Proof of Proposition 1

Let H be a complete graph over set of nodes V, where the weight of edges are represented by function ω, that maps each edge $e = (x, y)$ to RSD(x, y).

Lemma 1. *Let $\emptyset \neq W \subseteq V$. If $\{W_1, \ldots, W_\ell\}$ is an arbitrary partition of W, with $W_i \neq \emptyset$ and $\ell \geq 2$, then for every $1 \leq i \leq \ell$ there exists $j \neq i$, $1 \leq j \leq \ell$, such that there is an edge e connecting W_i and W_j with $\omega(e) \leq D(W)$.*

Proof. Denote by $H_W(W, E_W, \omega)$ the subgraph induced by W in H. Suppose there exists i such that for all $j \neq i$ and for any edge e connecting W_i to W_j it is the case that $\omega(e) > D(W)$. Let $e' = (x, y)$ be the edge that minimizes ω with $x \in W_i$ and $y \in W_j$ $(j \neq i)$. Hence $\omega(e') > D(W)$. Then we have,

$$\begin{aligned} D(W) &= \max_{W' \subsetneq W} \min_{\substack{u \in W' \\ v \in W \setminus W'}} \omega((u, v)) \geq \min_{\substack{u \in W_i \\ v \in W \setminus W_i}} \omega((u, v)) \\ &= \omega(e') > D(W), \end{aligned}$$

which is a contradiction. $\square$

Lemma 2. *Let $\emptyset \neq W \subsetneq V$. In any Minimum Spanning Tree T_H of H all the edges $e = (u, v)$, belonging to the tree, with $u \in W$ and $v \in V \setminus W$ are such that $w(e) \geq J(W)$.*

Proof. Suppose there exists a minimum spanning tree T_H of H that contains an edge $e = (u, v)$ such that, $u \in W$, $v \in V \setminus W$ but $\omega(e) < J(W)$. Since e is in T_H, it is also in H, hence

$$\omega(e) < J(W) = \min_{\substack{x \in W \\ y \notin W}} \omega((x, y)),$$

which is a contradiction. $\square$

Lemma 3. *Let $\emptyset \neq W \subsetneq V$. In any Minimum Spanning Tree T_H of H, if $e = (u, v)$, $u \in W$ and $v \in V \setminus W$, and e has minimum weight among edges connecting W to $V \setminus W$ in T_H, then $\omega(e) = J(W)$.*

Proof. Suppose there exists a minimum spanning tree T_H of H with an edge $e = (u, v) \in T_H$ such that, $u \in W$, $v \in V \setminus W$, $\omega(e)$ has minimum weight among edges connecting W to $V \setminus W$ and $\omega(e) > J(W)$ (from Lemma 2 it cannot be smaller). Pick $e' = (u', v')$ in H, with $u' \in W$ and $v' \in V \setminus W$, such that $\omega(e') = J(W)$ (see that by definition of $J(W)$ such edge exists).

Add e' to T_H. By doing that a cycle is formed in the tree, and in this cycle we have another edge e'' that connects W to $V \setminus W$ in T_H. Remove e'' from T_H obtaining a new spanning tree T'_H. See that $\omega(e'') \geq \omega(e) > \omega(e')$. We have

$$\sum_{e \in T'_H} \omega(e) = \sum_{e \in T_H} \omega(e) - \omega(e'') + \omega(e') < \sum_{e \in T_H} \omega(e),$$

which is a contradiction because T_H is a MST, and hence it has minimum cost. The last inequality comes from the fact that $\omega(e'') > \omega(e')$. $\square$

Lemma 4. *Let $\emptyset \neq W \subsetneq V$. If $J(W) > D(W)$, then W induces a connected subgraph in any MST of H.*

Proof. Suppose $\emptyset \neq W \subsetneq V$ with $J(W) > D(W)$, and that there exists a MST T_H of H in which W does not induce a connected subgraph. Say W induces l connected components in T_H denoted by $\{W_1, \ldots, W_l\}$. Pick W_1, by Lemma 2 we have that W_1 connects to $V \setminus W_1$ in T_H through an edge e_1 such that $\omega(e_1) \geq J(W)$.

By Lemma 1 we have that there is an edge, in H, e_2 that connects W_1 to W_j for some $2 \leq j \leq l$ and $\omega(e_2) \leq D(W)$.

Now, remove e_1 and add e_2 to T_H, obtaining T'_H. Then, since $J(W) > D(W)$ we have

$$\sum_{e \in T'_H} \omega(e) = \sum_{e \in T_H} \omega(e) - \omega(e_1) + \omega(e_2) < \sum_{e \in T_H} \omega(e),$$

which is a contradiction, since T_H is a MST of H it has minimum cost. The last inequality comes from the fact that $\omega(e_1) > \omega(e_2)$, once $J(W) > D(W)$. $\square$

With the above lemmas we move to prove Proposition 1. Let T_H be the MST computed in Step 2 of Algorithm 1. Also, let $\mathcal{C} = \{C_1, \ldots C_k\}$ be the output of Algorithm 1, and assume $J(C_1) \geq \cdots \geq J(C_k)$. Similarly, consider $\mathcal{B} = \{B_1, \ldots B_k\}$ be the optimal solution for Problem 1, and assume $J(B_1) \geq \cdots \geq J(B_k)$.

Suppose now that $\mathcal{B}$ is a better solution than $\mathcal{C}$, i.e., $J(B_k) > J(C_k)$.

From Lemma 4, the B_i's and C_i's induce connected subgraphs in T_H. As a consequence, all the B_i's (C_i's) can be obtained by removing edges from T_H. Moreover, from Lemma 2 those edges have weight greater than $J(B_k)$ ($J(C_k)$).

Now, let e^{b_i} and e^{c_i} be edges in T_H such that $\omega(e^{b_i}) = J(B_i)$ and $\omega(e^{c_i}) = J(C_i)$ respectively (From Lemma 3 we know that such edges exist).

Since $J(B_k) > J(C_k)$, we have $\omega(e^{b_k}) > \omega(e^{c_k})$. Now see that by the construction of Algorithm 1 this is a contradiction. Because the algorithm would have inspected all edges heavier than e^{b_k} (including edges with weight $\omega(e^{b_k})$) before reaching edge e^{c_k}, yielding a valid solution, and its output would not be $\mathcal{C}$.

Proof of proposition 2

For G_1: the next-hop from any source towards any destination x is always x itself. Hence, between two destinations x and y, the next-hops from all sources will always differ, which implies $\text{RSD}(x, y) = 1$.

For G_2: proof by induction on the length of the path between x and y. In the base case consider that x and y are adjacent nodes. Then for any z $N(z, x) = N(z, y)$, except when $z = x$ or $z = y$, where the equality never holds. So $\text{RSD}(x, y) = \frac{2}{|V|} = \frac{1+\ \text{treeDist}(x,y)}{|V|}$. Assume that the proposition is true for any two nodes u and v which $\text{treeDist}(u, v) = l$. Take now x and y, such that $\text{treeDist}(x, y) = l+1$, and let z be the adjacent neighbor of y in the shortest path from x to y. We have $\text{treeDist}(x, z) = l$, and hence $\text{RSD}(x, z) = \frac{1+l}{|V|}$. See now that if the next-hops towards x and z differ for $1 + l$ sources, then they will differ in $1 + (l + 1)$ for x and y, because of z. Hence, $\text{RSD}(x, y) = \frac{1+(l+1)}{|V|} = \frac{1+\text{ treeDist(x, y)}}{|V|}$.