is sometimes more informative.

$$\text{Budget} = \varepsilon_{global} - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 - \varepsilon_6 - \varepsilon_7 - \varepsilon_8$$

$$\varepsilon_{global} = \varepsilon+\varepsilon+\varepsilon+\varepsilon+\varepsilon+\varepsilon+\varepsilon+\varepsilon = 8\varepsilon$$

$$\text{Budget} = \varepsilon_{global} - \varepsilon_1 - \varepsilon_2 - \varepsilon_3$$

$$\varepsilon_{global} = \varepsilon+\varepsilon+\varepsilon = 3\varepsilon$$



	D1	D2	D3
I1	0	0	0
I2	1	0	1
I3	0	1	0
I4	1	0	1
I5	0	0	0
I6	0	0	1
I7	1	1	0
I8	0	0	0
I9	0	1	0
I10	1	0	1
margin	$.4+Y_1$	$.3+Y_2$	$.4+Y_3$