CS 591: Formal Methods in Security and Privacy

Hoare Logic

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LfA

Formal Semantics

We need to assign a formal meaning to the different components:

```
Precondition
Program
Postcondition
```

formal semantics of specification conditions

formal semantics of programs

formal semantics of specification conditions

We also need to describe the rules which combine program and specifications.

Semantics of Commands

This is defined on the structure of commands:

$\{\texttt{abort}\}_m = \bot$

$\{\texttt{skip}\}_m = m$

$\{\texttt{x:=e}\}_m = m[x \leftarrow \{e\}_m]$

$\{c;c'\}_m = \{c'\}_{m'}$ If $\{c\}_m = m'$

$\{c;c'\}_m = \bot$ If $\{c\}_m = \bot$

$\{\texttt{if e then } c_t \texttt{ else } c_f\}_m = \{c_t\}_m$ If $\{e\}_m = \texttt{true}$

$\{\texttt{if e then } c_t \texttt{ else } c_f\}_m = \{c_f\}_m$ If $\{e\}_m = \texttt{false}$

$\{\texttt{while e do c}\}_m = \sup_{n \in Nat}\{\texttt{while}_n \texttt{ e do c}\}_m$

where

$\texttt{while}_n \texttt{ e do c} = \texttt{while}^n \texttt{ e do c;if e then abort else skip}$

and

Semantics of Commands

This is defined on the structure of commands:

$$\{\mathtt{abort}\}_m = \bot$$

$$\{\mathtt{skip}\}_m = m$$

$$\{\mathtt{x:=e}\}_m = m[x \leftarrow \{e\}_m]$$

$$\{c;c'\}_m = \{c'\}_{m'} \quad \text{If} \quad \{c\}_m = m'$$

$$\{c;c'\}_m = \bot \quad \text{If} \quad \{c\}_m = \bot$$

$$\{\mathtt{if}\ e\ \mathtt{then}\ c_t\ \mathtt{else}\ c_f\}_m = \{c_t\}_m \quad \text{If } \{e\}_m = \mathtt{true}$$

$$\{\mathtt{if}\ e\ \mathtt{then}\ c_t\ \mathtt{else}\ c_f\}_m = \{c_f\}_m \quad \text{If } \{e\}_m = \mathtt{false}$$

$$\{\mathtt{while}\ e\ \mathtt{do}\ c\}_m = \sup_{n \in Nat}\{\mathtt{while}_n\ e\ \mathtt{do}\ c\}_m$$

where

$$\mathtt{while}_n\ e\ \mathtt{do}\ c = \mathtt{while}^n\ e\ \mathtt{do}\ c;\mathtt{if}\ e\ \mathtt{then}\ \mathtt{abort}\ \mathtt{else}\ \mathtt{skip}$$

and

$$\mathtt{while}^0\ e\ \mathtt{do}\ c = \mathtt{skip}$$

$$\mathtt{while}^{n+1}\ e\ \mathtt{do}\ c = \mathtt{if}\ e\ \mathtt{then}\ (c;\mathtt{while}^n\ e\ \mathtt{do}\ c)\ \mathtt{else}\ \mathtt{skip}$$

Program Specifications (Hoare Triples)

Specifications - Hoare triple

Precondition
(a logical formula)

```
Precondition
Program
Postcondition
```

$$c : P \Rightarrow Q$$

Program

Postcondition
(a logical formula)

Some examples

Precondition

$$x := z + 1 : \{z = n\} \Rightarrow \{x = n + 1\}$$

Postcondition

Is it a good specification?

Some examples

Precondition

$$x := z + 1 : \{z = n\} \Rightarrow \{x = n + 1\}$$

Postcondition

Is it a good specification?

✓

Specification can also be imprecise.

Some examples

Precondition

$$x := z + 1 : \{z > 0\} \Rightarrow \{x > 0\}$$

Postcondition

Is it a good specification?

Some examples

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$$x := z + 1 : \{z > 0\} \Rightarrow \{x > 0\}$$

Postcondition

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Some examples

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Some examples

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$$x := z + 1 : \{z < 0\} \Rightarrow \{x < 0\}$$

Postcondition

Is it a good specification?

Some examples

Precondition

$$x := z + 1 : \{z < 0\} \Rightarrow \{x < 0\}$$

Postcondition

Is it a good specification?

✗

Some examples

Precondition

$$x := z + 1 : \{z < 0\} \Rightarrow \{x < 0\}$$

Postcondition

Is it a good specification?

$$m_{in} = [z = -1, x = 2] \qquad m_{out} = [z = -1, x = 0]$$

Some examples

```
i:=0;
r:=1;
while(i≤k)do
  r:=r * n;
  i:=i + 1
```

Precondition

$$: \{0 \leq k\} \Rightarrow \{r = n^k\}$$

Postcondition

Is it a good specification?

Some examples

```
i:=0;
r:=1;
while(i≤k)do
  r:=r * n;
  i:=i + 1
```

Precondition

$$: \{0 \leq k\} \Rightarrow \{r = n^k\}$$

Postcondition

Is it a good specification?

✗

Some examples

```
i:=0;
r:=1;
while(i≤k)do
  r:=r * n;
  i:=i + 1
```

Precondition

$$: \{0 \leq k\} \Rightarrow \{r = n^k\}$$

Postcondition

Is it a good specification? ✗

$$m_{in} = [k = 0, n = 2, i = 0, r = 0]$$

$$m_{out} = [k = 0, n = 2, i = 1, r = 2]$$

Some examples

```
i:=0;
r:=1;
while(i≤k)do
  r:=r * n;
  i:=i + 1
```

Precondition

$$: \{0 < k\} \Rightarrow \{r = n^k\}$$

Postcondition

Is it a good specification?

Some examples

```
i:=0;
r:=1;
while(i≤k)do
  r:=r * n;
  i:=i + 1
```

Precondition

$$: \{0 < k\} \Rightarrow \{r = n^k\}$$

Postcondition

Is it a good specification?

✗

Some examples

```
i:=0;
r:=1;
while(i≤k)do
  r:=r * n;
  i:=i + 1
```

Precondition

$$: \{0 < k\} \Rightarrow \{r = n^k\}$$

Postcondition

Is it a good specification?

✗

$$m_{in} = [k = 1, n = 2, i = 0, r = 0]$$

$$m_{out} = [k = 1, n = 2, i = 2, r = 4]$$

Some examples

```
i:=0;
r:=1;
while(i<k)do
  r:=r * n;
  i:=i + 1
```

Precondition

$$: \{0 \leq k\} \Rightarrow \{r = n^k\}$$

Postcondition

Is it a good specification?

Some examples

```
i:=0;
r:=1;
while(i<k)do
  r:=r * n;
  i:=i + 1
```

Precondition

$$: \{0 \le k\} \Rightarrow \{r = n^k\}$$

Postcondition

Is it a good specification?

✓

Some examples

```
i:=0;
r:=1;
while(i≤k)do
  r:=r * n;
  i:=i + 1
```

Precondition

$$: \{0 \leq k\} \Rightarrow \{r = n^i\}$$

Postcondition

Is it a good specification?

Some examples

```
i:=0;
r:=1;
while(i≤k)do
  r:=r * n;
  i:=i + 1
```

Precondition

$$: \{0 \leq k\} \Rightarrow \{r = n^i\}$$

Postcondition

Is it a good specification?

✓

Some examples

```
i:=0;
r:=1;
while(i≤k)do
  r:=r * n;
  i:=i + 1
```

Precondition

$$: \{0 < k \wedge k < 0\} \Rightarrow \{r = n^k\}$$

Postcondition

Is it a good specification?

Some examples

```
i:=0;
r:=1;
while(i≤k)do
  r:=r * n;
  i:=i + 1
```

Precondition

$$: \{0 < k \wedge k < 0\} \Rightarrow \{r = n^k\}$$

Postcondition

Is it a good specification?

✓

Some examples

```
i:=0;
r:=1;
while(i≤k)do
  r:=r * n;
  i:=i + 1
```

Precondition

$$: \{0 < k \wedge k < 0\} \Rightarrow \{r = n^k\}$$

Postcondition

Is it a good specification?

✓

This is good because there is no memory that satisfies the precondition.

How do we determine the validity of an Hoare triple?

Validity of Hoare triple

Precondition
(a logical formula)

$$c : P \Rightarrow Q$$

Program

Postcondition
(a logical formula)

Validity of Hoare triple

Precondition
(a logical formula)

$$c : P \Rightarrow Q$$

Program

Postcondition
(a logical formula)

We are interested only in inputs that meets P and we want to have outputs satisfying Q.

Validity of Hoare triple

Precondition
(a logical formula)

$$c : P \Rightarrow Q$$

Program

Postcondition
(a logical formula)

We are interested only in inputs that meets P and we want to have outputs satisfying Q.

How shall we formalize this intuition?

Validity of Hoare triple

We say that the triple $c : P \Rightarrow Q$ is valid

if and only if

for every memory m such that $P(m)$

and memory m' such that $\{c\}_m = m'$

we have $Q(m')$.

Validity of Hoare triple

We say that the triple `c:P⇒Q` is valid

if and only if

for every memory `m` such that `P(m)`

and memory m' such that $\{c\}_m = m'$

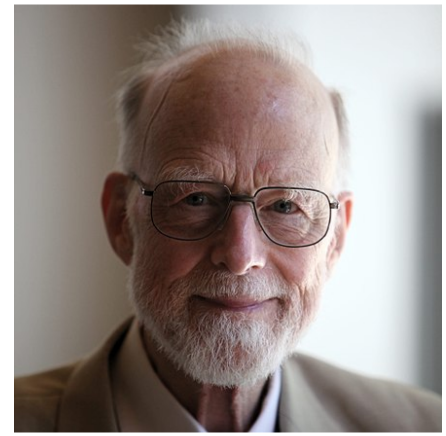
we have `Q(m')`.

$\boxed{\text{Is this condition easy to check?}}$

Hoare Logic

Floyd-Hoare reasoning


Robert W Floyd


Tony Hoare

A *verification* of an interpretation of a flowchart is a proof that for every command c of the flowchart, if control should enter the command by an entrance a_i with P_i true, then control must leave the command, if at all, by an exit b_j with Q_j true. *A semantic definition* of a particular set of command types, then, is a rule for constructing, for any command c of one of these types, a *verification condition* $V_c(\mathbf{P};\mathbf{Q})$ on the antecedents and consequents of c. This verification condition must be so constructed that a proof that the verification condition is satisfied for the antecedents and consequents of each command in a flowchart is a verification of the interpreted flowchart.

Rules of Hoare Logic: Skip

$$\frac{}{\vdash \texttt{skip: } P \Rightarrow P}$$

Rules of Hoare Logic: Skip

$$\frac{}{\vdash \texttt{skip: } P \Rightarrow P}$$

Is this correct?

Correctness of an axiom

$$\frac{}{\vdash c \; : \; P \Rightarrow Q}$$

We say that an axiom is correct if we can prove the validity of each triple which is an instance of the conclusion.

Correctness of Skip Rule

$$\overline{\vdash \texttt{skip: } P \Rightarrow P}$$

To show this rule correct we need to show the validity of the triple `skip: P⇒P`.

Correctness of Skip Rule

$$\overline{\qquad\qquad\qquad\qquad\qquad}$$
$$\vdash \texttt{skip: } P \Rightarrow P$$

To show this rule correct we need to show the validity of the triple $\texttt{skip: } P \Rightarrow P$.

For every $\texttt{m}$ such that $\texttt{P(m)}$ and m' such that $\texttt{\{skip\}}_{\texttt{m}}\texttt{=m'}$ we need $\texttt{P(m')}$.

Correctness of Skip Rule

$$\frac{}{\vdash \texttt{skip: } P \Rightarrow P}$$

To show this rule correct we need to show the validity of the triple $\texttt{skip: } P \Rightarrow P$.

For every m such that $P(m)$ and m' such that $\{\texttt{skip}\}_m = m'$ we need $P(m')$.

Follow easily by our semantics:

$\{\texttt{skip}\}_m = m$

Rules of Hoare Logic: Assignment

$$\frac{}{\vdash \texttt{x:=e}: \; P \Rightarrow P[e/x]}$$

Rules of Hoare Logic: Assignment

$$\frac{}{\vdash \texttt{x:=e: } P \Rightarrow P[e/x]}$$

Is this correct?

Some instances

$$x := x + 1 : \{x < 0\} \Rightarrow \{x + 1 < 0\}$$

Is this a valid triple?

Some instances

$$x := x + 1 : \{x < 0\} \Rightarrow \{x + 1 < 0\}$$

Is this a valid triple? ✗

Some instances

$$x := z + 1 : \{x > 0\} \Rightarrow \{z + 1 > 0\}$$

Is this a valid triple?

Some instances

$$x := z + 1 : \{x > 0\} \Rightarrow \{z + 1 > 0\}$$

Is this a valid triple? ✗

Rules of Hoare Logic: Assignment

$$\frac{}{\vdash \texttt{x:=e} \; : \; P[e/x] \Rightarrow P}$$

Rules of Hoare Logic: Assignment

$$\overline{\vdash \texttt{x:=e} \ : \ \texttt{P[e/x]} \Rightarrow \texttt{P}}$$

Is this correct?

Some instances

$$x := z + 1 : \{z + 1 > 0\} \Rightarrow \{x > 0\}$$

Is this a valid triple?

Some instances

$$x := z + 1 : \{z + 1 > 0\} \Rightarrow \{x > 0\}$$

Is this a valid triple? ✓

Some instances

$$x := x + 1 : \{x + 1 < 0\} \Rightarrow \{x < 0\}$$

Is this a valid triple?

Some instances

$$x := x + 1 : \{x + 1 < 0\} \Rightarrow \{x < 0\}$$

Is this a valid triple? ✓

Correctness Assignment Rule

$$\frac{}{\vdash \texttt{x:=e} \ : \ \texttt{P[e/x]} \Rightarrow \texttt{P}}$$

To show this rule correct we need to show the validity $\texttt{x:=e:P[e/x]} \Rightarrow \texttt{P}$ for every $\texttt{x}$, $\texttt{e}$, $\texttt{P}$.

Correctness Assignment Rule

$$\dfrac{}{\vdash \texttt{x:=e} \; : \; \texttt{P[e/x]} \Rightarrow \texttt{P}}$$

To show this rule correct we need to show the validity $\texttt{x:=e:P[e/x]} \Rightarrow \texttt{P}$ for every $\texttt{x}, \texttt{e}, \texttt{P}$.

For every $\texttt{m}$ such that $\texttt{P[e/x](m)}$ and m' such that $\texttt{\{x:=e\}}_{\texttt{m}}\texttt{=m'}$ we need $\texttt{P(m')}$.

Correctness Assignment Rule

$$\frac{}{\vdash x{:=}e \;:\; P[e/x] \Rightarrow P}$$

To show this rule correct we need to show the validity $x{:=}e : P[e/x] \Rightarrow P$ for every x, e, P.

For every m such that $P[e/x](m)$ and m' such that $\{x{:=}e\}_m = m'$ we need $P(m')$.

By our semantics: $\{x{:=}e\}_m = m[x = \{e\}_m]$ and we can show $P[e/x](m) = P(m[x = \{e\}_m])$

Rules of Hoare Logic
Composition

$$\frac{}{\vdash \texttt{c;c'}:\ P \Rightarrow Q}$$

Rules of Hoare Logic
Composition

$$\frac{\vdash \texttt{c} : P \Rightarrow R}{\vdash \texttt{c};\texttt{c}' : \; P \Rightarrow Q}$$

Rules of Hoare Logic Composition

$$\frac{\vdash \mathtt{c} : P \Rightarrow R \qquad\qquad \vdash \mathtt{c'} : R \Rightarrow Q}{\vdash \mathtt{c;c'} : \ P \Rightarrow Q}$$

Rules of Hoare Logic
Composition

$$\frac{\vdash \texttt{c} : P \Rightarrow R \qquad \vdash \texttt{c}' : R \Rightarrow Q}{\vdash \texttt{c;c}' : \ P \Rightarrow Q}$$

Is this correct?

Some Instances

$$\vdash x := z * 2; z := x * 2$$

$$: \{(z * 2) * 2 = 8\} \Rightarrow \{z = 8\}$$

Is this a valid triple?

Some Instances

$$\vdash x := z * 2; z := x * 2$$

$$: \{(z * 2) * 2 = 8\} \Rightarrow \{z = 8\}$$

Is this a valid triple?

✓

Some Instances

How can we prove it?

$$\vdash x := z * 2; z := x * 2 : \{(z * 2) * 2 = 8\} \Rightarrow \{z = 8\}$$

Some Instances

How can we prove it?

$$\frac{}{\vdash x := z*2 : \{(z*2)*2 = 8\} \Rightarrow \{x*2 = 8\}} \qquad \frac{}{\vdash z := x*2 \colon \{x*2 = 8\} \Rightarrow \{z = 8\}}$$

$$\vdash x := z*2; z := x*2 : \{(z*2)*2 = 8\} \Rightarrow \{z = 8\}$$

Correctness Composition Rule

$$\frac{\vdash \texttt{c} : \texttt{P} \Rightarrow \texttt{R} \qquad\qquad \vdash \texttt{c}' : \texttt{R} \Rightarrow \texttt{Q}}{\vdash \texttt{c};\texttt{c}' : \ \texttt{P} \Rightarrow \texttt{Q}}$$

To show this rule correct we need to show the validity $\texttt{c};\texttt{c}' : \texttt{P} \Rightarrow \texttt{Q}$ for every $\texttt{c}, \texttt{c}', \texttt{P}, \texttt{Q}$.

Correctness Composition Rule

$$\frac{\vdash \texttt{c} : P \Rightarrow R \qquad \vdash \texttt{c}' : R \Rightarrow Q}{\vdash \texttt{c};\texttt{c}' : P \Rightarrow Q}$$

To show this rule correct we need to show the validity $\texttt{c};\texttt{c}' : P \Rightarrow Q$ for every $\texttt{c}, \texttt{c}', P, Q$.

For every $\texttt{m}$ such that $P(\texttt{m})$ and m' such that $\{\texttt{c},\texttt{c}'\}_{\texttt{m}} = \texttt{m}'$ we need $Q(\texttt{m}')$.

Correctness Composition Rule

$$\frac{\vdash \texttt{c} : P \Rightarrow R \qquad \vdash \texttt{c'} : R \Rightarrow Q}{\vdash \texttt{c;c'} : P \Rightarrow Q}$$

Correctness Composition Rule

$$\frac{\vdash \texttt{c} : P \Rightarrow R \qquad \vdash \texttt{c'} : R \Rightarrow Q}{\vdash \texttt{c;c'} : P \Rightarrow Q}$$

By our semantics: $\{\texttt{c;c'}\}_{\texttt{m}} = \texttt{m'}$ if and only if there is $\texttt{m''}$ such that $\{\texttt{c}\}_{\texttt{m}} = \texttt{m''}$ and $\{\texttt{c'}\}_{\texttt{m''}} = \texttt{m'}$.

Correctness Composition Rule

$$\frac{\vdash \mathtt{c}:\mathtt{P}\Rightarrow\mathtt{R} \qquad \vdash \mathtt{c'}:\mathtt{R}\Rightarrow\mathtt{Q}}{\vdash \mathtt{c;c'}:\ \mathtt{P}\Rightarrow\mathtt{Q}}$$

By our semantics: $\{\mathtt{c;c'}\}_{\mathtt{m}}=\mathtt{m'}$ if and only if there is $\mathtt{m''}$ such that $\{\mathtt{c}\}_{\mathtt{m}}=\mathtt{m''}$ and $\{\mathtt{c'}\}_{\mathtt{m''}}=\mathtt{m'}$.

Assuming $\mathtt{c}:\mathtt{P}\Rightarrow\mathtt{R}$ and $\mathtt{c'}:\mathtt{R}\Rightarrow\mathtt{Q}$ valid, if $\mathtt{P(m)}$ we can show $\mathtt{R(m'')}$ and if $\mathtt{R(m'')}$ we can show $\mathtt{Q(m')}$, hence since we have $\mathtt{P(m)}$ we can conclude $\mathtt{Q(m')}$.

Correctness Composition Rule

$$\frac{\vdash \texttt{c}:\texttt{P}\Rightarrow\texttt{R} \qquad \vdash \texttt{c}':\texttt{R}\Rightarrow\texttt{Q}}{\vdash \texttt{c};\texttt{c}':\ \texttt{P}\Rightarrow\texttt{Q}}$$

By our semantics: $\{\texttt{c};\texttt{c}'\}_{\texttt{m}}=\texttt{m}'$ if and only if there is $\texttt{m}''$ such that $\{\texttt{c}\}_{\texttt{m}}=\texttt{m}''$ and $\{\texttt{c}'\}_{\texttt{m}''}=\texttt{m}'$.

Assuming $\texttt{c}:\texttt{P}\Rightarrow\texttt{R}$ and $\texttt{c}':\texttt{R}\Rightarrow\texttt{Q}$ valid, if $\texttt{P}(\texttt{m})$ we can show $\texttt{R}(\texttt{m}'')$ and if $\texttt{R}(\texttt{m}'')$ we can show $\texttt{Q}(\texttt{m}')$, hence since we have $\texttt{P}(\texttt{m})$ we can conclude $\texttt{Q}(\texttt{m}')$. ✓

Some examples

$$\vdash x := z * 2; z := x * 2$$

$$: \{z * 4 = 8\} \Rightarrow \{z = 8\}$$

Is this a valid triple?

Some examples

$$\vdash x := z * 2; z := x * 2$$

$$: \{z * 4 = 8\} \Rightarrow \{z = 8\}$$

Is this a valid triple? ✓

Some examples

$$\vdash x := z * 2; z := x * 2$$

$$: \{z * 4 = 8\} \Rightarrow \{z = 8\}$$

Is this a valid triple?

✓

Can we prove it with the rules that we have?

Some examples

$$\vdash x := z * 2; z := x * 2$$
$$: \{z * 4 = 8\} \Rightarrow \{z = 8\}$$

Is this a valid triple?	✓
Can we prove it with the rules that we have?	✗

Some Instances

What is the issue?

$$\frac{}{\vdash x := z * 2; z := x * 2 : \{z * 4 = 8\} \Rightarrow \{z = 8\}}$$

Some Instances

What is the issue?

$$\frac{}{\vdash x := z * 2 : \{z * 4 = 8\} \Rightarrow \{x * 2 = 8\}} \qquad \frac{}{\vdash z := x * 2 : \{x * 2 = 8\} \Rightarrow \{z = 8\}}$$

$$\frac{}{\vdash x := z * 2; z := x * 2 : \{z * 4 = 8\} \Rightarrow \{z = 8\}}$$

Some Instances

What is the issue?

$$
\dfrac{\dfrac{}{\vdash x := z * \cancel{2} : \{z * 4 = 8\} \Rightarrow \{x * 2 = 8\}} \qquad \dfrac{}{\vdash z := x * 2 : \{x * 2 = 8\} \Rightarrow \{z = 8\}}}{\vdash x := z * 2; z := x * 2 : \{z * 4 = 8\} \Rightarrow \{z = 8\}}
$$

Rules of Hoare Logic
Consequence

$$\frac{P \Rightarrow S \qquad \vdash c : S \Rightarrow R \qquad R \Rightarrow Q}{\vdash c : \; P \Rightarrow Q}$$

Some examples

$$\vdash x := z * 2; z := x * 2$$

$$: \{z * 4 = 8\} \Rightarrow \{z = 8\}$$

Is this a valid triple?

Some examples

$$\vdash x := z * 2; z := x * 2$$
$$: \{z * 4 = 8\} \Rightarrow \{z = 8\}$$

Is this a valid triple? ✓

Some examples

$$\vdash x := z * 2; z := x * 2$$
$$: \{z * 4 = 8\} \Rightarrow \{z = 8\}$$

Is this a valid triple?

Can we prove it with the rules that we have?

Some examples

$$\vdash x := z * 2; z := x * 2$$
$$: \{z * 4 = 8\} \Rightarrow \{z = 8\}$$

Is this a valid triple? ✓

Can we prove it with the rules that we have? ✓

Some Instances

$$\dfrac{\dfrac{\dfrac{}{\vdash x := z * 2 \; \{(z*2)*2 = 8\} \Rightarrow \{x * 2 = 8\}} \quad \{z * 4 = 8\} \Rightarrow \{(z*2)*2 = 8\}}{\vdash x := z * 2 \colon \{z * 4 = 8\} \Rightarrow \{x * 2 = 8\}} \qquad \dfrac{}{\vdash z := x * 2 \colon \{x * 2 = 8\} \Rightarrow \{z = 8\}}}{\vdash x := z * 2; z := x * 2 \colon \{z * 4 = 8\} \Rightarrow \{z = 8\}}$$

Rules of Hoare Logic
If then else

$$\frac{}{\vdash \texttt{if e then } c_1 \texttt{ else } c_2 \texttt{ : } P \Rightarrow Q}$$

Rules of Hoare Logic
If then else

$$\frac{\vdash \texttt{c}_1 : \texttt{P} \Rightarrow \texttt{Q} \qquad\qquad \vdash \texttt{c}_2 : \texttt{P} \Rightarrow \texttt{Q}}{\vdash \texttt{if e then c}_1 \texttt{ else c}_2 : \texttt{P} \Rightarrow \texttt{Q}}$$

Rules of Hoare Logic
If then else

$$\frac{\vdash \texttt{c}_1 : \texttt{P} \Rightarrow \texttt{Q} \qquad \vdash \texttt{c}_2 : \texttt{P} \Rightarrow \texttt{Q}}{\vdash \texttt{if e then c}_1 \texttt{ else c}_2 : \texttt{P} \Rightarrow \texttt{Q}}$$

Is this correct?

Some examples

$$\vdash \texttt{if } \texttt{y} = 0 \texttt{ then skip else } x := x + 1; x := x - 1$$

$$: \{x = 1\} \Rightarrow \{x = 1\}$$

$\boxed{\text{Is this a valid triple?}}$

Some examples

$$\vdash \texttt{if y} = 0 \texttt{ then skip else } x := x + 1; x := x - 1$$
$$: \{x = 1\} \Rightarrow \{x = 1\}$$

Is this a valid triple?

✓

Some examples

$$\vdash \texttt{if } \texttt{y} = 0 \texttt{ then skip else } x := x + 1; x := x - 1$$

$$: \{x = 1\} \Rightarrow \{x = 1\}$$

Is this a valid triple? ✓

Can we prove it with the rules that we have?

Some examples

$$\vdash \texttt{if y} = 0 \texttt{ then skip else } x := x + 1; x := x - 1$$

$$: \{x = 1\} \Rightarrow \{x = 1\}$$

Is this a valid triple?

✓

Can we prove it with the rules that we have?

✓

Some Instances

$$\dfrac{\dfrac{}{\vdash \texttt{skip}: \{x = 1\} \Rightarrow \{x = 1\}} \quad \dfrac{\vdots}{\vdash x := x + 1; x := x - 1 : \{x = 1\} \Rightarrow \{x = 1\}}}{\vdash \texttt{if y} = \texttt{0 then skip else } x := x + 1; x := x - 1 \quad : \{x = 1\} \Rightarrow \{x = 1\}}$$

Rules of Hoare Logic
If then else

$$\frac{\vdash \mathtt{c}_1 : \mathtt{P} \Rightarrow \mathtt{Q} \qquad \vdash \mathtt{c}_2 : \mathtt{P} \Rightarrow \mathtt{Q}}{\vdash \mathtt{if\ e\ then\ c}_1\ \mathtt{else\ c}_2 : \mathtt{P} \Rightarrow \mathtt{Q}}$$

Rules of Hoare Logic
If then else

$$\frac{\vdash c_1 : P \Rightarrow Q \qquad\qquad \vdash c_2 : P \Rightarrow Q}{\vdash \texttt{if e then } c_1 \texttt{ else } c_2 : P \Rightarrow Q}$$

Is this strong enough?

Some examples

$$\vdash \texttt{if false then skip else } x = x + 1$$
$$: \{x = 0\} \Rightarrow \{x = 1\}$$

Is this a valid triple?

Some examples

$$\vdash \texttt{if false then skip else } x = x + 1$$
$$: \{x = 0\} \Rightarrow \{x = 1\}$$

Is this a valid triple? ✓

Some examples

$$\vdash \texttt{if false then skip else } x = x + 1$$
$$: \{x = 0\} \Rightarrow \{x = 1\}$$

Is this a valid triple? ✓

Can we prove it with the rules that we have?

Some examples

$$\vdash \texttt{if false then skip else } x = x + 1$$
$$: \{x = 0\} \Rightarrow \{x = 1\}$$

Is this a valid triple? ✓

Can we prove it with the rules that we have? ✗

Rules of Hoare Logic
If then else

$$\frac{\vdash c_1 : e \land P \Rightarrow Q \qquad \vdash c_2 : \neg e \land P \Rightarrow Q}{\vdash \texttt{if } e \texttt{ then } c_1 \texttt{ else } c_2 : P \Rightarrow Q}$$

Is this correct?

Rules of Hoare Logic
If then else

$$\frac{\vdash c_1 : e \wedge P \Rightarrow Q \qquad \vdash c_2 : \neg e \wedge P \Rightarrow Q}{\vdash \texttt{if } e \texttt{ then } c_1 \texttt{ else } c_2 : P \Rightarrow Q}$$

Is this correct?

Homework

Rules of Hoare Logic: Abort

$$\frac{}{\vdash \texttt{Abort: } ? \Rightarrow ?}$$

Rules of Hoare Logic: Abort

$$\frac{}{\vdash \texttt{Abort: } ? \Rightarrow ?}$$

What can be a good specification?

Validity of Hoare triple

We say that the triple $\texttt{c:P}\Rightarrow\texttt{Q}$ is valid

if and only if

for every memory $\texttt{m}$ such that $\texttt{P(m)}$

and memory m' such that $\{\texttt{c}\}_{\texttt{m}}=\texttt{m'}$

we have $\texttt{Q(m')}$.

Rules of Hoare Logic: Abort

$$\frac{}{\vdash \texttt{Abort} : P \Rightarrow Q}$$

Is this correct?

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$$\frac{}{\vdash \mathtt{Abort} : P \Rightarrow Q}$$

Is this correct?

Homework