Fixed-parameter parallelizability

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March 9, 2016

Definition 1. A Boolean circuit, or simply a circuit, $C$, is an directed acyclic graph. The size of a circuit, denoted $\text{size}(C)$, is the number of vertices in the underlying graph. The depth of a circuit, denoted $\text{depth}(C)$, is the length of a longest path from the root to a sink.

Definition 2. A function $f$ is circuit-computable if there is a nonuniform family of Boolean circuits $\{C_n\}_{n \in \mathbb{N}}$ such that for each $x$ we have $f(x) = C_n(x)$, where $n = |x|$.

A language is circuit-decidable if it has a circuit-computable characteristic function. We may also require that the size and depth of each circuit $C_n$ in the family be circuit-computable from just $n$, the length of the input. In this case, we say the language is circuit-decidable with uniform size and depth.

Nonuniformity is required in Theorem 7 and Theorem 8, for example, in which the size of the input relative to the size of the parameter for an instance of the parameterized problem selects which of two circuits to use. If the circuits were uniform, we would need to include both alternatives, thereby making the circuits too deep to meet the required circuit depth bounds.

Definition 3. A language is a set of binary strings. A parameterization is a computable function $\kappa$ from binary strings to natural numbers. A parameterized problem is a pair $(Q, \kappa)$, where $Q$ is a language and $\kappa$ is a parameterization.

Definition 4 ($\text{NNC}^d$ and $\text{NC}^d$). Let $d$ be a natural number. A language $Q$ is in the class $\text{NNC}^d[\omega(n)]$ if there is a nondeterministic circuit family $\{C_n\}$ such that for each string $x$ of length $n$,

- $x \in Q$ if and only if $C_n(x) = 1$,
- $\text{size}(C_n) \leq n^{O(1)}$,
• \( \text{depth}(C_n) \leq O(\log^d n) \),
• \( \text{nondet}(C_n) \leq b(n) \).

If \( b \) is the zero function, then the language is in the class \( \text{NC}^d \).

**Definition 5** (\( \text{WP}^d[P] \) and \( \text{FPP}^d \)). Let \( d \) be a natural number. A parameterized problem \( (Q, \kappa) \) is in the class \( \text{WP}^d[P] \) if there are circuit-computable functions \( f, g, \) and \( h \), and a nondeterministic circuit family \( \{ C_{n,k} \} \) such that for each string \( x \),

- \( x \in Q \) if and only if \( C_{n,k}(x) = 1 \), where \( n = |x| \) and \( k = \kappa(x) \),
- \( \text{size}(C_{n,k}) \leq f(k)n^{O(1)} \),
- \( \text{depth}(C_{n,k}) \leq g(k)\log^d n \),
- \( \text{nondet}(C_{n,k}) \leq h(k)\log n \).

If \( h \) is the zero function, then the parameterized problem is in the class \( \text{FPP}^d \).

**Definition 6.** Suppose \( d \) is a natural number and \( (Q, \kappa) \) is a parameterized problem. We say the problem \( (Q, \kappa) \) is \textit{eventually in} \( \text{NC}^d \) if there is a circuit-
computable function \( h \) and a \( \text{NC}^d \) family of circuits \( \{ C_n \}_{n \in \mathbb{N}} \) such that for each \( x \) of length \( n \), if \( n \geq h(\kappa(x)) \), then \( x \in Q \) if and only if \( C_n(x) = 1 \).

Throughout we will often assume without loss of generality that functions like circuit size and depth bounds, nondeterminism bounds, and polynomials, are increasing.

This theorem is a characterization of parameterized problems in the class \( \text{FPP}^d \) as problems for which the parameter suggests an input size threshold, below which it suffices to perform an exhaustive search to solve the problem and after which we require an efficient algorithm. It is the adaptation of [4, Theorem 1.37].

**Theorem 7.** Suppose \( d \) is a natural number and \( (Q, \kappa) \) is a parameterized problem. If \( Q \) is circuit-decidable with uniform size and depth and \( (Q, \kappa) \) is eventually in \( \text{NC}^d \), then \( (Q, \kappa) \) is in \( \text{FPP}^d \).

**Proof.** Let \( h \) be the function that defines the lower bound on the length after which \( (Q, \kappa) \) is in \( \text{NC}^d \). The nonuniform family of circuits \( \{ A_{n,k} \} \) that decides \( (Q, \kappa) \) is defined by

\[
A_{n,k} = \begin{cases} 
C_{n}^1 & \text{if } n < h(k) \\
C_{n}^2 & \text{otherwise},
\end{cases}
\]

where \( \{ C_{n}^1 \} \) is the family of circuits that decides \( Q \) and \( \{ C_{n}^2 \} \) is the family of \( \text{NC}^d \) circuits that eventually decides \( (Q, \kappa) \). The correctness of \( A_{n,k} \) follows from the correctness of \( C_{n}^1 \) and \( C_{n}^2 \).
Let $S$ and $D$ be the circuit-computable function that give the size and depth, respectively, of $C_{n}^1$ from $n$. If $n < h(k)$, then

$$
\text{size}(A_{n,k}) = \text{size}(C_{n}^1) = S(n) \leq S(h(k)),
$$
$$
\text{depth}(A_{n,k}) = \text{depth}(C_{n}^1) = D(n) \leq D(h(k)).
$$

If $n \geq h(k)$, then

$$
\text{size}(A_{n,k}) = \text{size}(C_{n}^2) = n^{O(1)},
$$
$$
\text{depth}(A_{n,k}) = \text{depth}(C_{n}^2) = O(\log d n).
$$

An upper bound for the size of $A_{n,k}$ in either case is $S(h(k)) n^{O(1)}$ and for the depth $O(D(h(k)) \log d n)$. Since $S$ and $D$ are computable, composing each with $h$ yields another computable function, so these size and depth bounds meet the requirements of the definition of $FPP^d$.

The next theorem is similar in that it is a characterization of $	ext{WP}^d[P]$, a nondeterministic extension of $FPP^d$. It is the adaptation of [4, Proposition 3.7]. The parameterized problems in $	ext{WP}^d[P]$ of most interest to us are those for which there is a nondeterministic NC algorithm with nondeterminism limited by a function of the parameter.

TODO There’s some wiggle room with this theorem: if we change the definition of $\text{WP}^d[P]$ to have depth $g(k) + \log d n$, then we can get $\text{NNC}^d$ instead of $\text{NNC}^{d+c}$ in the conclusion. This sort of depth bound appears in [1].

**Theorem 8.** Suppose $d$ is a natural number, $e$ is a positive integer, and $(Q, \kappa)$ is a parameterized problem. If $(Q, \kappa) \in \text{WP}^d[P]$ and $Q \in \text{NNC}^d[\text{poly}]$, then there is a nondeterministic NC$^{d+c}$ circuit family $\{C_{n,k}\}$ and a circuit-computable function $h$ such that $\{C_{n,k}\}$ decides $Q$ and $\text{nondet}(C_{n,k}) \leq h(k) \log n$.

**Proof.** Assume $(Q, \kappa) \in \text{WP}^d[P]$ and $Q \in \text{NNC}^d[\text{poly}]$. Define the circuit family $\{A_{n,k}\}$ by

$$
A_{n,k} = \begin{cases} 
C_{n,k}^1 & \text{if } f_1(k) \leq n \text{ and } g_1(k) \leq \log^e n \\
C_{n,k}^2 & \text{otherwise},
\end{cases}
$$

where

- $\{C_{n,k}^1\}$ is the $\text{WP}^d[P]$ circuit family,
- $f_1$ and $g_1$ are the circuit-computable functions on the parameter that appear in the size and depth bounds, respectively, for $C_{n,k}^1$,
- $\{C_{n,k}^2\}$ is the $\text{NNC}^d[\text{poly}]$ circuit family.

The fact that $A_{n,k}$ correctly decides $Q$ follows from the correctness of the circuits $C_{n,k}^1$ and $C_{n,k}^2$. It remains to show that $A_{n,k}$ has the required size, depth, and nondeterminism bounds.
If $f_1(k) \leq n$ and $g_1(k) \leq \log^e n$, then
\[
\text{size}(A_{n,k}) \leq \text{size}(C_{n,k}^1) \leq f_1(k)n^{O(1)} \leq n^{O(1)}
\]
\[
\text{depth}(A_{n,k}) \leq \text{depth}(C_{n,k}^1) \leq g_1(k) \log^d n \leq \log^{d+e} n,
\]
and otherwise
\[
\text{size}(A_{n,k}) \leq \text{size}(C_{n}^2) \leq n^{O(1)}
\]
\[
\text{depth}(A_{n,k}) \leq \text{depth}(C_{n}^2) \leq \log^d n.
\]
The overall size and depth upper bounds for $A_{n,k}$ are thus $n^{O(1)}$ and $O(\log^{d+e} n)$, respectively.

For the nondeterminism, there are three cases. First, if $f_1(k) \leq n$ and $g_1(k) \leq \log^e n$, then
\[
\text{nondet}(A_{n,k}) \leq \text{nondet}(C_{n,k}^1) \leq h_1(k) \log n,
\]
where $h_1$ is the circuit-computable of the parameter that appears in the nondeterminism bound for $C_{n,k}^1$. Second, if $f_1(k) > n$, then
\[
\text{nondet}(A_{n,k}) \leq \text{nondet}(C_{n}^2) \leq n^{O(1)} \leq f_1(k)^{O(1)}.
\]
Finally, if $g_1(k) > \log^e n$, then $2^{(g_1(k))^{1/e}} > n$, which implies
\[
\text{nondet}(A_{n,k}) \leq \text{nondet}(C_{n}^2) \leq n^{O(1)} \leq 2^{O((g_1(k))^{1/e})}.
\]
If we choose $h(k) = \max\{h_1(k), f_1(k)^{O(1)}, 2^{O((g_1(k))^{1/e})}\}$, then we can conclude $\text{nondet}(A_{n,k}) \leq h(k) \log n$. \hfill $\square$

**Assumption 9.** There is a parameterized problem $(Q, \kappa)$ complete for $\text{WP}^d[P]$ under $\text{FPP}^d$ many-one reductions such that $Q \in \text{NNC}^d[\text{poly}]$.

This theorem is an adaptation of one direction of [4, Theorem 3.29].

**Theorem 10** (Theorem 3.29, part i). Assume Assumption 9. Suppose $d$ is a natural number and $e$ is a positive integer. If there is a circuit-computable, nondecreasing, unbounded function $i$ such that $\text{NC}^d = \text{NNC}^{d+e}[i(n) \log n]$, then $\text{FPP}^d = \text{WP}^d[P]$.

**Proof.** Assume $\text{NC}^d = \text{NNC}^{d+e}[i(n) \log n]$. By Assumption 9, let $(Q, \kappa)$ be a parameterized problem complete for $\text{WP}^d[P]$ with $Q \in \text{NNC}^d[\text{poly}]$. By Theorem 8, there is a circuit-computable function $h$ and an $\text{NNC}^{d+e}$ circuit family $\{C_{n,k}\}$ such that $\text{nondet}(C_{n,k}) \leq h(k) \log n$.

Let $f$ be a circuit-computable function such that $n \geq f(k)$ implies $h(k) \leq i(n)$. **TODO Explain why such an $f$ must exist.** Now consider the set $Q^+$, defined by
\[
Q^+ = Q \cap \{x \mid |x| \geq f(\kappa(x))\}.
\]
Since $|x| \geq f(\kappa(x))$ for each $x \in Q^+$, we have $h(\kappa(x)) \leq i(|x|)$. Thus $Q^+$ is in $\text{NNC}^{d+e}[i(n) \log n]$. By assumption, $Q^+$ is therefore also in $\text{NC}^d$. This means $Q$ is eventually in $\text{NC}^d$, so we have $Q \in \text{FPP}^d$ by Theorem 7. \hfill $\square$
This lemma is an adaptation of [4, Lemma 3.24].

**Lemma 11.** Suppose $e$ is a positive integer and $f$ and $g$ are increasing, circuit-computable functions. There are functions $i_f$ and $i_{g,e}$ such that

- $f(i_f(n)) \leq n$ for each $n \geq f(1)$,
- $g(i_{g,e}(n)) \leq \log^e n$ for each $n \geq g(1)$.

Furthermore, these functions are circuit-computable, nondecreasing, unbounded.

**Proof.** Define $i_f$ by

$$i_f(n) = \begin{cases} \max \{j \in \mathbb{N} \mid f(j) \leq n\} & \text{if } n \geq f(1) \\ 1 & \text{otherwise}, \end{cases}$$

and $i_{g,e}$ by

$$i_{g,e}(n) = \begin{cases} \max \{j \in \mathbb{N} \mid g(j) \leq \log^e n\} & \text{if } n \geq g(1) \\ 1 & \text{otherwise}. \end{cases}$$

It is straightforward to prove that these functions are nondecreasing and unbounded. To compute $i_f$ (computing $i_{g,e}$ is similar), we compute $f(1), \ldots, f(n)$ in parallel, filter by only those values that are at most $n$, and choosing the index of the rightmost value that passes the filter.  

This is the $\text{NC}^d$ bounded version of the parameterized problem from [4, Lemma 3.26].

**Definition 12** ($p$-log-$\text{NC}^d$-Circuit SAT).

- **Instance:** Boolean circuit $C$ of on $n$ inputs, with size $m$ and depth $O(\log^d n)$.
- **Parameter:** $n/\log m$.
- **Question:** Is $C$ satisfiable?

**Lemma 13.** $p$-log-$\text{NC}^d$-Circuit SAT is in $\text{WP}^d[P]$.

**Proof.** The depth-universal circuit $U$ of [3] proves membership of this problem in $\text{WP}^d[P]$. The number of nondeterministic bits required by the universal circuit is simply $n$, the size of the input to the circuit $C$, which is of the form $h(k) \log m$ if we choose $h(k) = k = n/\log m$.

**TODO** Ideally, it should be complete for it too (this would prove the assumption above), but the problem comes in the reduction from an arbitrary parameterized problem to this one: the circuit depth needs to be polylogarithmic, but it may be larger. Perhaps we can use the other complete problems from [2] here?

This is an adaptation of the other direction of [4, Theorem 3.29].
Theorem 14. Suppose $e$ is a positive integer. If $\text{FPP}^d = \text{WP}^d[P]$, then there is a circuit-computable, nondecreasing, unbounded function $i$ such that $\text{NC}^{d+e} = \text{NNC}^d[i(n) \log n]$.

Proof. Assume $\text{FPP}^d = \text{WP}^d[P]$. Since $p\text{-log-NC}^d$-Circuit SAT is in $\text{WP}^d[P]$, it is now in $\text{FPP}^d$ as well. Thus there is a deterministic circuit family $\{C_{m,k'}\}$ and circuit-computable functions $f$ and $g$ such that

- for each instance $D$, we have $D \in \text{NC}^d$-Circuit SAT if and only if $C_{m,k'}(D) = 1$,
- $\text{size}(C_{m,k'}) \leq f(k')m^{O(1)}$,
- $\text{depth}(C_{m,k'}) \leq g(k') \log^d m$.

Assume without loss of generality that $f$ and $g$ are increasing. Let $i_f$ and $i_{g,e}$ be the functions corresponding to $f$ and $g$, respectively, guaranteed by Lemma 11.

Let $Q \in \text{NNC}^d[i(n) \log n]$ Suppose $\{D_n\}$ is the family of $\text{NNC}^d[i(n) \log n]$ circuits that decides $Q$. For each input $x$, let $D_x$ denote $D_n$ with $x$ hardcoded.

Let $R_n$ be the circuit-computable function $x \mapsto D_x$, where $n$ denotes the length of $x$. Then

- for each $x$, we have $x \in Q$ if and only if $D_x$ is satisfiable,
- $\text{size}(R_n) \leq O(\text{size}(D_x)) = O(\text{size}(D_n)) = n^{O(1)}$,
- $\text{depth}(R_n) \leq O(1)$.

If $\kappa$ denotes the parameter function for $p\text{-log-NC}^d$-Circuit SAT, we can now define the family $\{A_n\}$ as $A_n = C_{m,k'} \circ R_n$, where

$$m = |R_n(x)| = O(\text{size}(R_n)) = n^{O(1)},$$

and

$$k' = \kappa(R_n(x)).$$

We claim that $\{A_n\}$ is the $\text{NC}^{d+e}$ circuit family that decides $Q$.

The correctness of $A_n$ follows from the correctness of both $R_n$ and $C_{m,k'}$. The size of $R_n$ is $n^{O(1)}$ and the size of $C_{m,k'}$ is $f(k')m^{O(1)}$. Assuming $n \leq m \leq n^{O(1)}$,

$$\text{size}(A_n) = n^{O(1)} + f(k')m^{O(1)} \leq f(k')n^{O(1)} \leq f\left(\frac{i_f(n) \log n}{\log m}\right)n^{O(1)} \leq f(i_f(n))n^{O(1)} \leq n^{O(1)}.$$
The depth of \( R_n \) is \( O(1) \) and the depth of \( C_{m,k'} \) is \( g(k') \log^d m \). Thus,

\[
\text{depth}(A_n) = O(1) + g(k') \log^d m \\
\leq O(g(k') \log^d n) \\
= O(g \left( \frac{i(n) \log n}{\log m} \right) \log^d n) \\
\leq O(g \left( \frac{i_{g,e}(n) \log n}{\log n} \right) \log^d n) \\
\leq O(g(i_{g,e}(n)) \log^d n) \\
\leq O((\log^e n)(\log^d n)) \\
\leq O(\log^{d+e} n).
\]

We have shown that \( \{ A_n \} \) is a \( \text{NC}^{d+e} \) circuit family that decides \( Q \), thus \( Q \in \text{NC}^{d+e} \). Since \( Q \) was an arbitrary element of \( \text{NNC}^{d \lceil i(n) \log n \rceil} \), we conclude that \( \text{NC}^{d+e} = \text{NNC}^{d \lceil i(n) \log n \rceil} \). \( \qed \)

**References**


