Monotonicity testing and isoperimetric inequalities

"On monotonicity testing and Boolean isoperimetric type theorems"
[Khot, Minzer, Safra '15]

"Improved testing algorithms for monotonicity"
[Dodis, Goldreich, Lehman, Raskhodnikova, Ron, Samorodnitsky '99]

"Testing monotonicity"
[Goldreich, Goldwasser, Lehman, Ron, Samorodnitsky '00]

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The problem

- Query a function \( f: \{0, 1\}^n \rightarrow \{0, 1\} \) at a few points and decide if the function is monotone or far from monotone.
- First introduced by: Goldreich, Goldwasser, Lehman, Ron ‘98

- “few” = sublinear in the size of the domain
- property testing, sublinear algorithms
Some definitions & Background
Monotonicity on hypercube

- $f: \{0, 1\}^n \rightarrow \{0, 1\}$
- $x \rightarrow y$ is an edge if:
  - $x_i = 0, y_i = 1$
  - $x_j = y_j$ for all $j \in [n] - i$

- $2^n$ vertices and $n \cdot 2^{n-1}$ edges in the hypercube
- $f$ is monotone if the value of $f$ along any edge is nondecreasing
Distance to monotonicity

- Let \( \varepsilon(f) \) denote the distance of \( f \) to monotonicity
- \( \varepsilon(f) = \) least fraction of values of \( f \) that need to be changed to make \( f \) monotone

\[
\begin{align*}
\text{dist}(f, \text{MONO}) &= 0 \\
\text{dist}(g, \text{MONO}) &= 3/8
\end{align*}
\]
Testing monotonicity

In this talk: tester always accepts if $f$ monotone (one-sided error)

\[ \epsilon \text{-Tester} \]

Input $f$

$\ x \rightarrow f(x) \rightarrow \epsilon \text{-Tester} \rightarrow \text{Output} \rightarrow \begin{cases} \text{Accept} & \text{if } f \text{ is monotone} \\ \text{Reject} & \text{if } f \text{ is } \epsilon \text{-far from monotone} \end{cases} \]

With constant $Pr$
Results on monotonicity testing

Variations of problem studied since the late ‘90s:
• on different ranges, different domains
• estimating distance to monotonicity

For Boolean functions on hypercube:
• $O\left(\frac{n}{\varepsilon}\right)$-query tester [Dodis, Goldreich, Lehman, Raskhodnikova, Ron, Samorodnitsky '99],
  [Goldreich, Goldwasser, Lehman, Ron, Samorodnitsky '00]
• $O\left(\frac{n^{7/8}}{\varepsilon^{3/2}}\right)$-query tester [Chakrabarty, Seshadri ‘13]
• $\tilde{O}\left(\frac{\sqrt{n}}{\varepsilon^2}\right)$-query tester [Khot, Minzer, Safra ‘15]

Lower bounds:
• $\Omega(\sqrt{n})$ queries for 1-sided, nonadaptive [Fischer, Lehman, Newman, Raskhodnikova, Rubinfeld, Samorodnitsky ‘02]:
• $\tilde{\Omega}(n^{1/3})$ queries for adaptive [Chen, Waingarten, Xie ‘17]
Plan for this talk

- $O\left(\frac{n}{\varepsilon}\right)$ query tester + analysis
- Overview of isoperimetric inequalities related to monotonicity testing
- Proof outline for isoperimetric inequality in $O\left(\frac{\sqrt{n}}{\varepsilon^2}\right)$-query tester
- Relationship between isoperimetric inequality and $\tilde{O}\left(\frac{\sqrt{n}}{\varepsilon^2}\right)$-query tester
Part 1: Edge tester

- Describe $\Theta \left( \frac{n}{\varepsilon} \right)$ query tester (a.k.a edge tester)
- Analysis of tester
The edge tester [Dodis, Goldreich, Lehman, Raskhodnikova, Ron, Samorodnitsy ’99]

<table>
<thead>
<tr>
<th>Given $n$ and $\epsilon$:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Repeat $0\left(\frac{n}{\epsilon}\right)$ times:</td>
</tr>
<tr>
<td>- Sample edge $x \rightarrow y$ from the hypercube</td>
</tr>
<tr>
<td>- Query $f(x)$ and $f(y)$</td>
</tr>
<tr>
<td>- Reject if and only if $f(x) &gt; f(y)$</td>
</tr>
<tr>
<td>Accept</td>
</tr>
</tbody>
</table>

- The tester is nonadaptive
- The tester always accepts when $f$ is monotone
- Need to show that tester rejects w.h.p if $f$ is $\epsilon$-far from monotone
The edge tester: analysis overview

- Want to show that tester rejects w.h.p
- Call edge $x \rightarrow y$ is **violated** if:
  - $f(x) = 1$, $f(y) = 0$, i.e. $f$ decreases along the edge
- We show there must be a lot of violated edges:
  
  $$\frac{\# \text{ violated edges}}{n \cdot 2^{n-1}} \geq \frac{\varepsilon(f)}{n}$$

- If $f$ is $\varepsilon$-far from monotone, tester finds a violated edge w.h.p
- Idea: Can repair $f$ by changing 2 values per violated edge
Switch operator

- $f \rightarrow S_i(f)$
- For all edges along dimension $i$:
  - if the edge $x \rightarrow y$ is violated:
    - switch around the values of $f(x)$ and $f(y)$

More precisely:

\[
\begin{align*}
  i &= 1 & f(y) &= b & S_i f(y) &= \max(a, b) \\
  i &= 0 & f(x) &= a & S_i f(x) &= \min(a, b)
\end{align*}
\]
Switch operator: example

- Switch the red edges
- Edges in - - - - are violated
Property of switch operator

**Lemma.**
Switching $f$ in dimension $i$:
- makes $f$ monotone in dimension $i$
- does not increase number of violated edges in dimension $j$

**Proof.** It suffices to look at squares in dimensions $i$ and $j$
Edge tester: analysis

- $S_1S_2 ... S_n(f)$ is monotone

- When switching $f$ in dimensions 1 through $n$ we change at most:
  \[ 2 \cdot (\# \text{ violated edges}) \text{ points} \]

- Therefore:
  \[
  \frac{2 \cdot (\# \text{ violated edges})}{2^n} \geq \text{dist}(f, S_1 S_2 ... S_n(f)) \geq \varepsilon(f)
  \]

- For a random edge $x \rightarrow y$:
  \[
  \text{Prob}[x \rightarrow y \text{ is violated}] \geq \frac{(\# \text{ violated edges})}{n \cdot 2^{n-1}} \geq \frac{\varepsilon(f)}{n}
  \]

- After $\frac{n}{\varepsilon(f)}$ rounds, w.h.p, we have drawn a violated edge $\rightarrow f$ is rejected
Part 2: Background on isoperimetric inequalities

- Describe isoperimetric inequality of this talk
- Some background on isoperimetric inequalities
Isoperimetric inequality in this talk

Bipartite graph of violated edges

\[ f(y) = 0 \]

\[ f(x) = 1 \]

“average square root degree”

Sum of square root of degrees of x

\[ 2^n \]

\[ \geq \]

Distance of f to monotonicity
Isoperimetric inequalities (undirected)

- An edge $x \to y$ is nonconstant if $f(x) \neq f(y)$

- Define

$$I_f(x) = \begin{cases} 
0 & \text{if } f(x) = 0 \\
\# \text{ nonconstant edges incident at } x & \text{if } f(x) = 1
\end{cases}$$

- Then:

$$E_x[I_f(x)] \geq \Omega(\text{var}(f)) \quad \text{[folklore]}$$

$$E_x[\sqrt{I_f(x)}] \geq \Omega(\text{var}(f)) \quad \text{[Talagrand '93]}$$

- Bipartite graph of nonconstant edges

$$I_f(y) = 0$$

$$f(y) = 0$$

$$I_f(x) = \deg(x)$$

$$f(x) = 1$$

- $\text{var}(f) = \text{fraction of ones} \cdot \text{fraction of zeroes}$
Isoperimetric inequalities (directed)

An edge $x \rightarrow y$ is violated if $f(x) > f(y)$

Define

$$I_f^-(x) = \begin{cases} 0 & \text{if } f(x) = 0 \\ \# \text{ violated edges} & \text{incident at } x \\ \text{if } f(x) = 1 \end{cases}$$

Then:

$$E_x[I_f^-(x)] \geq \Omega(\varepsilon(f))$$ [Edge tester]

$$E_x\left[\sqrt{I_f^-(x)}\right] \geq \tilde{\Omega}(\varepsilon(f))$$ [Khot, Minzer, Safra '15]
**Summary of isoperimetric inequalities**

<table>
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<tr>
<th>Expression</th>
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<tr>
<td>$\mathbb{E}_x[I_f(x)] \geq \Omega(\text{var}(f))$</td>
<td>$\mathbb{E}_x[I_f^-(x)] \geq \Omega(\varepsilon(f))$</td>
</tr>
<tr>
<td>$\mathbb{E}_x[\sqrt{I_f(x)}] \geq \Omega(\text{var}(f))$</td>
<td>$\mathbb{E}_x\left[\sqrt{I_f^-(x)}\right] \geq \tilde{\Omega}(\varepsilon(f))$</td>
</tr>
</tbody>
</table>

> $I_f(y) = 0$ 
> $I_f(x) = \deg(x)$ 
> $I_f^-(y) = 0$ 
> $I_f^-(x) = \deg(x)$
Part 3: Outline of proof of our isoperimetric inequality

- Outline of proof for isoperimetric inequality
  - Only main ideas, no actual proofs!
- But before: define a new operator similar to switch operator
- Relate isoperimetric inequality to analysis of $\sqrt{n}$ - tester
Split operator

- $f \rightarrow \nabla_i (f)$

$i = 1 \quad f(y) = b$

$i = 0 \quad f(x) = a$

- Non-decreasing (monotone) in dimension $i +$
- Non-increasing in dimension $i -$
- All violated edges will be along dimension $i -$
Split operator: example

Note:
# violated edges may increase,
But they will all be in the negative direction
Isoperimetric inequality

\[ E_x \left[ \sqrt{I_f(x)} \right] \geq \widetilde{\Omega}(\varepsilon(f)) \]

“average square root degree”

distance to monotonicity

Bipartite graph of violated edges

\[ f(y) = 0 \]
\[ I_f^-(y) = 0 \]

\[ f(x) = 1 \]
\[ I_f^-(x) = \deg(x) \]
Outline of proof: Attempt 1

Objective:

\[ E_x \left[ \sqrt{I_f(x)} \right] \geq \sqrt{I_g(x)} \geq \varepsilon(g) \geq \varepsilon(f) \]

Phase 1:
- totally split \( f \) to get \( g = \nabla_1 \nabla_2 \ldots \nabla_n(f) \)
- Splitting only decreases the objective

Phase 2:
- Inequality holds for a totally split function

Not true!!
Outline of proof: Attempt 2

Objective:

\[ \mathbb{E}_x \left[ \sqrt{I_f(x)} \right] \geq \mathbb{E}_x \left[ \sqrt{I_g(x)} \right] \]

Phase 1: **totally split** \( f \) to get \( g = \nabla_1 \nabla_2 \cdots \nabla_n(f) \)

Splitting only decreases the objective

Phase 2:

Inequality holds for a **totally split** function

Phase 3:

Splitting is like switching half the coordinates

\[ \mathbb{E}[\text{dist}(f, f \text{ switched in all the coordinates})] - \mathbb{E}[\text{dist}(f, f \text{ switched in half the coordinates})] \]
Outline of proof: Attempt 2, Generalized

\[ E_x \left[ \sqrt{I_f(x)} \right] \geq E_x \left[ \sqrt{I_g(x)} \right] \geq \varepsilon(g) \geq \varepsilon \log n \] such inequalities!

\[ g = \]

- \( f \) split **only in a subset** of coordinates
- restricted to coordinates that are split
- fixed on rest of coordinates

\[ g \text{ is totally split!} \]

Need to add expectation over order of splits
AND
expectation over values of fixed coordinates

\[ E[\text{dist}(f, f \text{ switched in } 1/2^i \text{ of the coordinates})] - E[\text{dist}(f, f \text{ switched in } 1/2^{i+1} \text{ of the coordinates})] \]
Phase 1 Idea

- **Phase 1:** Splitting only decreases our objective

\[
\mathbb{E}_x \left[ \sqrt{I_f^-(x)} \right] \geq \mathbb{E}_x \left[ \sqrt{I_{\nabla_i f}^-(x)} \right]
\]

- Let \( g = \nabla_1 \nabla_2 \ldots \nabla_n (f) \). Then:

\[
\mathbb{E}_x \left[ \sqrt{I_f^-(x)} \right] \geq \mathbb{E}_x \left[ \sqrt{I_g^-(x)} \right]
\]

- **Proof idea:**
  - Like case analysis for switch operator
  - Consider cube in dimensions \( i+, i-, j \) instead of a square
Phase 2 Idea

**Phase 2:** The inequality is true for a “totally split” function

\[
E_x \left[ \sqrt{I_g^- (x)} \right] \geq \varepsilon (g)
\]

where \( g = \nabla_1 \nabla_2 \ldots \nabla_n (f) \).

- \( g \) is “simple”: all the violated edges are in the negative coordinates, monotone in half the coordinates
- Use the “undirected” version of the isoperimetric inequality, i.e.:

\[
E_x \left[ \sqrt{I_g (x)} \right] \geq \Omega (\text{var} (g))
\]
Phase 3 Idea, part 1

- Fix the following order of coordinates: 1, 2, ..., n
- This is the order in which we split $f$ to obtain $g$

For each $i$, switch $f$ in dimension $i$, with probability $p$

- Then it holds that for $p = \frac{1}{2}$:

$$\varepsilon(g) \geq \mathbb{E}[\varepsilon(S_{1/2}f)]$$
Phase 3 Idea, part 2

Phase 3: Then it holds that:

\[
\epsilon(g) \geq \mathbb{E}[\epsilon(S_{1/2}f)] \\
\approx \mathbb{E}[\text{dist}(S_{1/2}f, S_1f)] \\
\geq \mathbb{E}[\text{dist}(f, S_1f)] - \mathbb{E}[\text{dist}(f, S_{1/2}f)]
\]

No dimension is switched

All dimensions are switched $\rightarrow$ monotone

distance $\geq \epsilon(f)$
Final step of proof

For $i = 0, 1, \ldots, 5 \log n$:

$$E_x \left[ \sqrt{I_f^-(x)} \right] \geq E[\text{dist} \left( f, S_{1/2^i} f \right)] - E[\text{dist} \left( f, S_{1/2^{i+1}} f \right)]$$

Telescoping sum:

For $i = 0$: $f$ is switched in every dimension, expected distance is $\varepsilon(f)$

For $i = 5 \log n$: w.h.p $f$ is not switched in any dimension, expected distance is $\approx 0$

Hence:

$$\log n \cdot E_x \left[ \sqrt{I_f^-(x)} \right] \geq \varepsilon(f)$$
The $\sqrt{n}$ – tester

Given $n$ and $\varepsilon$:

Repeat $\tilde{O}\left(\frac{\sqrt{n}}{\varepsilon^2}\right)$ times:

- Sample $x$ from $\{0, 1\}^n$
- Sample $k$ from $\{0, 1, 2, \ldots , \log \sqrt{n}\}$
- Obtain $z$ by changing $2^k$ coordinates of $x$ from 0 to 1
- Reject if $f(x) = 1$ and $f(z) = 0$

Accept
Analysis of the $\sqrt{n}$ tester

Bipartite graph of violated edges

$\begin{align*}
    f(y) &= 0 \\
    f(x) &= 1
\end{align*}$

Good bipartite subgraph

$\begin{align*}
    \deg(y) &\leq 2d \\
    \deg(x) &= d
\end{align*}$

From isoperimetric inequality

$|A| \sqrt{d} \geq \frac{\varepsilon(f)}{\log n}$

Either $A$ or $\sqrt{d}$ is big!
Analysis of the $\sqrt{n}$ tester

Good bipartite subgraph

$\deg(y) \leq 2d$

$f(y) = 0$

$f(x) = 1$

$\deg(x) = d$

$\frac{|A|}{2n} \sqrt{d} \geq \frac{\varepsilon(f)}{\log n}$

With high probability

With high probability

Unique!
Conclusion

- Showed an analysis of the edge tester
- Overview of isoperimetric inequalities
- Outlined proof of main inequality in the $\sqrt{n}$ - tester
- Related isoperimetric inequality to analysis of $\sqrt{n}$ - tester

Open problems:
- Gap between lower bound and upper bound for Boolean functions on hypergrid
  - $f: [n]^d \rightarrow \{0,1\}$
  - Lower bound is $\Omega(\sqrt{d})$ and upper bound is $\tilde{O}(d^{\frac{5}{6}})$ [Black, Chakrabarty, Seshadri ‘17]
- Better adaptive algorithm or better adaptive lower bound
  - Current lower bound: $\tilde{\Omega}(n^{\frac{1}{3}})$ [Chen, Waingarten, Xie ‘17]