CS 591: Formal Methods in Security and Privacy

Formal Proofs for Cryptography — Continued

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Symmetric Encryption from PRF + Randomness

- We are studying a symmetric encryption scheme built out of a pseudorandom function plus randomness.
    - Symmetric encryption means the same key is used for both encryption and decryption.
- We'll review the definition of when a symmetric encryption scheme is IND-CPA (indistinguishability under chosen plaintext attack) secure.
- We'll also review our instance of this scheme, and our informal analysis of adversaries' strategies for breaking security.
- You can find all the definitions and the proofs on GitHub:

https://github.com/alleystoughton/EasyTeach/tree/master/encryption

Symmetric Encryption Schemes

- Our treatment of symmetric encryption schemes is parameterized by three types:

```
type key.     (* encryption keys, key_len bits *)
type text.    (* plaintexts, text_len bits *)
type cipher.  (* ciphertexts - scheme specific *)
```

- An encryption scheme is a *stateless* implementation of this module interface:

```
module type ENC = {
  proc key_gen() : key  (* key generation *)
  proc enc(k : key, x : text) : cipher  (* encryption *)
  proc dec(k : key, c : cipher) : text  (* decryption *)
}.
```

Scheme Correctness

- An encryption scheme is *correct* if and only if the following procedure returns true with probability 1 for all arguments:

```
module Cor (Enc : ENC) = {
  proc main(x : text) : bool = {
    var k : key; var c : cipher; var y : text;
    k <@ Enc.key_gen();
    c <@ Enc.enc(k, x);
    y <@ Enc.dec(k, c);
    return x = y;
  }
}.
```

- The module `Cor` is parameterized (may be applied to) an arbitrary encryption scheme, `Enc`.

Encryption Oracles

- To define IND-CPA security of encryption schemes, we need the notion of an *encryption oracle*, which both the adversary and IND-CPA game will interact with:

```
module type EO = {
  (* initialization - generates key *)
  proc * init() : unit
  (* encryption by adversary before game's encryption *)
  proc enc_pre(x : text) : cipher
  (* one-time encryption by game *)
  proc genc(x : text) : cipher
  (* encryption by adversary after game's encryption *)
  proc enc_post(x : text) : cipher
}.
```

Standard Encryption Oracle

- Here is the standard encryption oracle, parameterized by an encryption scheme, `Enc`:

```
module EncO (Enc : ENC) : EO = {
  var key : key
  var ctr_pre : int
  var ctr_post : int

  proc init() : unit = {
    key <@ Enc.key_gen();
    ctr_pre <- 0; ctr_post <- 0;
  }
```

Standard Encryption Oracle

```
proc enc_pre(x : text) : cipher = {
  var c : cipher;
  if (ctr_pre < limit_pre) {
    ctr_pre <- ctr_pre + 1;
    c <@ Enc.enc(key, x);
  }
  else {
    c <- ciph_def;  (* default result *)
  }
  return c;
}
```

Standard Encryption Oracle

```
proc genc(x : text) : cipher = {
  var c : cipher;
  c <@ Enc.enc(key, x);
  return c;
}
```

Standard Encryption Oracle

```
  proc enc_post(x : text) : cipher = {
    var c : cipher;
    if (ctr_post < limit_post) {
      ctr_post <- ctr_post + 1;
      c <@ Enc.enc(key, x);
    }
    else {
      c <- ciph_def;  (* default result *)
    }
    return c;
  }
}.
```

Encryption Adversary

- An *encryption adversary* is parameterized by an encryption oracle:

```
module type ADV (EO : EO) = {
  (* choose a pair of plaintexts, x1/x2 *)
  proc * choose() : text * text {EO.enc_pre}

  (* given ciphertext c based on a random boolean b
     (the encryption using EO.genc of x1 if b = true,
      the encryption of x2 if b = false), try to guess b
  *)
  proc guess(c : cipher) : bool {EO.enc_post}
}.
```

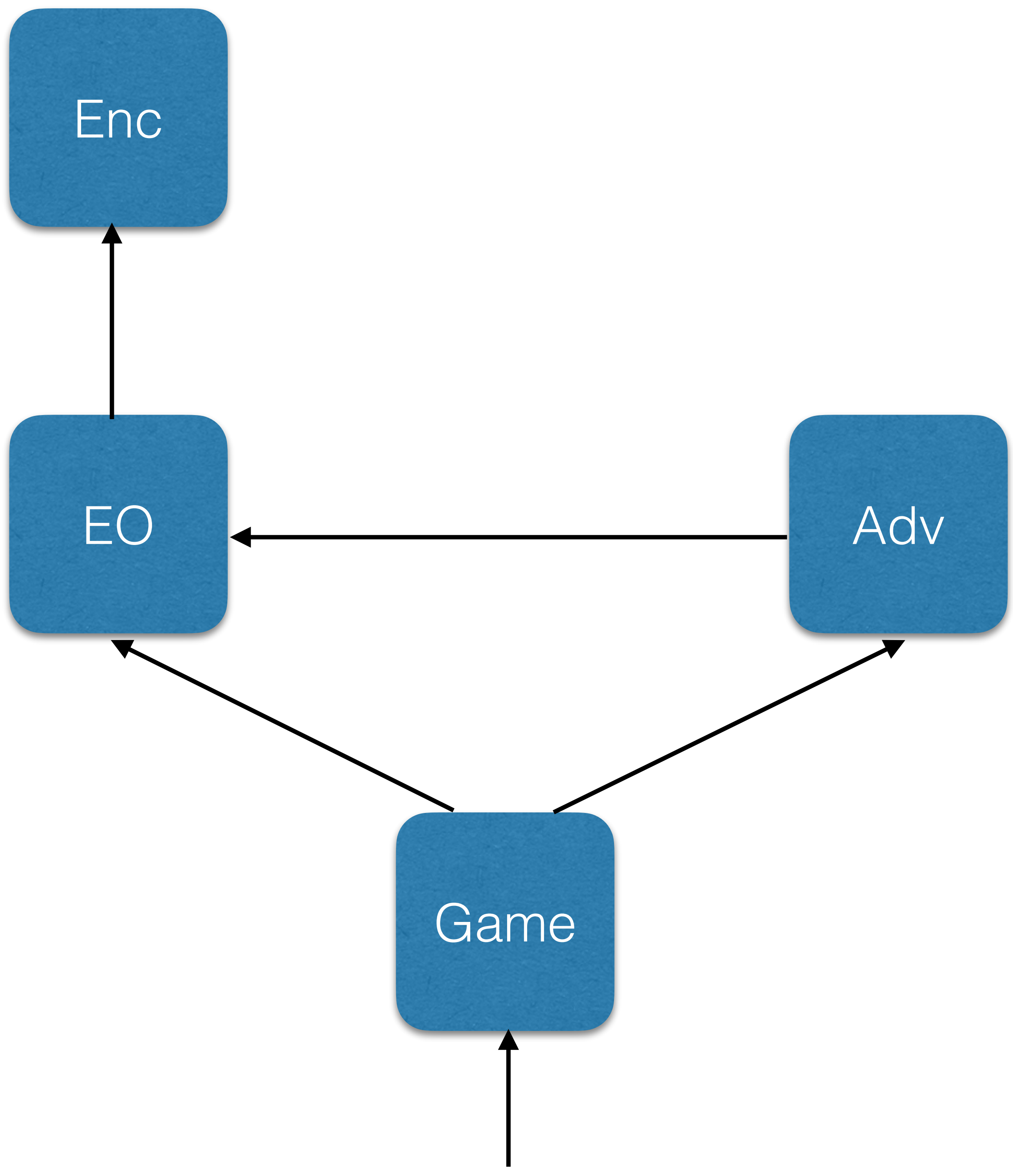
- Adversaries may be probabilistic.

IND-CPA Game

- The IND-CPA Game is parameterized by an encryption scheme and an encryption adversary:

```
module INDCPA (Enc : ENC, Adv : ADV) = {
  module EO = EncO(Enc)          (* make EO from Enc *)
  module A = Adv(EO)             (* connect Adv to EO *)
  proc main() : bool = {
    var b, b' : bool; var x1, x2 : text; var c : cipher;
    EO.init();                   (* initialize EO *)
    (x1, x2) <@ A.choose();      (* let A choose x1/x2 *)
    b <$ {0,1};                  (* choose boolean b *)
    c <@ EO.genc(b ? x1 : x2);   (* encrypt x1 or x2 *)
    b' <@ A.guess(c);            (* let A guess b from c *)
    return b = b';               (* see if A won *)
  }
}.
```

IND-CPA Game



IND-CPA Security

- In our security theorem for a given encryption scheme `Enc` and adversary Adv, we prove an upper bound on the absolute value of the difference between the probability that `Adv` wins the game and 1/2:

```
`|Pr[INDCPA(Enc, Adv).main() @ &m : res] - 1%r / 2%r|
  <= … Adv …
```

- Ideally, we'd like the upper bound to be 0, so that the probability that `Enc` wins is exactly 1/2, but this won't be possible.

- The upper bound may also be a function of the number of bits `text_len` in `text` and the encryption oracle limits `limit_pre` and `limit_post`.

Pseudorandom Functions

- Our pseudorandom function (PRF) is an operator `F` with this type:

```
op F : key -> text -> text.
```

- For each value `k` of type key, `(F k)` is a function from text to text.
- Since key is a bitstring of length `key_len`, then there are at most $2^{\texttt{key\_len}}$ of these functions.
- If we wanted, we could try to spell out the code for `F`, but we choose to keep `F` abstract.
- How do we know if `F` is a “good” PRF?

Pseudorandom Functions

- We will assume that dtext (dkey) is a sub-distribution on text (key) that is a distribution (is "lossless"), and where every element of text (key) has the same non-zero value:

```
op dtext : text distr.
op dkey  : key distr.
```

- A *random function* is a module with the following interface:

```
module type RF = {
  (* initialization *)
  proc * init() : unit
  (* application to a text *)
  proc f(x : text) : text
}.
```

Pseudorandom Functions

- Here is a random function made from our PRF **F**:

```
module PRF : RF = {
  var key : key
  proc init() : unit = {
    key <$ dkey;
  }
  proc f(x : text) : text = {
    var y : text;
    y <- F key x;
    return y;
  }
}.
```

Pseudorandom Functions

- Here is a random function made from true randomness:

```
module TRF : RF = {
  (* mp is a finite map associating texts with texts *)
  var mp : (text, text) fmap
  proc init() : unit = {
    mp <- empty;  (* empty map *)
  }
  proc f(x : text) : text = {
    var y : text;
    if (! x \in mp) {   (* give x a random value in *)
      y <$ dtext;  (* mp if not already in mp's domain *)
      mp.[x] <- y;
    }
   return oget mp.[x];  (* return value of x in mp *)
  }
}.
```

Pseudorandom Functions

- A *random function adversary* is parameterized by a random function module:

```
module type RFA (RF : RF) = {
  proc * main() : bool {RF.f}
}.
```

Pseudorandom Functions

- Here is the random function game:

```
module GRF (RF : RF, RFA : RFA) = {
  module A = RFA(RF)
  proc main() : bool = {
    var b : bool;
    RF.init();
    b <@ A.main();
    return b;
  }
}.
```

- A random function adversary RFA tries to tell the PRF and true random functions apart, by *returning true with different probabilities.*

Pseudorandom Functions

- Our PRF F is “good” if and only if the following is small, whenever RFA is limited in the amount of computation it may do (maybe we say it runs in polynomial time):

```
`|Pr[GRF(PRF, RFA).main() @ &m : res] -
  Pr[GRF(TRF, RFA).main() @ &m : res]|
```

- `RFA` must be limited, because there will typically be many more true random functions than functions of the form `(F k)`, where `k` is a key (there are at most 2^{key_len} such functions).
  - Since `text_len` is the number of bits in `text`, there will be 2^{text_len} ^ 2^{text_len} distinct maps from `text` to `text` (e.g., 2^8 = 256, 2^8 ^ 2^8 ~= 10^{617}).
  - Thus, with enough running time, `RFA` may be able to tell with reasonable probability if it’s interacting with a PRF random function or a true random function.

Our Symmetric Encryption Scheme

- We construct our encryption scheme `Enc` out of `F`:

```
(+^) : text -> text -> text  (* bitwise exclusive or *)

type cipher = text * text.  (* ciphertexts *)

module Enc : ENC = {
  proc key_gen() : key = {
    var k : key;
    k <$ dkey;
    return k;
  }
```

Our Symmetric Encryption Scheme

```
  proc enc(k : key, x : text) : cipher = {
    var u : text;
    u <$ dtext;
    return (u, x +^ F k u);
  }

  proc dec(k : key, c : cipher) : text = {
    var u, v : text;
    (u, v) <- c;
    return v +^ F k u;
  }
}.
```

Correctness

- Suppose that `enc(k, x)` returns `c` = `(u, x +^ F k u)`, where `u` is randomly chosen.
- Then `dec(k, c)` returns `(x +^ F k u) +^ F k u` = `x`.

Adversarial Attack Strategy

- Before picking its pair of plaintexts, the adversary can call `enc_pre` some number of times with the same argument, `text0` (the bitstring of length `text_len` all of whose bits are `0`).
- This gives us …, $(u_i,$ `text0 +^ F key` $u_i)$, …, i.e., …, $(u_i,$ `F key` $u_i)$, …
- Then, when `genc` encrypts one of x_1/x_2, it *may happen* that we get a pair $(u_i, x_j$ `+^ F key` $u_i)$ for one of them, where u_i appeared in the results of calling `enc_pre`.
- But then

`F key` u_i `+^` $(x_j$ `+^ F key` $u_i)$ = `text0 +^` x_j = x_j

Adversarial Attack Strategy

- Similarly, when calling `enc_post`, before returning its boolean judgement `b` to the game, a collision with the left-side of the cipher text passed from the game to the adversary will allow it to break security.
- Suppose, again, that the adversary repeatedly encrypts `text0` using `enc_pre`, getting …, `(`u_i`, F key `u_i`)`, …
- Then by *experimenting directly* with `F` with different keys, it may learn enough to guess, with reasonable probability, `key` itself.
- This will enable it to decrypt the cipher text `c` given it by the game, also breaking security.
- Thus we must assume some bounds on how much work the adversary can do (we can't tell if it's running `F`).

IND-CPA Security for Our Scheme

- Our security upper bound

```
`|Pr[INDCPA(Enc, Adv).main() @ &m : res] - 1%r / 2%r|
   <= …
```

will be a function of:

(1) the ability of a random function adversary constructed from `Adv` to tell the PRF random function from the true random function; and

(2) the number of bits `text_len` in `text` and the encryption oracles limits `limit_pre` and `limit_post`.

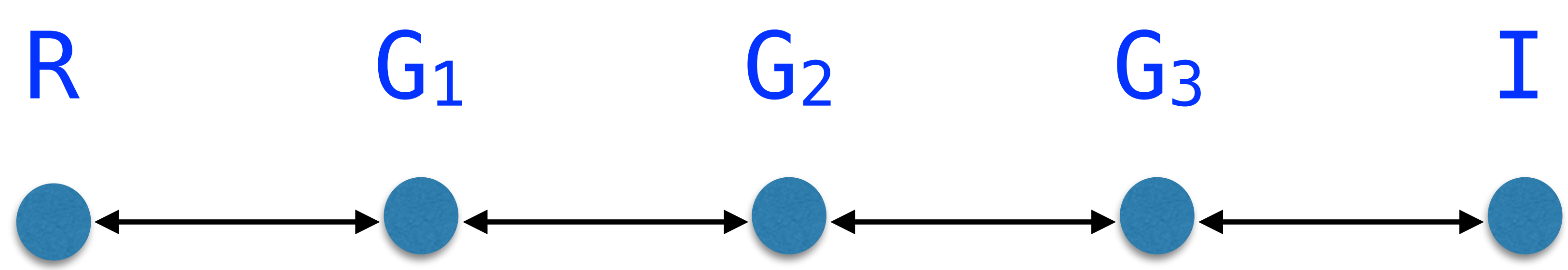
- Q: Why doesn't the upper bound also involve `ken_len`, the number of bits in `key`?
    - A: that's part of (1).

Next: Proof of IND-CPA Security

Sequence of Games Approach

- Our proof of IND-CPA security uses the *sequence of games approach*, which is used to connect a “real” game `R` with an *“ideal”* game `I` via a sequence of intermediate games.
- Each of these games is parameterized by the adversary, and each game has a `main` procedure returning a boolean.
- We want to establish an upper bound for

```
`| Pr[R.main() @ &m : res] - Pr[I.main() : res] |
```



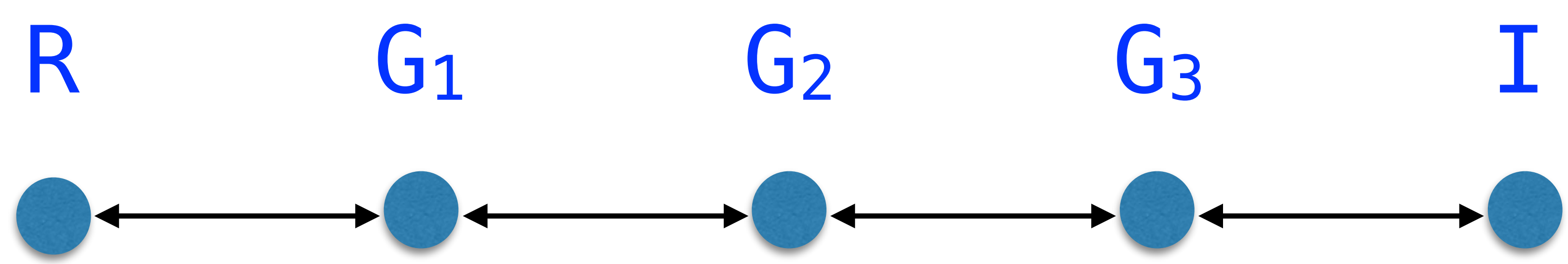
Sequence of Games Approach

- Suppose we can prove

```
`| Pr[R.main() @ &m : res] - Pr[G1.main() : res] | <= b1
`| Pr[G1.main() @ &m : res] - Pr[G2.main() : res] | <= b2
`| Pr[G2.main() @ &m : res] - Pr[G3.main() : res] | <= b3
`| Pr[G3.main() @ &m : res] - Pr[I.main() : res] | <= b4
```

for some b_1, b_2, b_3 and b_4. Then we can conclude

```
`| Pr[R.main() @ &m : res] - Pr[I.main() @ &m : res] | <=
  ??
```



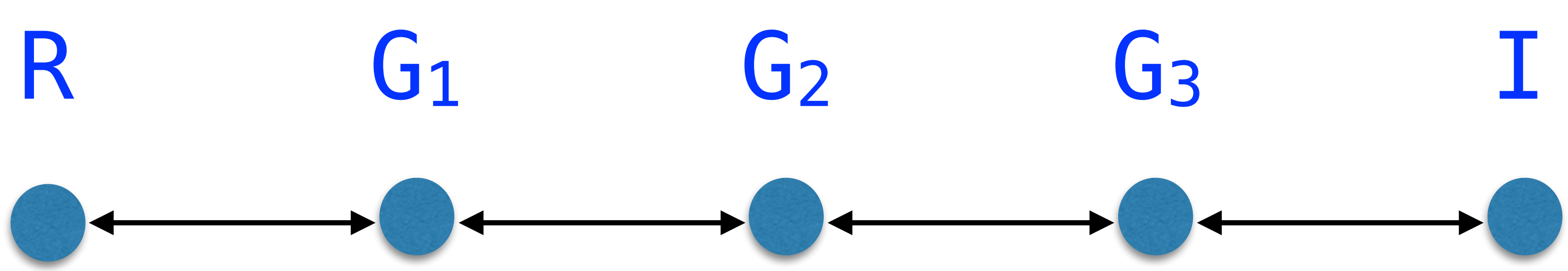
Sequence of Games Approach

- Suppose we can prove

```
`| Pr[R.main() @ &m : res] - Pr[G1.main() : res] | <= b1
`| Pr[G1.main() @ &m : res] - Pr[G2.main() : res] | <= b2
`| Pr[G2.main() @ &m : res] - Pr[G3.main() : res] | <= b3
`| Pr[G3.main() @ &m : res] - Pr[I.main() : res] | <= b4
```

for some b_1, b_2, b_3 and b_4. Then we can conclude

```
`| Pr[R.main() @ &m : res] - Pr[I.main() @ &m : res] | <=
   b1 + b2 + b3 + b4
```



Sequence of Games Approach

- This follows using the triangular inequality:

```
`| x - z | <= `| x - y | + `| y - z |.
```

- Q: what can our strategy be to establish an upper bound for the following?

```
`|Pr[INDCPA(Enc, Adv).main() @ &m : res] - 1%r / 2%r|
```

- A: We can use a sequence of games to connect `INDCPA(Enc, Adv)` to an ideal game `I` such that

```
Pr[I.main() @ &m : res] = 1%r / 2%r.
```

- The overall upper bound will be the sum $b_1 + \ldots + b_n$ of the sequence $b_1, \ldots, b_n$ of upper bounds of the steps of the sequence of games.

Sequence of Games Approach

- Q: But how do we know what this `I` should be?
- A: We start with `INDCPA(Enc, Adv)` and make a sequence of simplifications, hoping to get to such an `I`.
- Some simplifications work using code rewriting, like inlining. (The upper bound for such a step is 0.)
- Some simplifications work using cryptographic reductions, like the reduction to the security of PRFs.
  - The upper bound for such a step involves a constructed adversary for the security game of the reduction.
- Some simplifications make use of "up to bad" reasoning, meaning they are only valid when a bad event doesn't hold.
  - The upper bound for such a step is the probability of the bad event happening.

Starting the Proof in a Section

- First, we enter a “section”, and declare our adversary `Adv` as not interfering with certain modules and as being lossless:

```
section.

declare module Adv : ADV{EncO, PRF, TRF, Adv2RFA}.

axiom Adv_choose_ll :

  forall (EO <: EO{Adv}),

  islossless EO.enc_pre => islossless Adv(EO).choose.

axiom Adv_guess_ll :

  forall (EO <: EO{Adv}),

  islossless EO.enc_post => islossless Adv(EO).guess.
```

Step 1: Replacing PRF with TRF

- In our first step, we switch to using a true random function instead of a pseudorandom function in our encryption scheme.
  - We have an exact model of how the TRF works.
- When doing this, we inline the encryption scheme into a new kind of encryption oracle, `EO_RF`, which is parameterized by a random function.
- We also instrument `EO_RF` to detect two kinds of “clashes” (repetitions) in the generation of the inputs to the random function.
  - This is in preparation for Steps 2 and 3.

Step 1: Replacing PRF with TRF

```
local module EO_RF (RF : RF) : EO = {
  var ctr_pre : int
  var ctr_post : int
  var inps_pre : text fset
  var clash_pre : bool
  var clash_post : bool
  var genc_inp : text

  proc init() = {
    RF.init();
    ctr_pre <- 0; ctr_post <- 0; inps_pre <- fset0;
    clash_pre <- false; clash_post <- false;
    genc_inp <- text0;
  }
```

finite set

Step 1: Replacing PRF with TRF

```
proc enc_pre(x : text) : cipher = {
  var u, v : text; var c : cipher;
  if (ctr_pre < limit_pre) {
    ctr_pre <- ctr_pre + 1;
    u <$ dtext;
    inps_pre <- inps_pre `|` fset1 u;
    v <@ RF.f(u);
    c <- (u, x +^ v);
  }
  else {
    c <- (text0, text0);
  }
  return c;
}
```

size of `inps_pre` is at most `limit_pre`

Step 1: Replacing PRF with TRF

```
proc genc(x : text) : cipher = {
  var u, v : text; var c : cipher;
  u <$ dtext;
  if (mem inps_pre u) {
    clash_pre <- true;
  }
  genc_inp <- u;
  v <@ RF.f(u);
  c <- (u, x +^ v);
  return c;
}
```

Step 1: Replacing PRF with TRF

```
  proc enc_post(x : text) : cipher = {
    var u, v : text; var c : cipher;
    if (ctr_post < limit_post) {
      ctr_post <- ctr_post + 1;
      u <$ dtext;
      if (u = genc_inp) {
        clash_post <- true;
      }
      v <@ RF.f(u);
      c <- (u, x +^ v);
    }
    else {
      c <- (text0, text0);
    }
    return c;
  }
}.
```

Step 1: Replacing PRF with TRF

- Now, we define a game `G1` using `E0_RF`:

```
local module G1 (RF : RF) = {
  module E = E0_RF(RF)
  module A = Adv(E)

  proc main() : bool = {
    var b, b' : bool; var x1, x2 : text; var c : cipher;
    E.init();
    (x1, x2) <@ A.choose();
    b <$ {0,1};
    c <@ E.genc(b ? x1 : x2);
    b' <@ A.guess(c);
    return b = b';
  }
}.
```

Step 1: Replacing PRF with TRF

- Then it is easy to prove:

```
local lemma INDCPA_G1_PRF &m :
  Pr[INDCPA(Enc, Adv).main() @ &m : res] =
  Pr[G1(PRF).main() @ &m : res].
```

- To upper-bound

```
`| Pr[G1(PRF).main() @ &m : res] -
   Pr[G1(TRF).main() @ &m : res] |,
```

we need to construct a module `Adv2RFA` that transforms `Adv` into a random function adversary:

```
module Adv2RFA(Adv : ADV, RF : RF) = {
 …
  proc main() : bool = { … }
}.
```

`Adv2RFA(Adv)` is a random function adversary

Step 1: Replacing PRF with TRF

- Our goal in defining `Adv2RFA` is for this lemma to be provable:

```
local lemma G1_GRF (RF <: RF{EO_RF, Adv, Adv2RFA}) &m :
  Pr[G1(RF).main() @ &m : res] =
  Pr[GRF(RF, Adv2RFA(Adv)).main() @ &m : res].
```

- Recall the definition of `GRF`:

```
module GRF (RF : RF, RFA : RFA) = {
  module A = RFA(RF)
  proc main() : bool = {
    var b : bool;
    RF.init();
    b <@ A.main();
    return b;
  }
}.
```

Step 1: Replacing PRF with TRF

```
module Adv2RFA(Adv : ADV, RF : RF) = {
  module EO : EO = {  (* uses RF *)
    var ctr_pre : int
    var ctr_post : int

    proc init() : unit = {
      (* RF.init will be called by GRF *)
      ctr_pre <- 0; ctr_post <- 0;
    }
```

Step 1: Replacing PRF with TRF

```
proc enc_pre(x : text) : cipher = {
  var u, v : text; var c : cipher;
  if (ctr_pre < limit_pre) {
    ctr_pre <- ctr_pre + 1;
    u <$ dtext;
    v <@ RF.f(u);
    c <- (u, x +^ v);
  }
  else {
    c <- (text0, text0);
  }
  return c;
}
```

identical to EO_RF (minus instrumentation)

Step 1: Replacing PRF with TRF

```
proc genc(x : text) : cipher = {
  var u, v : text; var c : cipher;
  u <$ dtext;
  v <@ RF.f(u);
  c <- (u, x +^ v);
  return c;
}
```

identical to
EO_RF
(minus
instrumentation)

Step 1: Replacing PRF with TRF

```
  proc enc_post(x : text) : cipher = {
    var u, v : text; var c : cipher;
    if (ctr_post < limit_post) {
      ctr_post <- ctr_post + 1;
      u <$ dtext;
      v <@ RF.f(u);
      c <- (u, x +^ v);
    }
    else {
      c <- (text0, text0);
    }
    return c;
  }
}
```

identical to EO_RF (minus instrumentation)

Step 1: Replacing PRF with TRF

```
  module A = Adv(EO)

  proc main() : bool = {
    var b, b' : bool; var x1, x2 : text; var c : cipher;
    EO.init();
    (x1, x2) <@ A.choose();
    b <$ {0,1};
    c <@ EO.genc(b ? x1 : x2);
    b' <@ A.guess(c);
    return b = b';
  }
}.
```

Like `G1`, except `Adv` and `main` use `EO` instead of `EncO(RF)`

Step 1: Replacing PRF with TRF

- From

```
local lemma G1_GRF (RF <: RF{EO_RF, Adv, Adv2RFA}) &m :
  Pr[G1(RF).main() @ &m : res] =
  Pr[GRF(RF, Adv2RFA(Adv)).main() @ &m : res].
```

we can conclude

```
  Pr[INDCPA(Enc, Adv).main() @ &m : res] =
  Pr[G1(PRF).main() @ &m : res] =
  Pr[GRF(PRF, Adv2RFA(Adv)).main() @ &m : res]
```

and

```
  Pr[G1(TRF).main() @ &m : res] =
  Pr[GRF(TRF, Adv2RFA(Adv)).main() @ &m : res]
```

Step 1: Replacing PRF with TRF

- Thus

```
local lemma INDCPA_G1_TRF &m :
  `|Pr[INDCPA(Enc, Adv).main() @ &m : res] -
    Pr[G1(TRF).main() @ &m : res]| =
  `|Pr[GRF(PRF, Adv2RFA(Adv)).main() @ &m : res] -
    Pr[GRF(TRF, Adv2RFA(Adv)).main() @ &m : res]|.
```

- Here, we have an exact upper bound.

Next: Handling the Clashes