CSE660
Differential Privacy

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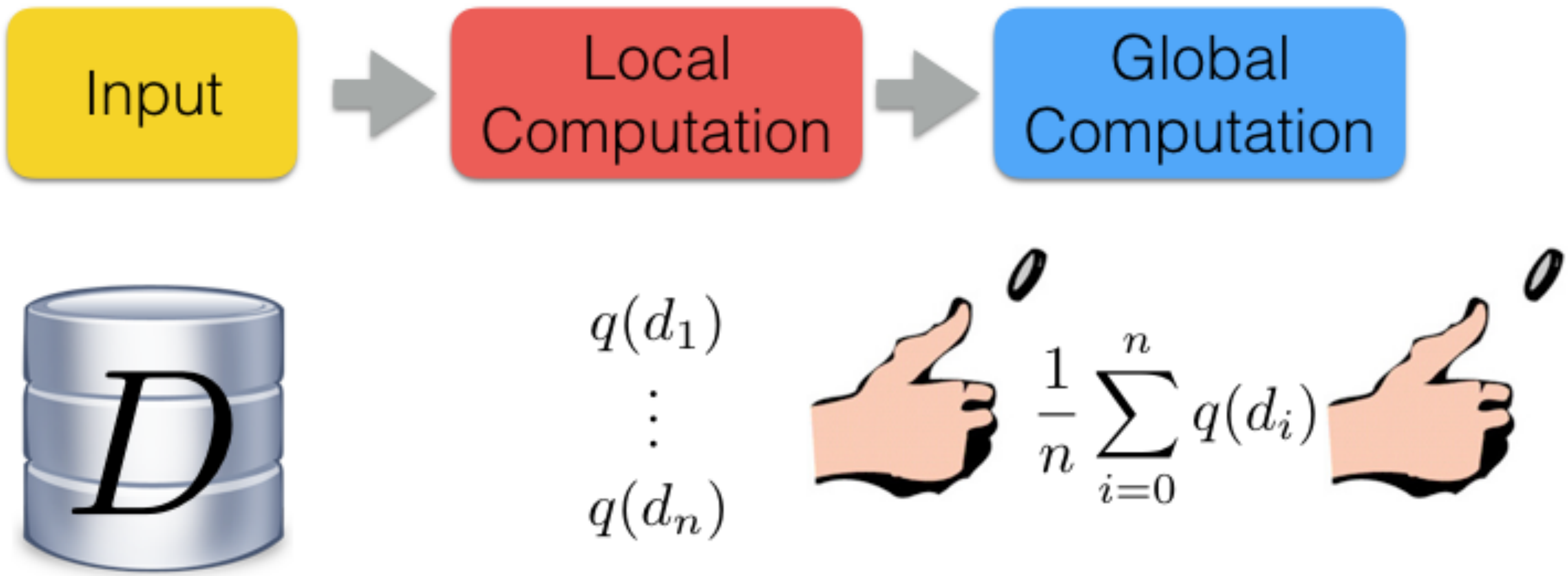
Differential privacy

Definition

Given $\varepsilon, \delta \geq 0$, a probabilistic query $Q: X^n \rightarrow R$ is (ε,δ)-differentially private iff for all adjacent database b_1, b_2 and for every $S \subseteq R$:

$$\Pr[Q(b_1) \in S] \leq \exp(\varepsilon)\Pr[Q(b_2) \in S] + \delta$$

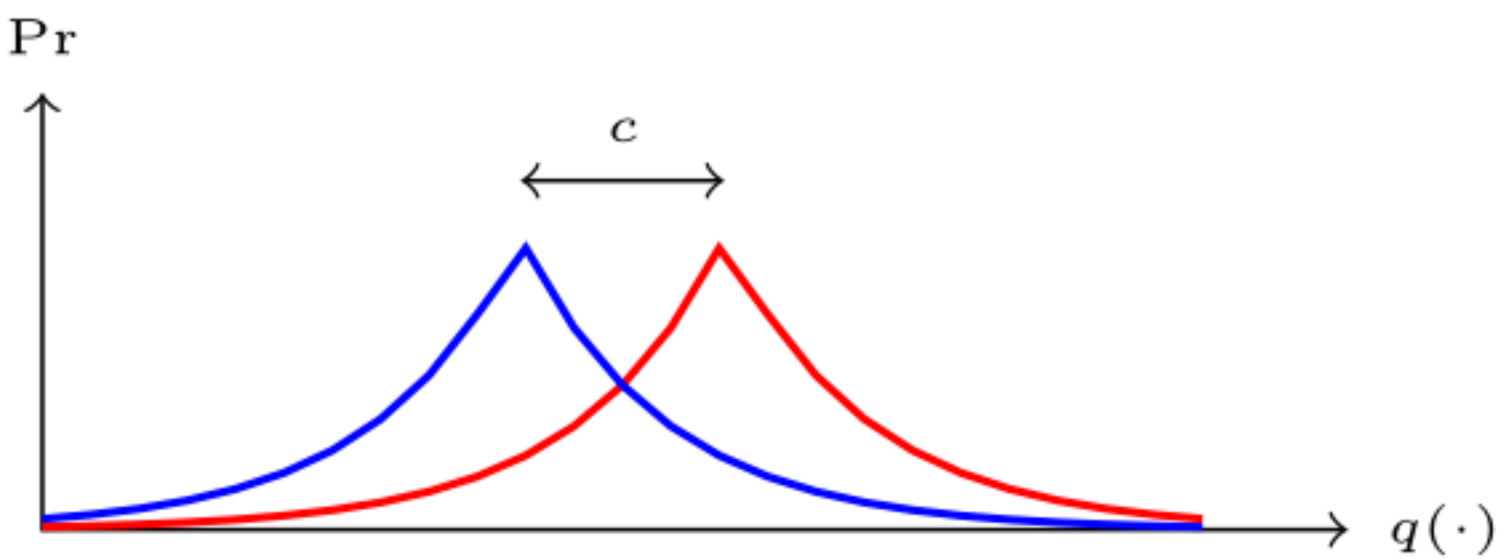
Noise on Input vs Noise on Output



Laplace Mechanism

Algorithm 2 Pseudo-code for the Laplace Mechanism

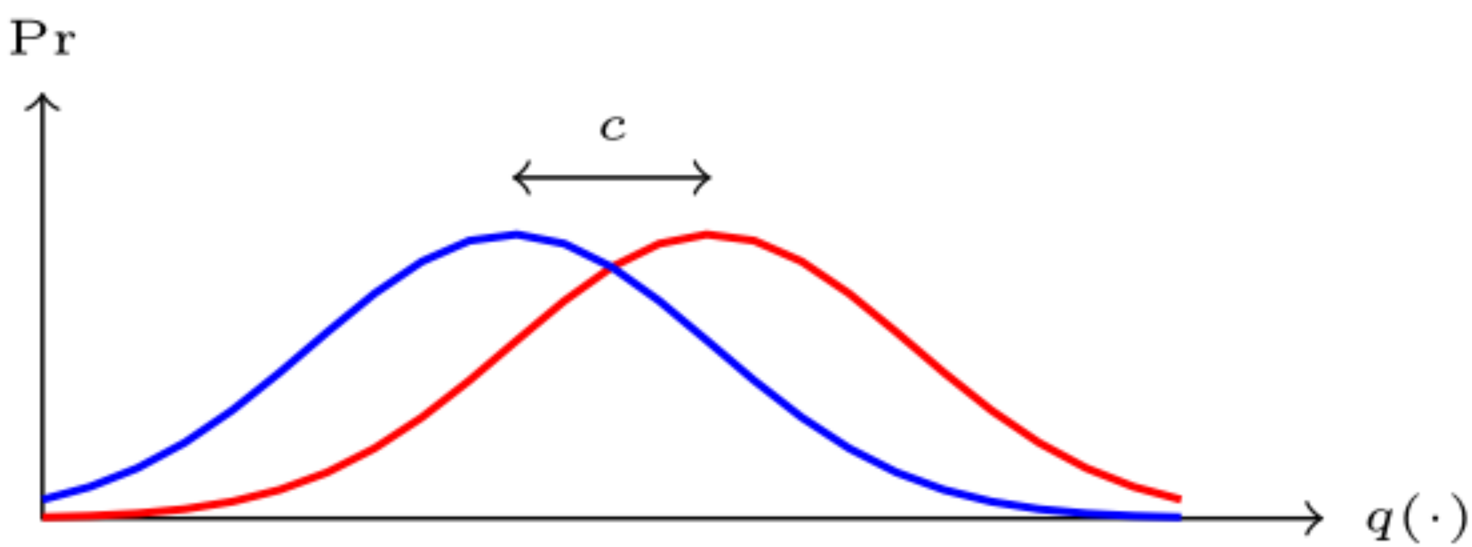
function $\textsc{LapMech}(D, q, \epsilon)$
  $Y \xleftarrow{\$} \mathsf{Lap}(\frac{\Delta q}{\epsilon})(0)$
  return $q(D) + Y$
end function



Gaussian Mechanism

Algorithm 14 Pseudo-code for the Gaussian Mechanism

1: **function** GaussMech(D, q, ϵ)
2: $\quad Y \xleftarrow{\$} \mathsf{Gauss}(0, \frac{2\ln(\frac{1.25}{\delta})(\Delta_2 q)^2}{\epsilon^2})$
3: $\quad$ **return** $q(D) + Y$
4: **end function**



Exponential Mechanism

Exponential Mechanism:

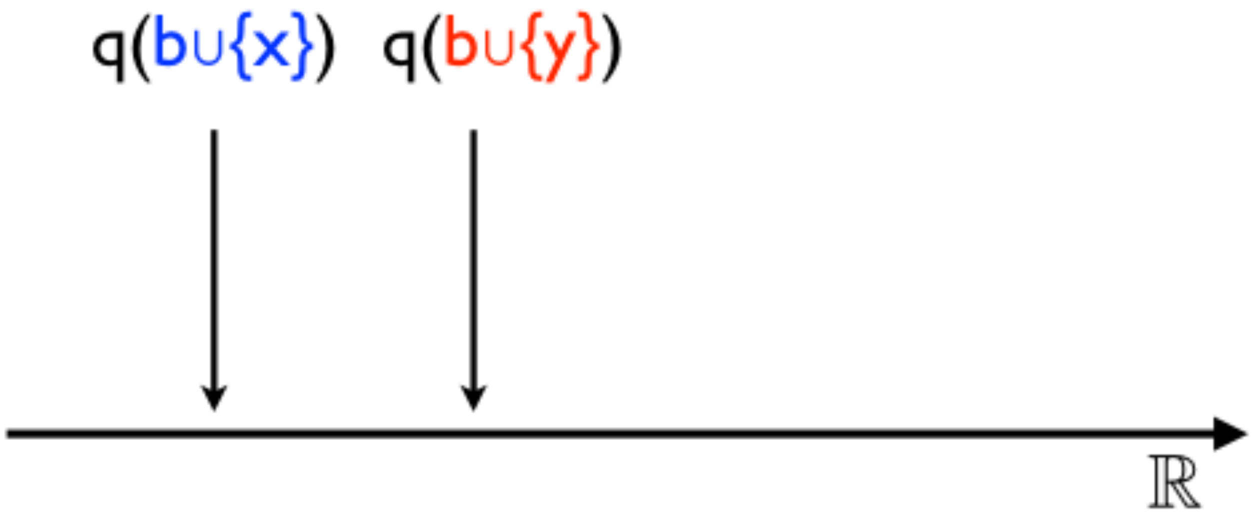
$\mathcal{M}_E(x, u, \mathcal{R})$

return $r \in \mathcal{R}$ with prob. $\dfrac{\exp(\frac{\varepsilon u(x,r)}{2\Delta u})}{\sum_{r' \in \mathcal{R}} \exp(\frac{\varepsilon u(x,r')}{2\Delta u})}$

Global Sensitivity

Definition 1.8 (Global sensitivity). The *global sensitivity* of a function $q : \mathcal{X}^n \to \mathbb{R}$ is:

$$\Delta q = \max \left\{ |q(D) - q(D')| \;\middle|\; D \sim_1 D' \in \mathcal{X}^n \right\}$$



Local sensitivity

Definition 1.8 (Global sensitivity)**.** The *global sensitivity* of a function $q : \mathcal{X}^n \to \mathbb{R}$ is:

$$\Delta q = \max \left\{ |q(D) - q(D')| \;\middle|\; D \sim_1 D' \in \mathcal{X}^n \right\}$$

Definition 1.14 (Local sensitivity)**.** The *local sensitivity* of a function $q : \mathcal{X}^n \to \mathbb{R}$ at $D \in \mathcal{X}^n$ is:

$$\ell\Delta q(D) = \max \left\{ |q(D) - q(D')| \;\middle|\; D \sim_1 D',\; D' \in \mathcal{X}^n \right\}$$

Calibrating noise to the local sensitivity

We may add noise proportional to the local sensitivity (LS).

Unfortunately, this does not guarantee privacy.

Suppose that for a given D we have LS(D)=0 but that we also have D~D' with LS(D')=10^9.

We will see that we can do anyway better than GS.

Some methods

- Smooth Sensitivity
- Propose-Test-Release
- Releasing Stable Values

Smooth Sensitivity

Definition 2.2 (Smooth sensitivity)**.** *For $\beta > 0$, the β-smooth sensitivity of f is*

$$S^*_{f,\beta}(x) = \max_{y \in D^n} \left(LS_f(y) \cdot e^{-\beta d(x,y)} \right).$$

Definition 2.1 (A Smooth Bound on LS)**.** *For $\beta > 0$, a function $S : D^n \to \mathbb{R}^+$ is a β-smooth upper bound on the local sensitivity of f if it satisfies the following requirements:*

$$\forall x \in D^n : \qquad S(x) \geq LS_f(x) \,; \tag{1}$$

$$\forall x, y \in D^n, d(x,y) = 1 : \qquad S(x) \leq e^{\beta} \cdot S(y) \,. \tag{2}$$

What kind of noise can we add?

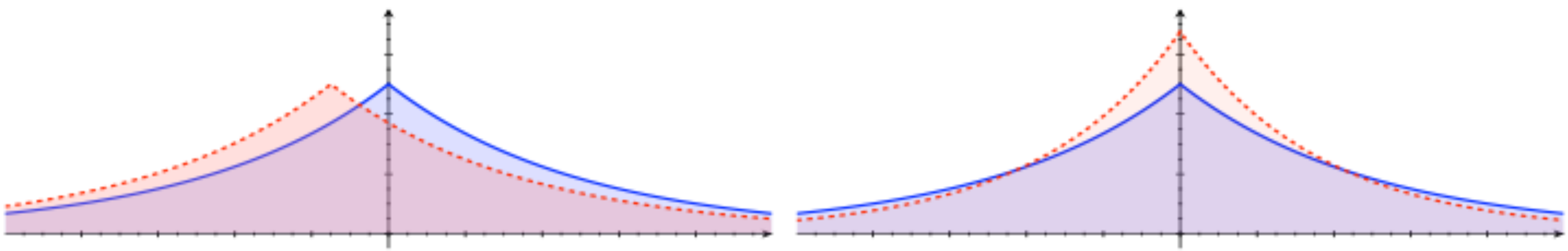
[Nissim, Raskhodnikova, Smith '06]

Admissible Noise

Definition 2.5 (Admissible Noise Distribution). *A probability distribution on $\mathbb{R}^d$, given by a density function h, is (α, β)-*admissible *(with respect to ℓ_1) if, for $\alpha = \alpha(\epsilon, \delta)$, $\beta = \beta(\epsilon, \delta)$, the following two conditions hold for all $\Delta \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$ satisfying $\|\Delta\|_1 \leq \alpha$ and $|\lambda| \leq \beta$, and for all measurable subsets $\mathcal{S} \subseteq \mathbb{R}^d$:*

Sliding Property: $$\Pr_{Z \sim h}\Big[Z \in \mathcal{S}\Big] \leq e^{\frac{\epsilon}{2}} \cdot \Pr_{Z \sim h}\Big[Z \in \mathcal{S} + \Delta\Big] + \frac{\delta}{2}.$$

Dilation Property: $$\Pr_{Z \sim h}\Big[Z \in \mathcal{S}\Big] \leq e^{\frac{\epsilon}{2}} \cdot \Pr_{Z \sim h}\Big[Z \in e^{\lambda} \cdot \mathcal{S}\Big] + \frac{\delta}{2}.$$

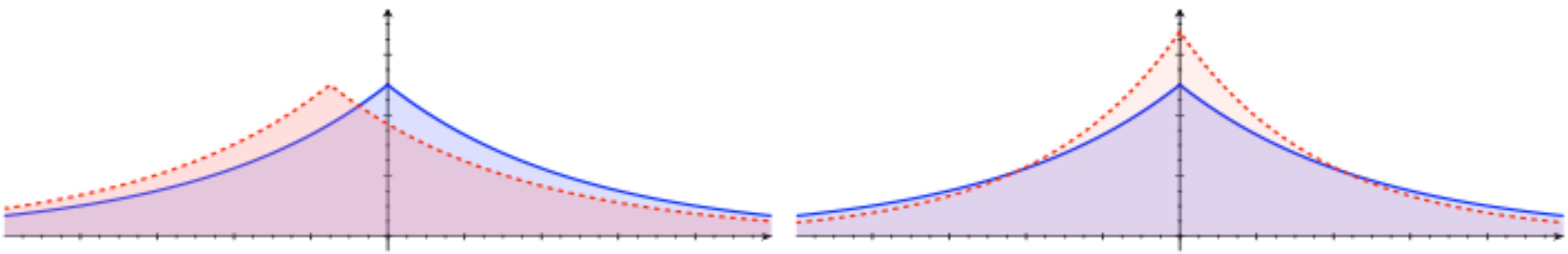


[Nissim, Raskhodnikova, Smith '06]

Calibrating noise to the smooth sensitivity

Lemma 2.6. *Let h be an (α, β)-admissible noise probability density function, and let Z be a fresh random variable sampled according to h. For a function $f : D^n \to \mathbb{R}^d$, let $S : D^n \to \mathbb{R}$ be a β-smooth upper bound on the local sensitivity of f. Then algorithm $\mathcal{A}(x) = f(x) + \frac{S(x)}{\alpha} \cdot Z$ is (ϵ, δ)-differentially private.*

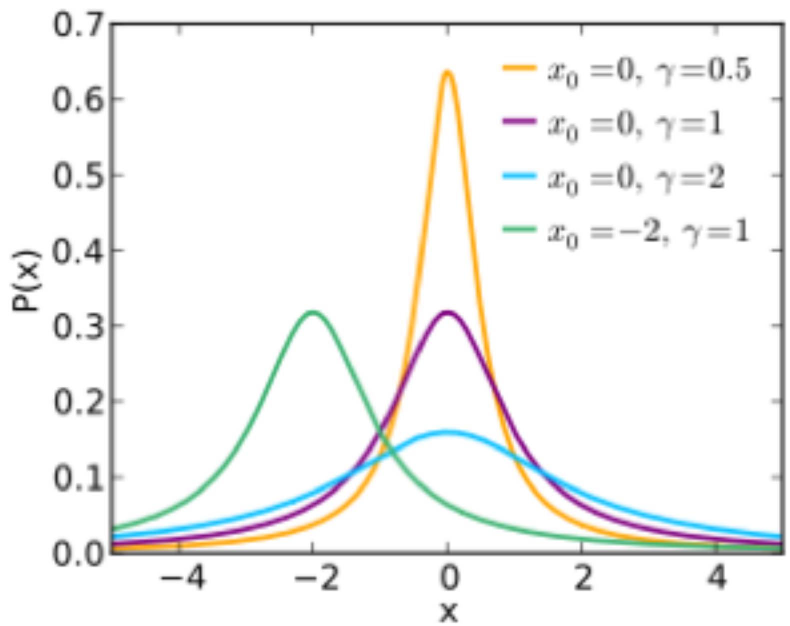
For two neighbor databases x and y, the output distribution $\mathcal{A}(y)$ is a shifted and scaled version of $\mathcal{A}(x)$. The sliding and dilation properties ensure that $\Pr[\mathcal{A}(x) \in \mathcal{S}]$ and $\Pr[\mathcal{A}(y) \in \mathcal{S}]$ are close for all sets $\mathcal{S}$ of outputs.



[Nissim, Raskhodnikova, Smith '06]

Admissible Noise

Adding noise $O(\mathrm{SS}_q^{\varepsilon}(x)/\varepsilon)$ (according to a Cauchy distribution) is sufficient for ε-differential privacy.



Laplace and Gauss give (ε,δ)-DP

Computing the Smooth Sensitivity can be intractable.

[Nissim, Raskhodnikova, Smith '06]

Propose Test Release

Propose-test-release Given $q : \mathcal{X}^n \to \mathbb{R}$, $\epsilon, \delta, \beta \geq 0$

1. Propose a target bound β on local sensitivity.
2. Let $\hat{d} = d(x, \{x' : \mathrm{LS}_q(x') > \beta\}) + \mathrm{Lap}(1/\varepsilon)$, where d denotes Hamming distance.
3. If $\hat{d} \leq \ln(1/\delta)/\varepsilon$, output $\perp$.
4. If $\hat{d} > \ln(1/\delta)/\varepsilon$, output $q(x) + \mathrm{Lap}(\beta/\varepsilon)$.

Propose Test Release

Proposition 3.2 (propose-test-release [33]). *For every query $q : \mathcal{X}^n \to \mathbb{R}$ and $\varepsilon, \delta, \beta \geq 0$, the above algorithm is $(2\varepsilon, \delta)$-differentially private.*

Proof. Consider any two neighboring datasets $x \sim x'$. Because of the Laplacian noise in the definition of $\hat{d}$ and the fact that Hamming distance has global sensitivity at most 1, it follows that

$$\Pr[\mathcal{M}(x) = \perp] \in [e^{-\varepsilon} \cdot \Pr[\mathcal{M}(x') = \perp], e^{\varepsilon} \cdot \Pr[\mathcal{M}(x') = \perp]]. \tag{3}$$

Case 1: $\mathrm{LS}_q(x) > \beta$. In this case, $d(x, \{x'' : \mathrm{LS}_q(x'') > \beta\}) = 0$, so the probability that $\hat{d}$ will exceed $\ln(1/\delta)/\varepsilon$ is at most δ. Thus, for every set $T \subseteq \mathbb{R} \cup \{\perp\}$, we have:

$$\begin{aligned} \Pr[\mathcal{M}(x) \in T] &\leq \Pr[\mathcal{M}(x) \in T \cap \{\perp\}] + \Pr[\mathcal{M}(x) \neq \perp] \\ &\leq e^{\varepsilon} \cdot \Pr[\mathcal{M}(x') \in T \cap \{\perp\}] + \delta \\ &\leq e^{\varepsilon} \cdot \Pr[\mathcal{M}(x') \in T] + \delta, \end{aligned}$$

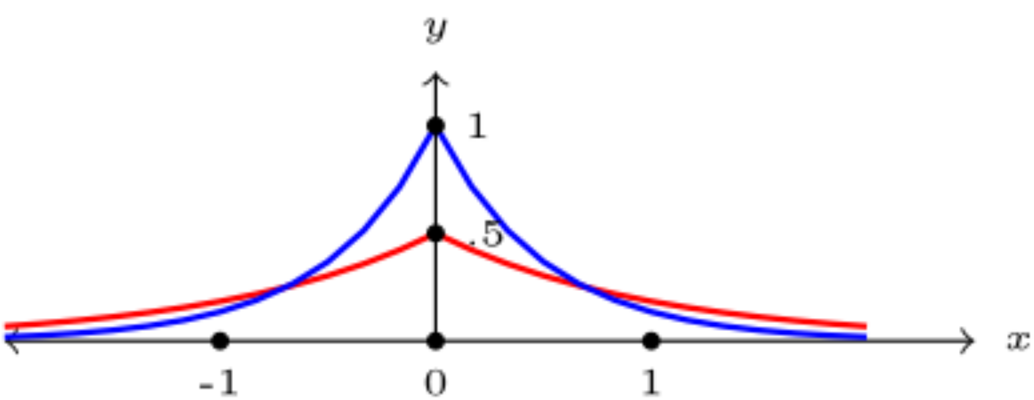
where the second inequality follows from (3), noting that $T \cap \{\perp\}$ equals either $\{\perp\}$ or $\emptyset$.

Case 2: $\mathrm{LS}_q(x) \leq \beta$. In this case, $|q(x) - q(x')| \leq \beta$, which in turn implies the $(\varepsilon, 0)$-indistinguishability of $q(x) + \mathrm{Lap}(\beta/\varepsilon)$ and $q(x') + \mathrm{Lap}(\beta/\varepsilon)$. Thus, by (3) and Basic Composition, we have $(2\varepsilon, 0)$-indistinguishability overall. $\square$

Laplace Mechanism

Accuracy Theorem: let $r = \mathsf{LapMech}(D, q, \epsilon)$

$$\Pr\left[|q(D) - r| \geq \left(\frac{\Delta q}{\epsilon}\right) \ln\left(\frac{1}{\beta}\right)\right] = \beta$$



$$\mathsf{Lap}(b, \mu)(X) = \frac{1}{2b} \exp\left(-\frac{|\mu - X|}{b}\right)$$

$$\Pr\Big[|X| \geq b\, t\Big] = \exp(-t)$$