Clarification of a proof of completeness in classes of approximable optimization problems

Jeffrey Finkelstein
Computer Science Department, Boston University
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1 Introduction

This work presents a more precise proof of a theorem in [5] that provides sufficient conditions for an optimization problem to be hard for a class of approximable optimization problems under a certain type of reduction. For consistency, we use definitions and notation that match and extend the definitions from that work.

2 Preliminaries

We denote the real numbers by \( \mathbb{R} \) and the positive real numbers by \( \mathbb{R}^+ \). Similarly we denote the rational numbers by \( \mathbb{Q} \) and the positive rational numbers by \( \mathbb{Q}^+ \). We denote the natural numbers (including 0) by \( \mathbb{N} \). For all \( a \) and \( b \) in \( \mathbb{R} \) with \( a < b \), we denote the open interval between \( a \) and \( b \) by \( (a,b) \) and the closed interval by \( [a,b] \).

If \( S \) is a set, \( S^* \) denotes the set of all finite sequences consisting of elements of \( S \). For each finite sequence \( s \), we denote the length of \( s \) by \( |s| \). We denote the set of all finite binary strings by \( \Sigma^* \).

Definition 2.1. Suppose \( f \) and \( g \) are functions with domain \( \mathbb{N} \) and codomain \( \mathbb{R} \). We denote by \( O(g(n)) \) the set of all functions \( f: \mathbb{N} \to \mathbb{R} \) such that there exists a \( k \in \mathbb{Q}^+ \) and an \( N \in \mathbb{N} \) such that for all \( n \in \mathbb{N} \) we have \( n > N \) implies \( f(n) \leq kg(n) \).

We will use the following lemma to simplify the proof of the main theorem.
Lemma 2.2. Suppose \( f \) and \( g \) are two functions with domain \( \mathbb{N} \) and codomain \( \mathbb{R} \). If \( g(n) > 1 \) for all sufficiently large \( n \), then \( f(n) \) is in \( O(g(n)) \) if and only if \( f(n) - 1 \) is in \( O(g(n) - 1) \).

In this work we wish to determine the approximability of optimization problems. Therefore, we require rigorous definitions for both optimization problems and approximations for them.

Definition 2.3. An optimization problem is given by \( (I, S, m, t) \), where \( I \) is called the instance set, \( S \subseteq I \times \Sigma^* \) and is called the solution relation, \( m : I \times \Sigma^* \to \mathbb{N} \) and is called the measure function, and \( t \in \{ \text{max}, \text{min} \} \) and is called the type of the optimization.

1. The set of solutions corresponding to an instance \( x \in I \) is denoted \( S_P(x) \).
2. The optimal measure for any \( x \in I \) is denoted \( m^*(x) \). Observe that for all \( x \in I \) and all \( y \in \Sigma^* \), if \( t = \text{max} \) then \( m(x, y) \leq m^*(x) \) and if \( t = \text{min} \) then \( m(x, y) \geq m^*(x) \).
3. The performance ratio, \( \rho \), of a solution \( y \) for \( x \) is defined by
   \[
   \rho(x, y) = \begin{cases} 
   \frac{m(x, y)}{m^*(x)} & \text{if } t = \text{max} \\
   \frac{m^*(x)}{m(x, y)} & \text{if } t = \text{min}
   \end{cases}
   \]
4. A function \( f : I \to \Sigma^* \) is an \( r(n) \)-approximator if \( \rho(x, f(x)) \geq \frac{1}{r(|x|)} \), for some function \( r : \mathbb{N} \to \mathbb{Q}^+ \).

Just as a focus of computational complexity for decision problems is on the relationship between \( \mathbb{P} \) and \( \mathbb{NP} \), so too do we study the relationship between optimization problems in \( \mathbb{NP} \).

Definition 2.4. The complexity class \( \mathbb{NP} \) is the class of all optimization problems \( (I, S, m, t) \) such that the following conditions hold.

1. The instance set \( I \) is decidable by a deterministic polynomial time Turing machine.
2. The solution relation \( S \) is decidable by a deterministic polynomial time Turing machine and is polynomially bounded (that is, there is a polynomial \( p \) such that for all \( (x, y) \in S \), we have \( |y| \leq p(|x|) \)).
3. The measure function \( m \) is computable by a deterministic polynomial time Turing machine.

Some optimization problems in \( \mathbb{NP} \) are approximable within some ratio, so we wish to classify those problems as well.

Definition 2.5. If \( \mathcal{F} \) is a collection of functions, \( \mathcal{F}-\text{APX} \) is the subclass of \( \mathbb{NP} \) in which for each optimization problem \( P \) there exists a polynomial time computable \( r(n) \)-approximator for \( P \), where \( r(n) \in \mathcal{F} \).
In this notation, the class usually identified as APX is written $O(1)$-APX.

The following definitions are required for producing complete problems for $\mathcal{F}$-APX.

**Definition 2.6.** A collection of functions $\mathcal{F}$ is *downward closed* if for all functions $g \in \mathcal{F}$, all $c \in \mathbb{R}^+$, and all functions $h$ we have $h(n) \in O(g(nc))$ implies $h \in \mathcal{F}$.

**Definition 2.7.** A function $g$ is *hard* for the collection of functions $\mathcal{F}$ if for all $h \in \mathcal{F}$ there is a $c \in \mathbb{R}^+$ such that $h(n) \in O(g(nc))$.

**Definition 2.8.** Suppose $L$ is a decision problem in $\mathsf{NP}$, $P$ is a maximization problem, and $\mathcal{F}$ is a downward closed collection of functions. There is an *enhanced $\mathcal{F}$-gap-introducing reduction from $L$ to $P$* if there are functions $f: (\Sigma^* \times \mathbb{Q}^+) \to I$ and $g: (\Sigma^* \times \Sigma^* \times \mathbb{Q}^+) \to \Sigma^+$, constants $n_0 \in \mathbb{R}^+$ and $c \in \mathbb{R}^+$, and a function $F: \mathbb{Q}^+ \to \mathbb{Q}^+$ such that

1. $F$ is hard for $\mathcal{F}$,
2. $F$ is non-decreasing for all sufficiently large $n$,
3. $F(n) > 1$ for all sufficiently large $n$,

and for all $x \in \Sigma^+$ with $|x| \geq n_0$ and for all $N \in \mathbb{Q}^+$ such that $N \geq |x|^c$,

1. if $x \in L$ then $m_P(f(x,N)) \geq N$,
2. if $x \notin L$ then $m_P(f(x,N)) \leq \frac{1}{F(N)}N$,
3. for all $y \in S_P(f(x,N))$ such that $m_P(x,y) > \frac{1}{F(N)}N$, we have $g(x,y,N)$ is a witness that $x \in L$, and
4. $f$, $g$, and $F$ are computable in deterministic time polynomial in $|x|$.

The above definition can be modified to allow $P$ to be a minimization problem by swapping the direction of the inequalities and using $F(N) \cdot N$ instead of $\frac{1}{F(N)}N$.

**Definition 2.9.** Suppose $\mathcal{F}$ is a downward closed family of functions and suppose $P$ is a maximization problem in $\mathsf{NPO}$, with $P = (I_P, S_P, m_P, \text{max})$. The maximization problem $P$ is *canonically hard for $\mathcal{F}$-APX* if for all decision problems $L \in \mathsf{NP}$ there is an enhanced $\mathcal{F}$-gap-introducing reduction from $L$ to $P$.

One of the main tools for studying the structural complexity of class of approximable optimization problems is the approximation preserving reduction. One such reduction is the PTAS reduction.

**Definition 2.10.** Suppose $P$ and $Q$ are maximization problems in $\mathsf{NPO}$ with $P = (I_P, S_P, m_P, \text{max})$ and $Q = (I_Q, S_Q, m_Q, \text{max})$. There is a *PTAS reduction* from $P$ to $Q$ if there are functions $f: (I_P \times \mathbb{Q}^+) \to I_Q$, $g: (I_P \times \Sigma^*) \to \Sigma^*$, and $c: (0,1) \to (0,1)$ such that for all $x \in I_P$ and all $\epsilon \in (0,1)$,
1. for all $y \in S_Q(f(x, \epsilon))$, we have $g(x, y, \epsilon) \in S_P(x)$,

2. for all $y \in S_Q(f(x, \epsilon))$, if $\rho_Q(f(x, \epsilon), y) \geq 1 - c(\epsilon)$ then $\rho_P(x, g(x, y, \epsilon)) \geq 1 - \epsilon$, and

3. $f$ and $g$ are computable in deterministic time polynomial in $|x|$. 

The PTAS reduction has been successful in proving completeness in certain classes of approximable optimization problems. In [5], the authors introduce the following relaxation of the PTAS reduction in order to allow more general completeness results.

**Definition 2.11.** Suppose $P$ and $Q$ are maximization problems in NPO with $P = (I_P, S_P, m_P, \text{max})$ and $Q = (I_Q, S_Q, m_Q, \text{max})$. There is an MPTAS reduction from $P$ to $Q$ if there are a polynomial $p$, functions $f : (I_P \times \mathbb{Q}^+) \rightarrow I_Q^*$, $g : (I_P \times (\Sigma^*)^+) \rightarrow \Sigma^*$, and $c : (0, 1) \rightarrow (0, 1)$ such that for all $x \in I_P$ and all $\epsilon \in (0, 1)$,

1. $|f(x, \epsilon)| \leq p(|x|)$,

2. if $f(x, \epsilon) = (x'_1, \ldots, x'_{p(|x|)})$ then for all $y \in S_Q(x'_1) \times \cdots \times S_Q(x'_{p(|x|)})$, we have $g(x, y, \epsilon) \in S_P(x)$,

3. if $f(x, \epsilon) = (x'_1, \ldots, x'_{p(|x|)})$ then for all $y \in S_Q(x'_1) \times \cdots \times S_Q(x'_{p(|x|)})$ there is a $j \in \{1, \ldots, p(|x|)\}$ such that $\rho_Q(x'_j, y_j) \geq 1 - c(\epsilon)$ implies $\rho_P(x, g(x, y, \epsilon)) \geq 1 - \epsilon$, where $y = (y_1, \ldots, y_{p(|x|)})$, and

4. $f$ and $g$ are computable in deterministic polynomial time with respect to $x$.

The “M” in “MPTAS” stands for “multi-valued”, because $f$ produces a finite sequence of instances instead of a single instance, as in the PTAS reduction.

**Lemma 2.12.** Suppose $P$ and $Q$ are two optimization problems. If there is a PTAS reduction from $P$ to $Q$ then there is an MPTAS reduction from $P$ to $Q$.

**Proof.** Suppose $(f, g, c)$ is the PTAS reduction from $P$ to $Q$. If we consider the output of $f$ and the second input to $g$ to be sequences of length one, then $(f, g, c)$ are also an MPTAS reduction from $P$ to $Q$. \qed

The proof of the next lemma is straightforward but tedious, so it is omitted.

**Lemma 2.13.** MPTAS reductions are closed under composition.

### 3 Results

**Theorem 3.1** ([5] Theorem 2). Suppose $F$ is a class of downward closed functions and $Q$ is a maximization problem. If $Q$ is canonically hard for $F$-APX then $Q$ is hard for the class of maximization problems in $F$-APX under MPTAS reductions.
Proof. Suppose $P$ is a maximization problem in $F$-APX, where $P$ is defined by $P = (I_P, S_P, m_P, \text{max})$. By the definition of $F$-APX, we know the instance set $I_P$ is in $\mathcal{P}$, the solutions set $S_P$ is in $\mathcal{P}$ and is polynomially bounded, the measure function $m_P$ is computable in deterministic polynomial time, and there is an $r(n)$-approximator $A$ for $P$ for some $r \in F$.

Assume without loss of generality that $m_P(x, y) \geq 1$ for all $x$ and $y$ (if $m_P(x, y) < 1$ for some $x$ and $y$, just consider a new maximization problem whose measure function is $m'_P(x, y) = m_P(x, y) + 1$; the approximation results will be the same). Since $m_P$ is computable in deterministic polynomial time, the length of its output is bounded by a polynomial $p$ with respect to the length of its first input, $x$, so the value of $m_P(x, y)$ is in the interval $[1, 2^{p(|x|)}]$. Our goal is to construct functions $f, g,$ and $c$ satisfying the conditions of the MPTAS reduction. We do this by partitioning the interval $[1, 2^{p(|x|)}]$ into subintervals, each of “multiplicative size” $\frac{1}{1-\epsilon}$, in order to find a solution that is within a $\frac{1}{1-\epsilon}$ multiplicative factor of the optimal solution, then using the enhanced $F$-gap-introducing reduction to reconstruct a candidate solution.

For any $\epsilon \in (0, 1)$, consider the subintervals $\left[\left(\frac{1}{1-\epsilon}\right)^{i-1}, \left(\frac{1}{1-\epsilon}\right)^{i}\right]$ for each $i \in \{1, \ldots, M\}$ where $M = \left\lceil \frac{p(|x|)}{\log_2 \frac{1}{1-\epsilon}} \right\rceil$. We choose $M$ this way so that $\left(\frac{1}{1-\epsilon}\right)^M \geq 2^{p(|x|)} \geq m_P(x)$ for all $x \in I_P$. Since $\epsilon$ is a constant independent of $x$, the value of $M$ is a polynomial in $|x|$.

Define $L_i = \left\{ x \in I_P \mid m'_P(x) \geq \left(\frac{1}{1-\epsilon}\right)^{i-1} \right\}$ for each $i \in \{1, \ldots, M\}$. Each of the languages $L_i$ is in NPO: the witness is a candidate solution $y$ for $x$ and the verification procedure checks (in polynomial time) that $m'_P(x, y) \geq \left(\frac{1}{1-\epsilon}\right)^{i-1}$ (using the fact that $m'_P(x, y) \geq m_P(x, y)$ for all $y$). By hypothesis there is an enhanced $F$-gap-introducing reduction from $L_i$ to $Q$ for each $i \in \{1, \ldots, M\}$: let $f_i$, $g_i$, $F_i$, $n_i$, $c_i$ be the functions and constants satisfying the definition of that reduction. For each $i \in \{1, \ldots, M\}$, since $r \in F$ and $F_i$ is hard for $F$, we have $r(n) \in O(F_i(n^{d_i}))$ for some $d_i \in \mathbb{R}^+$. In order to facilitate some inequalities in the proof below, we invoke Lemma 2.2 to yield the equivalent statement $r(n) - 1 \in O(F_i(n^{d_i}) - 1)$. More specifically, there is a $k \in \mathbb{Q}^+$ and an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n > N$, we have $r(n) - 1 \leq k(F_i(n^{d_i}) - 1)$, or in other words,

$$ r(n) \leq k(F_i(n^{d_i}) - 1) + 1. \tag{1} $$

At this point we can construct the MPTAS reduction from $P$ to $Q$. Let $f(x, \epsilon) = (f_1(x, N_1), \ldots, f_M(x, N_M))$ for all $x \in I_P$ and all $\epsilon \in (0, 1)$, where $N_i = |x|^{\max\{c_i, d_i\}}$ for each $i \in \{1, \ldots, M\}$. Define $g$ as in Algorithm 1. Let $c(\epsilon) = \frac{\epsilon}{\epsilon + k(\epsilon - 1)}$.

Condition 1 of the reduction is satisfied because $M$ is bounded above by a polynomial in $|x|$, as previously stated. Condition 2 of the reduction is satisfied because $g$ outputs either $A(x)$ or $g_i(x, y_i, N_i)$ for some $i$. The output of the approximation algorithm $A$ when run on input $x$ is a solution in $S_P(x)$ by
We will show that this
−1
Algorithm 1

function $g(x, y, \epsilon)$:
for $i \in \{1, \ldots, M\}$ do
    $N_i \leftarrow |x|^{|\max(c_i, d_i)|}$
    if $m_Q(f_i(x, N_i), y_i) > \frac{N_i}{F_i(N_i)}$ then
        $z_i \leftarrow g_i(x, y_i, N_i)$
    else
        $z_i \leftarrow A(x)$
    $Z \leftarrow \{z_1, \ldots, z_M\}$
    $z_j \leftarrow \arg\max_{z_i \in Z}\{m_P(x, z_i)\}$
return $z_j$

Let $x \in I_F$, let $\epsilon \in (0, 1)$, and let $y \in (S_Q(f_1(x, N_1)) \times \cdots \times S_Q(f_M(x, N_M)))$, where $y = (y_1, \ldots, y_M)$. Assume without loss of generality that $x$ has a solution in $P$ (if $x$ had no solution in $P$, an approximation preserving reduction is meaningless anyway). Thus $m_p^{*}(x)$ exists and there must exist a $j \in \{1, \ldots, M\}$ such that
\[
\left(\frac{1}{1 - \epsilon}\right)^{j-1} \leq m_p^{*}(x) \leq \left(\frac{1}{1 - \epsilon}\right)^{j}.
\] (2)

We will show that this $j$ satisfies condition 3. Suppose $\rho_Q(f_j(x, N_j), y_j) \geq 1 - \epsilon(c)$. If $m_Q(f_j(x, N_j), y_j) > \frac{N_j}{F_j(N_j)}$ then $g(x, y, \epsilon)$ outputs $g_j(x, y_j, N_j)$. By the construction of $g$ and the definition of $g_j$,
\[
m_p(x, g_j(x, y_j, N_j)) \geq \left(\frac{1}{1 - \epsilon}\right)^{j-1}.
\] (3)

Combining Equation 2 and Equation 3 we have
\[
\left(\frac{1}{1 - \epsilon}\right)^{j-1} \leq m_p(x, g_j(x, y_j, N_j)) = m_Q(x, g(x, y, \epsilon)) \leq m_p^{*}(x) \leq \left(\frac{1}{1 - \epsilon}\right)^{j},
\]
which implies
\[
\rho_p(x, g(x, y, \epsilon)) = \frac{m_p(x, g(x, y, \epsilon))}{m_p^{*}(x)} \geq 1 - \epsilon.
\]
This is true even without the premise \( \rho_Q(f_j(x,N_j),y_j) \geq 1 - c(\epsilon) \).

Suppose now that \( m_Q(f_j(x,N_j),y_j) \leq \frac{N_j}{F_j(N_j)} \), so \( g(x,y,\epsilon) \) will yield \( A(x) \).

Since

\[
\rho_P(x,g(x,y,\epsilon)) = \frac{m_P(x,g(x,y,\epsilon))}{m_P(x)} = \frac{m_P(x,A(x))}{m_P(x)} \geq \frac{1}{r(|x|)},
\]

it suffices to show \( \frac{1}{r(|x|)} \geq 1 - \epsilon \). Since \( m_p(x) \geq (\frac{1}{1-\epsilon})^{j-1} \), we have \( x \in L_j \). Since \( f_j \) is part of a gap-introducing reduction from \( L_j \) to \( Q \), we have \( m_p^*(f_j(x,N_j)) \geq N_j \). Hence

\[
\rho_Q(f_j(x,N_j),y_j) = \frac{m_Q(f_j(x,N_j),y_j)}{m_p^*(f_j(x,N_j))} \leq \frac{N_j}{F_j(N_j)} = \frac{1}{F_j(N_j)}.
\]

This implies \( \frac{1}{F_j(N_j)} \geq 1 - c(\epsilon) \), or \( F_j(N_j) \leq \frac{1}{1-c(\epsilon)} \). By construction, \( N_j = |x|^{\max(c,d_j)} \), so \( N_j \geq |x|^{d_j} \). Since \( F_j \) is non-decreasing for sufficiently large inputs, \( F_j(N_j) \geq F_j(|x|^{d_j}) \). Thus \( F_j(|x|^{d_j}) \leq \frac{1}{1-c(\epsilon)} \). Combining this with Equation 1 yields \( r(|x|) \leq k \left( \frac{1}{1-c(\epsilon)} - 1 \right) + 1 \), for sufficiently long \( x \). Substituting the definition of \( c(\epsilon) \) and performing some algebraic manipulation yields \( r(|x|) \leq \frac{1}{1-\epsilon} \), which is equivalent to \( \frac{1}{r(|x|)} \geq 1 - \epsilon \). Therefore \( \rho_P(x,g(x,y,\epsilon)) \geq 1 - \epsilon \), completing the proof that \((f,g,c)\) is a correct MPTAS reduction from an arbitrary language \( P \) in \( \mathcal{F} \)-APX to \( Q \).

In some steps of the above proof we ignore strings \( x \) that were not “sufficiently long”. This is acceptable because a lookup table of finite size can store the optimal solutions to all instances of optimization problem \( P \) of length up to \( n_0 \), where \( n_0 \) is a constant (representing the “sufficient length”). The function \( g \) could then look up and output the optimal solution for all inputs \( x \) of length up to \( n_0 \).

**Corollary 3.2 ([5 Theorem 3]).** Suppose \( \mathcal{F} \) is a class of downward closed functions and suppose \( Q \) is an optimization problem. If \( Q \) is canonically hard for \( \mathcal{F} \)-APX then \( Q \) is hard for \( \mathcal{F} \)-APX under MPTAS reductions.

**Proof.** We already know from [Theorem 3.1] that there is an MPTAS reduction from every maximization problem in \( \mathcal{F} \)-APX to \( Q \). Consider an arbitrary maximization problem \( P \) in \( \mathcal{F} \)-APX. By [5], there is an “E reduction” (another type of approximation preserving reduction) from \( P \) to \( P' \) for some maximization problem \( P' \), and an E reduction implies a PTAS reduction (see, for example, [4, Figure 2]). By [Lemma 2.12] a PTAS reduction implies an MPTAS reduction. By [Lemma 2.13] an MPTAS reduction from \( P \) to \( P' \) and another from \( P' \) to \( Q \) implies an MPTAS reduction from \( P \) to \( Q \). Therefore \( Q \) is hard for \( \mathcal{F} \)-APX under MPTAS reductions.

According to [2] Section 4.6], the original proof of the PCP Theorem from [1] implies a \( O(1) \)-gap-introducing reduction from SATISFIABILITY to MAXIMUM 3-SATISFIABILITY. Since SATISFIABILITY is NP-complete, this implies MAXIMUM
3-Satisfiability is hard for $O(1)$-APX under MPTAS reductions. A well-known $\frac{1}{2}$-approximation for Maximum 3-Satisfiability proves the following.

**Theorem 3.3.** Maximum 3-Satisfiability is complete for APX under MPTAS reductions.

This is not a new result: the problem is known to be complete for APX under “AP reductions” [3, Corollary 8.8], and an AP reduction implies a PTAS reduction (see, for example, [4, Figure 2]), which in turn implies an MPTAS reduction.

**References**


