Supplementary Material I: Error Bound Proof

In this file, we provide the proof for the error bound result presented in our paper *Minimum Barrier Salient Object Detection at 80 FPS*. We start with a result about the distance transform on general graphs, which shows sufficient conditions for an locally equilibrial path map to be optimal (Section 1). Then, some topological preliminaries are introduced (Section 2). Finally, we give the proof for our error bound result (Section 3).

1. Distance Transform on Graph

We start with a result about the distance transform of general graphs. A graph $\mathbf{G} = (V, E)$ is characterized by a vertex set V and an edge set E. If $(v_1, v_2) \in E$, then there is an edge from v_1 to v_2. A path on $\mathbf{G}$ is a sequence of vertices $\langle v_0, \cdots, v_n \rangle$, where $(v_{i-1}, v_i) \in E$ for $i = 1, \cdots, n$. The graph under consideration can be directed or undirected.

Let $\Pi_{\mathbf{G}}$ denote the path set on $\mathbf{G}$. For a distance cost function $\mathcal{F} : \Pi_{\mathbf{G}} \rightarrow \mathbb{R}^+$, without loss of generality, we assume that $\mathcal{F}$ obeys the following condition regarding a seed set S:

$$\mathcal{F}(\pi) = \begin{cases} 0 & \pi = \langle t \rangle, t \in S \\ +\infty & \pi \text{ does not start from } S. \end{cases} \tag{1}$$

Definition 1. A path map $\mathcal{P}$ of a graph $\mathbf{G} = (V, E)$ is a map that records a path $\mathcal{P}(t)$ for each vertex t on the graph. Given a seed set S, we say $\mathcal{P}$ is in its *equilibrium* state in terms of a distance cost function $\mathcal{F}$, if $\mathcal{P}(t) = \langle t \rangle, \forall t \in S$, and

$$\mathcal{F}(\mathcal{P}(t)) \leq \min_{r:(r,t)\in E} \mathcal{F}(\mathcal{P}(r) \cdot \langle r, t \rangle), \forall t \in V \setminus S. \tag{2}$$

In other words, if a path map is in an equilibrium state, no local update will further reduce the distance cost of any vertex. Note that there can be more than one equilibrial path map.

Definition 2. For a graph $\mathbf{G}$, given a seed set S and a distance function $\mathcal{F}$, a path map $\mathcal{P}$ is *optimal*, if

$$\mathcal{F}(\mathcal{P}(t)) = \min_{\pi \in \Pi_t} \mathcal{F}(\pi), \forall t \in V. \tag{3}$$

where Π_t is the set of paths that end at vertex t.

Solving a distance transform problem can be reduced to finding an optimal (shortest) path map for a graph and a given seed set.

Next, we introduce two properties for a distance cost function.

Definition 3. (Non-Decreasing Property) For a given graph and a seed set, if $\mathcal{F}(\pi \cdot \tau) \geq \mathcal{F}(\pi)$ always holds, we say the distance function $\mathcal{F}$ is non-decreasing.

Definition 4. (Reduction Property) Let π_x^* denote the optimal path for a vertex x in terms of some distance function $\mathcal{F}$. For a given graph and a seed set, we say $\mathcal{F}$ is reducible, if for any non-trivial optimal path π_t^* , and any prefix π_p of π_t^*, *i.e.* $\pi_t^* = \pi_p \cdot \sigma$, we have $\mathcal{F}(\pi_t^*) = \mathcal{F}(\pi_p^* \cdot \sigma)$.

Non-trivial optimal paths are the optimal paths that have more than one vertex. The *non-decreasing* property and the *reduction* property are related to the smoothness conditions proposed in [5]. See [5] for a comparison. Many popular distance functions are reducible, *e.g.* geodesic distance and fuzzy distance [5]. Note that these two properties of a distance function can depend on the image and the seed set.

Lemma 1. *For a given graph and a seed set, if $\mathcal{F}$ is non-decreasing and reducible, any equilibrial path map is optimal.*

Proof. Given an equilibrial path map $\mathcal{P}$, let K denote set of all the vertices whose paths on $\mathcal{P}$ are optimal, and $M = V - K$ the set of all vertices whose paths are not optimal. It is easy to see that $K \neq \emptyset$ because $S \subset K$.

Suppose $M \neq \emptyset$. Let t denote a vertex of the smallest cost in M, and π_t^* its optimal path from S to t[1]. From vertex t, by tracing back along the optimal path π_t^*, we can always find a vertex $p \in M$ on π_t^*, whose preceding vertex r on π_t^* is in K, since $S \subset K$. Note that both p and t are in M, and t has the smallest cost in M, so we have

$$\mathcal{F}(\pi_t) \leq \mathcal{F}(\pi_p), \tag{4}$$

where $\pi_t := \mathcal{P}(t)$ and $\pi_p := \mathcal{P}(p)$ are the paths for p and t on the given path map $\mathcal{P}$. Since the map is in its equilibrium

[1] The optimal path π_t^* must be a valid path from the seed set S to t. If there is no path from S to t, then $\min_{\pi \in \Pi_t} \mathcal{F}(\pi) = +\infty$, and according to Definition 2, $p \in K$.

state, we have

$$\mathcal{F}(\pi_p) \leq \mathcal{F}(\pi_r \cdot \langle r, p \rangle), \tag{5}$$

where $\pi_r := \mathcal{P}(r)$.

Let $\pi^*_{r,t}$ denote the part from r to t on π^*_t. The non-decreasing property of $\mathcal{F}$ indicates that

$$\mathcal{F}(\pi_r \cdot \langle r, p \rangle) \leq \mathcal{F}(\pi_r \cdot \pi^*_{r,t}), \tag{6}$$

since $\pi_r \cdot \langle r, p \rangle$ is a prefix of $\pi_r \cdot \pi^*_{r,t}$. Furthermore, $r \in K$, so π_r is optimal. According to the reduction property, we also have

$$\mathcal{F}(\pi^*_t) = \mathcal{F}(\pi_r \cdot \pi^*_{r,t}). \tag{7}$$

Then combining Eqn. 4, 5, 7 and 6, we have

$$\mathcal{F}(\pi_t) \leq \mathcal{F}(\pi^*_t), \tag{8}$$

which contradicts our assumption that $t \in M$, *i.e.* π_t is not optimal. Therefore, $M = \emptyset$, which concludes our proof. □

Remark 1. Lemma 1 indicates that for a non-decreasing and reducible distance function, any algorithm that returns an equilibrial path map, *e.g.* Dijkstra algorithm, fast raster scanning and parallel updating scheme, can solve the shortest path problem exactly. This result is more general than the analysis in [5], since it does not rely on any specific algorithm.

It is easy to check that the cost function $\beta_{\mathcal{I}}$ of MBD is always non-decreasing, but generally it is not reducible (see the counter-example in [8]). In Section 3, we will show a sufficient condition when $\beta_{\mathcal{I}}$ is reducible, based on which Lemma 1 in our paper will be proved.

2. Some Topological Preliminaries

Before we go to the proof of Lemma 1 in our paper, we first introduce some topological results.

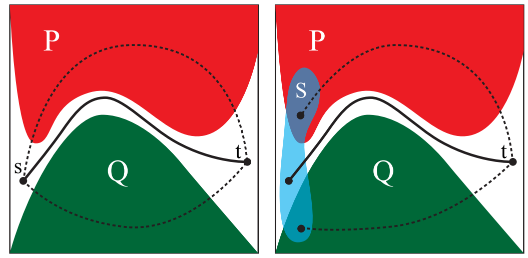
The following is a version of Alexander's lemma [7].

Lemma 2. *Let s and t be two points in the topological space $D = [0,1]^2$, and P and Q be two disjoint closed sets in D. If there exists a path from s to t in $D \setminus P$ and a path from s to t in $D \setminus Q$, then there exists a path from s to t in $D \setminus (P \cup Q)$.*

An illustration of Lemma 2 is given in Fig. 1(a). The proof can be found in [6] (also see Theorem 8.1, *p*. 100 in [7]).

Our error bound proof is based on a generalized version of the aforementioned Alexander's lemma [9].

Lemma 3. *Consider the topological space $D = [0,1]^2$. Let $S \subset D$ be a connected set, $t \in D$ a point, and P and Q two disjoint closed sets in D. If there exists a path from S to t in $D \setminus P$ and a path from S to t in $D \setminus Q$, then there exists a path from S to t in $D \setminus (P \cup Q)$.*



(a) (b)

Figure 1: An illustration of Lemma 2 and Lemma 3. P and Q are two disjoint closed sets in $[0,1]^2$. In (a), there is a path from s to t which does not meet P, and a path from s to t which does not meet Q (see the two dash lines). Then Lemma 2 says that there exist a path (the solid line) from s to t which does not meet $P \cup Q$. In (b), a generalized scenario is shown, where the seed set S is not necessarily singleton, but a connected set. If S and t can be connected by paths that do not meet P and Q respectively, then Lemma 3 guarantees that S and t can be connected by a path that does not not meet $P \cup Q$.

An illustration of Lemma 3 is shown in Fig. 1(b). For its proof, see [9].

Next, we introduce *hyxel* [1, 4], a useful concept in digital topology.

Definition 5. A *hyxel* H_x for a grid point x in a k-D image $\mathcal{I}$ is a unit k-D cube centered at t, *i.e.*

$$H_x = \left[x_1 - \frac{1}{2}, x_1 + \frac{1}{2}\right] \times \cdots \times \left[x_k - \frac{1}{2}, x_k + \frac{1}{2}\right],$$

where $x = (x_1, \cdots, x_k) \in \mathbb{Z}^k$. Hyxels are the generalization of pixels in the 2-D images.

Since hyxels and and grid points in $\mathbb{Z}^k$ have a one-to-one correspondence, the connectivity of a set of hyxels can be induced by the adjacency relationship between grid points.

Definition 6. A sequence of hyxels is a *n_{adj}-path* if the corresponding sequence of grid points is a path in terms of n-adjacency. A set of hyxels is *n_{adj}-connected* if any pair of hyxels in this set are connected by a n_{adj}-path contained in this set.

Definition 7. The *supercover* $\mathcal{S}(M)$ of a point set M in $\mathbb{R}^k$ is the set of all hyxels that meet M.

The following is a result given in [1].

Lemma 4. *Let $M \subset \mathbb{R}^k$ be a connected set. Then $\mathcal{S}(M)$ is $2k_{adj}$-connected.*

In what follows, we consider the case when $k = 2$.

3. The Proof

A digital image $\mathcal{I}$ can be thought of as a special vertex-weighted graph. Let V denote the set of the grid points of its hyxels. For a 2-D image, we assume $V = \{1, \cdots, W\} \times \{1, \cdots, H\}$ is a rectangular region of the integer grids. The edge set E is induced by a specific type of neighborhood adjacency. We consider 4-adjacency in the paper.

In what follows, we will first show that on the discretized image $\widetilde{\mathcal{I}}$ (see the definition in our paper), the path cost function of MBD is reducible when the seed set is connected. Therefore, the equilibrial path map for $\widetilde{\mathcal{I}}$ is optimal. Then we show the optimal path map for $\widetilde{\mathcal{I}}$ gives the MBD estimation for the original image $\mathcal{I}$ with errors bounded by $\varepsilon_{\mathcal{I}}$.

Definition 8. For an image $\mathcal{I}$, a value $u \in \mathbb{R}$ is *separating* if $A = \bigcup_{\mathcal{I}(t)<u} H_t$ and $B = \bigcup_{\mathcal{I}(t)>u} H_t$ are disjoint.

Remark 2. Note that A and B are both finite unions of closed sets (hyxels), so they are closed. The interior of A and B must be disjoint, but their boundaries may intersect. If u is separating, then the set $C = \bigcup_{\mathcal{I}(t)=u} H_t$ is in between A and B.

Lemma 5. *Let $\mathcal{I}$ be a 2-D image,and S be a seed set. Assuming 4-adjacency, if S is connected and $\mathcal{I}(t)$ is separating for each $t \in V$, then the MBD distance cost function $\beta_{\mathcal{I}}$ is reducible.*

Proof. Let $\pi_t^* \in \Pi_t$ be a non-trivial optimal path for point t, and $U^-_{\pi_t^*} = \min_i \mathcal{I}(\pi_t^*(i))$ and $U^+_{\pi_t^*} = \max_i \mathcal{I}(\pi_t^*(i))$, where $\mathcal{I}(\pi)$ denote the set of values on the path π. It suffices to show that for any prefix of π_t^*, say π_r such that $\pi_t^* = \pi_r \cdot \sigma$, $\pi_r^* \cdot \sigma$ is also an optimal path for t. Equivalently, we need to show $\mathcal{I}(\pi_r^*) \subset [U^-_{\pi_t^*}, U^+_{\pi_t^*}]$. In what follows, we will show that $\mathcal{I}(\pi_r^*) \subset [U^-_{\pi_r}, U^+_{\pi_r}]$, where $U^-_{\pi_r} = \min_i \mathcal{I}(\pi_r(i))$ and $U^+_{\pi_r} = \max_i \mathcal{I}(\pi_r(i))$. Since π_r is part of π_t^*, this will immediately conclude our proof.

Suppose $\mathcal{I}(\pi_r^*) \not\subset [U^-_{\pi_r}, U^+_{\pi_r}]$. Without loss of generality, we can assume that $U^+_{\pi_r^*} = \max_i \mathcal{I}(\pi_r^*) > U^+_{\pi_r}$. In this case, we also have $U^-_{\pi_r^*} = \min_i \mathcal{I}(\pi_r^*) > U^-_{\pi_r}$, otherwise π_r^* is not optimal. Then we show there exists a path π_r^{**} such that $\mathcal{I}(\pi_r^{**}) \subset [U^-_{\pi_r^*}, U^+_{\pi_r}]$, contradicting the fact that π_r^* is optimal.

Now we show why π_r^{**} exists under our assumptions. We do this by translating the problem to the continuous setting. Let $D \subset \mathbb{R}^2$ denote the counterpart of V in the continuous space, which is defined as $D = [1, W] \times [1, H]$[2]. Let s_1 and s_2 denote the starting point of π_r and π_r^* respectively. There is a path $\widetilde{\pi}_r$ joining s_1 and r, which is included in the interior of $H_{\pi_r} = \left(\bigcup_{x\in\pi_r} H_x\right) \cap D$ in the topological space D[3]. Similarly, we can find such $\widetilde{\pi}_r^*$ joining s_2 and r in the interior of $H_{\pi_r^*}$.

Let
$$A := \bigcup\{H_x : \mathcal{I}(x) < U^-_{\pi_r^*}\} \cap D,$$
and
$$B := \bigcup\{H_x : \mathcal{I}(x) > U^+_{\pi_r}\} \cap D.$$
It is easy to see that $\widetilde{\pi}_r^*$ does not meet A and $\widetilde{\pi}_r$ does not meet B. Note that $\mathcal{I}(r) \in [U^-_{\pi_r^*}, U^+_{\pi_r}]$ and it is separating, so A and B are disjoint closed sets. Furthermore, $H_S = \bigcup\{H_x : x \in S\}$ is connected, for S is 4_{adj}-connected. According to Lemma 3, there is a path $\widetilde{\pi}_r^{**}$ from H_S to point r which does not meet $A \cup B$.

According to Lemma 4, the supercover $\mathcal{S}(\widetilde{\pi}_r^{**})$ forms a 4_{adj}-path π_r^{**} in V. $\mathcal{S}(\widetilde{\pi}_r^{**})$ cannot contain any pixel outside the digital image because there is a 0.5 wide margin between the boundaries of D and $\bigcup_{x\in V} H_x$. Furthermore, $\mathcal{S}(\widetilde{\pi}_r^{**})$ cannot contain any pixel in A or B, for $\widetilde{\pi}_r^{**} \subset D \setminus (A \cup B)$. Therefore, $I(\pi_r^{**}) \subset [U^-_{\pi_r^*}, U^+_{\pi_r}]$, and we arrive at the contradiction. □

Let $\varepsilon_{\mathcal{I}}$ denote the *maximum neighbor difference* defined in our paper. Then we show that $\widetilde{\mathcal{I}}$, the discretized image using the discretization step $\varepsilon_{\mathcal{I}}$, has the property that each value of $\widetilde{\mathcal{I}}$ is separating.

Lemma 6. *Given an image $\mathcal{I}$, we define $\widetilde{\mathcal{I}}$, such that*

$$\widetilde{\mathcal{I}}(x) = \left\lfloor \frac{\mathcal{I}(x)}{\varepsilon_{\mathcal{I}}} \right\rfloor \varepsilon_{\mathcal{I}}$$

Then for each $x \in V$, $\widetilde{\mathcal{I}}(x)$ is separating w.r.t. $\widetilde{\mathcal{I}}$.

Proof. Suppose there exist a point $x \in V$ such that $\widetilde{\mathcal{I}}(x)$ is not separating on $\widetilde{\mathcal{I}}$. Then there exists a pair of pixels H_a and H_b that touch each other at their boundaries, *s.t.* $\widetilde{\mathcal{I}}(a) < \widetilde{\mathcal{I}}(x)$ and $\widetilde{\mathcal{I}}(b) > \widetilde{\mathcal{I}}(x)$. Since $\widetilde{\mathcal{I}}(x)$ and $\widetilde{\mathcal{I}}(a)$ are multiples of $\varepsilon_{\mathcal{I}}$, we have $\widetilde{\mathcal{I}}(x) - \widetilde{\mathcal{I}}(a) \geq \varepsilon_{\mathcal{I}}$. Similiarly, we have $\widetilde{\mathcal{I}}(b) - \widetilde{\mathcal{I}}(x) \geq \varepsilon_{\mathcal{I}}$. Therefore,

$$\widetilde{\mathcal{I}}(b) - \widetilde{\mathcal{I}}(a) \geq 2\varepsilon_{\mathcal{I}}. \tag{9}$$

Let $\widetilde{\mathcal{I}}(a) = n\varepsilon_{\mathcal{I}}$. Then

$$\mathcal{I}(a) < (n+1)\varepsilon_{\mathcal{I}}, \tag{10}$$
$$\mathcal{I}(b) \geq (n+2)\varepsilon_{\mathcal{I}}. \tag{11}$$

It follows that $\mathcal{I}(b) - \mathcal{I}(a) > \varepsilon_{\mathcal{I}}$. On the other hand,because H_a and H_b are intersecting at their boundaries, a and b must be 8-adjacent. Thus, we also have $|\mathcal{I}(b) - \mathcal{I}(a)| \leq \varepsilon_{\mathcal{I}}$, and a contradiction is reached. □

Theorem 7. *Let $\mathcal{I}$ be a 4-adjacent image, and $\varepsilon_{\mathcal{I}}$ be its maximum local difference. We define $\widetilde{\mathcal{I}}$, such that*

$$\widetilde{\mathcal{I}}(x) = \left\lfloor \frac{\mathcal{I}(x)}{\varepsilon_{\mathcal{I}}} \right\rfloor \varepsilon_{\mathcal{I}}.$$

[2] D is properly included in $\bigcup_{x\in V} H_x$.

[3] Note that π_r is 4-connected, so by simply linking the center points of the consecutive pixels, the resultant paths will be included in the interior of the corresponding hyxel sets.

Given a connected seed set S in terms of 4-adjacency, let $d_{\beta_{\mathcal{I}}}(S, t)$ denote the MBD for t w.r.t. the original image $\mathcal{I}$. If $\mathcal{P}$ is an equilibrial path map for $\widetilde{\mathcal{I}}$ w.r.t. $\beta_{\widetilde{\mathcal{I}}}$, then for each $t \in V$,

$$\left|\beta_{\widetilde{\mathcal{I}}}(\mathcal{P}(t)) - d_{\beta_{\mathcal{I}}}(S, t)\right| < \varepsilon_{\mathcal{I}}.$$

Proof. According to Lemma 1, 5 and 6, the equilibrial path map $\mathcal{P}$ is an exact solution for the MBD shortest path problem on $\widetilde{\mathcal{I}}$, *i.e.*

$$\beta_{\widetilde{\mathcal{I}}}(\mathcal{P}(t)) = \min_{\pi \in \Pi_{S,t}} \beta_{\widetilde{\mathcal{I}}}(\pi). \tag{12}$$

For a path π, $\beta_{\mathcal{I}}(\pi) = U^+ - U^-$, where $U^- = \min_i \mathcal{I}(\pi(i))$ and $U^+ = \max_i \mathcal{I}(\pi(i))$. Similarly, $\beta_{\widetilde{\mathcal{I}}}(\pi) = \widetilde{U}^+ - \widetilde{U}^-$, where $\widetilde{U}^- = \min_i \widetilde{\mathcal{I}}(\pi(i))$ and $\widetilde{U}^+ = \max_i \widetilde{\mathcal{I}}(\pi(i))$. According to the definition of $\widetilde{\mathcal{I}}$, we have

$$U^+ - \varepsilon_{\mathcal{I}} < \widetilde{U}^+ \leq U^+,$$
$$U^- - \varepsilon_{\mathcal{I}} < \widetilde{U}^- \leq U^-.$$

It follows that

$$U^+ - U^- - \varepsilon_{\mathcal{I}} < \widetilde{U}^+ - \widetilde{U}^- < U^+ - U^- + \varepsilon_{\mathcal{I}}$$
$$\Rightarrow |\beta_{\widetilde{\mathcal{I}}}(\pi) - \beta_{\mathcal{I}}(\pi)| < \varepsilon_{\mathcal{I}}, \text{ for any path } \pi \tag{13}$$

Based on Eqn. 12 and 13, it is easy to see that for each $t \in V$,

$$\left|\min_{\pi \in \Pi_{S,t}} \beta_{\widetilde{\mathcal{I}}}(\pi) - \min_{\pi \in \Pi_{S,t}} \beta_{\mathcal{I}}(\pi)\right| < \varepsilon_{\mathcal{I}}$$
$$\Rightarrow \left|\beta_{\widetilde{\mathcal{I}}}(\mathcal{P}(t)) - d_{\beta_{\mathcal{I}}}(S, t)\right| < \varepsilon_{\mathcal{I}}, \ \forall t \in V. \tag{14}$$

□

In our paper, the converged solution of `FastMBD`* is a distance map corresponding to an equilibrial path map for $\widetilde{\mathcal{I}}$. Thus, Lemma 1 in our paper is proved. It also follows that any Dijkstra-like algorithm that returns an equilibrial path map *w.r.t.* the MBD path cost function has the same error bound result if a discretization step is first applied. We conjecture that the discretization step is not necessary for the error bound to hold, but the proof seems much more challenging and is left for future work.

As discussed in [2], we can assume that the digital image $\mathcal{I}$ is a discrete sampling of an *idealized* image in the continuous domain $\mathbb{R}^2$ [3], and this idealized image is a continuous function due to the smoothing effect of the point spread function in a given imaging system. Under this assumption, $\varepsilon_{\mathcal{I}}$ will approach 0, as the sampling density of the digital image increases. Therefore, Lemma 1 in our paper indicates that the stable solution of `FastMBD`* is guaranteed to converge to the exact MBD transform of the idealized image, when the sampling density of an imaging system goes to infinity.

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