Structure of quantum Markov chains and its applications

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Abstract

This paper studies the state space decomposition and limit behavior of quantum Markov chains(qMC). We give the definition of aperiodic subspaces of qMCs and give the decomposition of irreducible subspaces. We also define pseudo-unitary-evolution subspace of a qMC and use it to express the structure of recurrent subspace of a qMC. Then we study the limit behavior of qMC based on these results. We also give the algorithms to solve these problems and study the application of these results.

Keywords: quantum Markov chains, period, limit behavior

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1 Introduction

In classical Markov chain, any BSCC has a period, and it could be decomposed to p subspaces that every subspaces are an aperiodic BSCC under $\mathcal{G}^p$. We give similar results for quantum Markov chains, and solve the problem of lacking uniqueness in the BSCC decomposition of quantum Markov chains. We have also developed a method to study the limit distribution and stationary distribution of quantum Markov chains.

Briefly speaking, we consider a trace-preserving completely-positive super-operator as a quantum Markov chain. This model is general enough. I studied the period of the irreducible subspaces, limit behavior and so on. Other researchers have proved that the recurrent subspace of a quantum Markov chain could be decomposed to irreducible subspaces two years ago but there were still some problems on the uniqueness of decomposition, which I have also solved. The results that I have got are strong enough(as strong as the theorems on classical Markov chains). The proofs of results on quantum Markov chains are much more complicated than that in classical Markov chains and although the results are similar in some aspects, they have subtle and beautiful differences.

This paper is organized as follows. TBA.

2 Preliminaries

2.1 Basic notations

We use Greek letters with Dirac notations to denote pure states, like $|\varphi\rangle$. We use Greek letters without Dirac notations to denote their corresponding density operators like $\varphi = |\varphi\rangle\langle\varphi|$

We use $a^\dagger$ to denote the conjugate of $a \in C$ or conjugate transposition of a complex matrix.

We use P_S to denote the projection operator on subspace S.

We use $\mathrm{supp}(\rho)$ to denote the support of a density matrix ρ.

We use $\mathrm{span}(\cdots)$ to express the spanning subspace of pure states, sequence of subspaces and so on. For example, if $\rho = \sum_{i=1}^n c_i\varphi_i$, then $\mathrm{supp}(\rho) = \mathrm{span}(\{|\varphi_i\rangle\}_{i=1}^n)$. Other examples include $\mathrm{span}(S_1, S_2)$, $\mathrm{span}(|\varphi\rangle)$ and so on.

We write $\dim(\rho)$ as an abbreviation of $\dim(\mathrm{supp}(\rho))$.

$F(\rho,\sigma)$ is the fidelity of ρ and σ. $\|\cdot\|_{tr}$ denotes the trace norm. $\|\rho-\sigma\|_{tr}$ is the trace distance of ρ and σ times 2.

We use $\rho \in S$ as an abbreviation of $\rho \in \mathcal{D}(S)$, where $\mathcal{D}(S)$ is the set of density operators in S.

We use Gothic font characters to denote a sequence of pure states, for example, $\mathfrak{s} = \{|0\rangle, |1\rangle\}$, where the right hand of equation is a sequence instead of a set. We also use the common notation $\{a_i\}_{i=0}^{+\infty}$.

We write imaginary unit as upright font i. And we will also avoid using i as index when there is i in the formula. But we won't avoid using i when there is no i.

Let's introduce a new symbol $\boxplus$ which operates on two sequences of states with the same length: For two sequences of states $\{|\varphi_i\rangle\}_n$ and $\{|\chi_i\rangle\}_n$, two coefficients α, β, introduce notation $\boxplus$ as follows:

$$\alpha\{|\varphi_i\rangle\}_n \boxplus \beta\{|\chi_i\rangle\}_n = \{\alpha|\varphi_i\rangle + \beta|\chi_i\rangle\}_n$$

This notation is convenient. We will often use the form $\mathrm{span}(\alpha\{|\varphi_i\rangle\}_n \boxplus \beta\{|\chi_i\rangle\}_n)$ to express a subspace. For example:

Example 2.1. Suppose $\{|\varphi_i\rangle\}_2 = \{|0\rangle, |1\rangle\}_2$, $\{|\chi_i\rangle\}_2 = \{|2\rangle, |3\rangle\}_2$, then we have $\mathrm{span}(\frac{1}{\sqrt{2}}\{|\varphi_i\rangle\}_2 \boxplus \frac{1}{\sqrt{2}}\{|\chi_i\rangle\}_2) = \mathrm{span}(\frac{|0\rangle+|2\rangle}{\sqrt{2}}, \frac{|1\rangle+|3\rangle}{\sqrt{2}})$

Then let's introduce a symbol $\boxed{DenM}$ to express density matrices:

$$\{c_i\}_n \boxed{DenM} \{|\varphi_i\rangle\}_n = \sum_{i=1}^{n} c_i \varphi_i$$

We will often use the form $\bullet \boxed{DenM} (\bullet \boxplus \bullet)$ to express density operators.

Example 2.2.

$$\{\frac{1}{2}, \frac{1}{2}\}_2 \boxed{DenM} (\frac{3}{5}\{|0\rangle, |1\rangle\}_2 \boxplus \frac{4}{5}\mathrm{i}\{|2\rangle, |3\rangle\}_2) = \frac{1}{2}(\frac{3}{5}|0\rangle + \frac{4}{5}\mathrm{i}|2\rangle)(\frac{3}{5}\langle 0| - \frac{4}{5}\mathrm{i}\langle 2|) + \frac{1}{2}(\frac{3}{5}|1\rangle + \frac{4}{5}\mathrm{i}|3\rangle)(\frac{3}{5}\langle 1| - \frac{4}{5}\mathrm{i}\langle 3|)$$

Suppose $\mathcal{E}$ is a superoperator, ρ_1, ρ_2 are density operators. Since when $\mathrm{supp}(\rho_1) = \mathrm{supp}(\rho_2)$, $\mathrm{supp}(\mathcal{E}(\rho_1)) = \mathrm{supp}(\mathcal{E}(\rho_2))$ must hold, then we could use $\mathcal{E}(S)$ to represent $\mathrm{supp}(\mathcal{E}(\rho))$, where ρ is a density operator that satisfies $\mathrm{supp}(\rho) = S$.

Following are the important new symbols and their meanings in this paper:

Symbol	Arguments	Meaning
$\alpha\{\|\varphi_i\rangle\}_n \boxplus \beta\{\|\chi_i\rangle\}_n$	sequences	$\{\alpha\|\varphi_i\rangle + \beta\|\chi_i\rangle\}_n$
$\{c_i\}_n \boxed{DenM} \{\|\varphi_i\rangle\}_n$	sequences	$\sum_{i=1}^{n} c_i\varphi_i$
$\prec, \succ, \sim$	density operators	an order structure on density operators
$MS(S_1, S_2)$	two subspaces	similarity
$MSPOS_{S_1}(S_2)$	two subspaces	maximum similarity subspace in S_1 towards S_2
$XD(\rho)$	density operator	orbit dimension function
$LF_k(\rho)$	integer and density operator	orbit fidelity function
$\rho_\blacklozenge$	-	A density operator that we construct in section 3.4
$MID(\|\varphi_1\rangle, \|\varphi_2\rangle)$ $MID(\rho_1, \rho_2)$	pure states or density operators	middle state
$AXLE(\|\varphi_1\rangle, \|\varphi_2\rangle)$ $AXLE(\rho_1, \rho_2)$	pure states or density operators	axle state

2.2 Basic quantum theory and lemmas

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Lemma 2.1. *Suppose $S = \mathrm{supp}(\rho_1)$, then $F(\rho_1, \rho_2) \le \sqrt{tr(P_S\rho_2)}$, the equality holds if and only if $\rho_1 \perp \rho_2$ or $\rho_1 = \frac{P_S\rho_2P_S}{tr(P_S\rho_2)}$*

Proof. Without loss of generality, suppose $\rho_1 = \sum_{i=1}^{k} c_i |i\rangle\langle i|$. Suppose one of its purification is $|\phi\rangle = \sum_{i=1}^{k}\sqrt{c_i}|i\rangle|i\rangle$. Then

$$F(\rho_1, \rho_2) = \max_{|\varphi\rangle} |\langle\varphi|\phi\rangle| = \max_{|\varphi\rangle} |\sum_i \sqrt{c_i}\langle ii|\varphi\rangle|$$

Where $|\varphi\rangle$ is a purification of ρ_2. Then

$$|\sum_i \sqrt{c_i}\langle ii|\varphi\rangle| \le \sum_i \sqrt{c_i}|\langle ii|\varphi\rangle| \le \sqrt{(\sum_i c_i)(\sum_i |\langle ii|\varphi\rangle|^2)} = \sqrt{\sum_i |\langle ii|\varphi\rangle|^2} \quad (2.1)$$

Suppose $|\varphi\rangle = \sum_{i=1}^{n}\sqrt{d_i}|i\rangle|\psi_i\rangle$. Because $|\varphi\rangle$ is a purification of ρ_2 we know $\rho_2 = \sum_{i=1}^{n} d_i|\psi_i\rangle\langle\psi_i|$. Then

$$\sum_{i=1}^{k} |\langle ii|\varphi\rangle|^2 = \sum_{i=1}^{k} |\sqrt{d_i}\langle i|\psi_i\rangle|^2 \le \sum_{i=1}^{k} tr(P_S d_i \psi_i) = tr(P_S\rho_2) \quad (2.2)$$

So these equality must hold. Which means

By equality condition of Equation (2.2)

$$|\langle i|\psi_i\rangle|^2 = tr(P_S\psi_i) \Rightarrow |\psi_i\rangle = e_i|i\rangle + f_i|\eta\rangle, |\eta\rangle \perp S$$

By equality condition of Equation (2.1)

$$\exists C, \forall i, \sqrt{c_i} = C|\langle ii|\varphi\rangle| = C|\sqrt{d_i}\langle i|\psi_i\rangle| \quad \text{or} \quad \forall i, e_i = 0$$

$$\therefore \rho_1 = \sum_{i=1}^{k} |i\rangle c_i \langle i| = C\sum_{i=1}^{k} d_i |i\rangle \langle i|\psi_i\rangle \langle\psi_i|i\rangle \langle i| = C\sum_{i=1}^{k} d_i P_S \psi_i P_S = CP_S\rho_2 P_S \quad \text{or} \quad \rho_1 \perp \rho_2$$

□

2.3 Overview of classical Markov chains

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2.4 Concept of quantum Markov chains

Definition 2.1. A quantum Markov chain is a pair $\mathcal{G} = (H, \mathcal{E})$, H is a finite-dimensional Hilbert space, $\mathcal{E}$ is a super-operator on H

This definition is general enough for many applications.

For $\mathcal{G} = (H, \mathcal{E})$, we use $\mathcal{G}^i$ to represent $(H, \mathcal{E}^i)$, where $\mathcal{E}^i = \mathcal{E} \circ \mathcal{E} \circ \cdots \mathcal{E}$.

Suppose $\mathcal{G} = (H, \mathcal{E})$ is a quantum Markov chain, we say a subspace X is bottom invariant if $\mathcal{E}(X) = X$.

Suppose $\mathcal{G} = (H, \mathcal{E})$ is a quantum Markov chain, S is a subspace of H, define $\mathcal{E}|S$ as quantum Markov chain $\mathcal{G}' = (S, \mathcal{E}')$, where $\mathcal{E}'$ is a super-operator on S defined by $\mathcal{E}'(\rho) = P_S\mathcal{E}(\rho)P_S$.

We will only consider trace-preserving operator $\mathcal{E}$.

2.5 Existing results on quantum Markov chains

Our problem is: how to describe the structure of quantum Markov chains? Does it have transient subspace, recurrent subspace, irreducible subspaces and decomposition theorems like classical Markov chains?

Definition 2.2. For a quantum Markov chain $\mathcal{G} = (H, \mathcal{E})$, define the reachable subspace of ρ as the subspace spanned by $\mathrm{supp}(\rho), \mathrm{supp}(\mathcal{E}(\rho)), \mathrm{supp}(\mathcal{E}^2(\rho)) \cdots$. Denote it as $\mathcal{R}_{\mathcal{G}}(\rho)$.

Definition 2.3. Consider a quantum Markov chain $\mathcal{G} = (H, \mathcal{E})$. We say a subspace S of H is a bottom strongly connected component(BSCC) of $\mathcal{G}$ if it satisfies $\forall \rho \in \mathcal{D}(S), \mathcal{R}_{\mathcal{G}}(\rho) = S$

BSCC is the quantum analog to "irreducible" in classical Markov chains.

I omit the properties of reachable subspaces and BSCCs for simplicity. You could see [5] for details. The definitions of "reachable subspace" and "BSCC" in [5] are different from the definitions here but they are equivalent.

And I will skip the formal definition of "transient subspace" and "recurrent subspace"(See [5] for details) and present the decomposition theorem directly.

Theorem 2.2. *Consider a quantum Markov chain $\mathcal{G} = (H, \mathcal{E})$. H could be decomposed to the direct sum of orthogonal subspaces $B_1, B_2 \cdots B_u, T$, where T is the transient subspace of $\mathcal{G}$, B_is are BSCCs of $\mathcal{G}$.*

Let's see an example, which is the example used in [5]:

Example 2.3. Consider quantum Markov chain $\mathcal{G} = (H, \mathcal{E})$. The state space is $H = \mathrm{span}(|0\rangle, |1\rangle, |2\rangle, |3\rangle, |4\rangle)$, the superoperator is

$$\mathcal{E} = \sum_{i=1}^{5} E_i \cdot E_i^\dagger$$

where

$$E_1 = \frac{1}{\sqrt{2}}(|1\rangle \langle 0+1| + |3\rangle \langle 2+3|)$$
$$E_2 = \frac{1}{\sqrt{2}}(|1\rangle \langle 0-1| + |3\rangle \langle 2-3|)$$
$$E_3 = \frac{1}{\sqrt{2}}(|0\rangle \langle 0+1| + |2\rangle \langle 2+3|)$$
$$E_4 = \frac{1}{\sqrt{2}}(|0\rangle \langle 0-1| + |2\rangle \langle 2-3|)$$
$$E_5 = \frac{1}{10}(|0\rangle \langle 4| + |1\rangle \langle 4| + |2\rangle \langle 4| + 4|3\rangle \langle 4| + 9|4\rangle \langle 4|)$$

where $|i \pm j\rangle = \frac{|i\rangle \pm |j\rangle}{\sqrt{2}}$

Then H has BSCC decomposition $H = B_1 \oplus B_2 \oplus T$, where $T = \mathrm{span}(|4\rangle)$ is the transient subspace, $B_1 = \mathrm{span}(|0\rangle, |1\rangle), B_2 = \mathrm{span}(|2\rangle, |3\rangle)$ are BSCCs of $\mathcal{G}$.

But this decomposition only has a very weak uniqueness: the dimension of B_i is unique.

3 Period of BSCCs and the period decomposition theorem

3.1 Introduction to the theorem

In classical Markov chains we have the concepts of irreducible components and aperiodic components[1]:

In classical Markov chains, the recurrent component could be decomposed to irreducible components uniquely[1]. Irreducible components could be analyzed and decomposed further: we could introduce the concept of "period" on an irreducible components as the greatest common divisor of all the possible length of cycles in it. Suppose the transition matrix of this Markov chain is P, the period of an irreducible component is p, then this component could be decomposed to p subcomponents that is aperiodic(also called ergodic) under the p-th power of this Markov chain. Furthermore, we could find p distributions $\pi_1, \pi_2 \cdots \pi_n$ on these components that satisfies $\pi_i P = \pi_{i+1 \mod n}$. We call this decomposition as the period decomposition.

It has been proven that quantum Markov chains have analogous structure[5]. The state space of a quantum Markov chain could be decomposed to transient subspace and recurrent subspace, and the recurrent subspace could be decomposed to irreducible subspaces. But there is no existing result on period and period decomposition of an irreducible component. In this section we will introduce the concept of aperiodic subspace and give the period decomposition theorem. We will see the theorem is analogous to classical case but still has many differences.

"Aperiodic BSCC" is defined as follows:

Definition 3.1. We say a BSCC S of quantum Markov chain $(H, \mathcal{E})$ is aperiodic if it satisfies $\forall |\varphi\rangle \in S, \exists N$, $\mathrm{supp}(\mathcal{E}^N(\varphi)) = S$

The following theorem is the period decomposition theorem for BSCCs, which is also the main theorem of this section:

Theorem 3.1 (Period decomposition theorem)**.** *A BSCC S must be one of the following two types:*

1. *Aperiodic BSCC*
2. *There exists a period $p \geq 2$ and a sequence of density operators $\rho_1, \rho_2, \cdots \rho_p$ that satisfy:*
   (a) $\mathrm{supp}(\rho_1) \oplus \mathrm{supp}(\rho_2) \oplus \cdots \mathrm{supp}(\rho_p) = S$ *and subspace sequence* $\{\mathrm{supp}(\rho_i)\}_{i=1}^p$ *are orthogonal to each other.*
   (b) $\mathcal{E}(\rho_i) = \rho_{i+1 \mod p}$
   (c) $\forall i \in \{1, \cdots p\}$, $\mathrm{supp}(\rho_i)$ *is an aperiodic BSCC of* $\mathcal{G}^p$

   For any decomposition that satisfies these conditions, the integer p, the set of $\dim(\rho_i)$ *and the number of occurrence of each element is unique.*

This section is organized as follows: section 3.1 is this introduction; section 3.2 is overview of our proof; the mainline of our proof is in section 3.4, 3.7, 3.9, 3.10, and section 3.3, 3.5, 3.6, 3.8 are basic concepts and lemmas. In section 3.3 we introduce the concept of "orbit" of a density operator under a quantum Markov chain and "weak fixed point state" of a quantum Markov chain, then we prove some of its properties which will be used in our proof. Then in section 3.4 we present the first part of mainline. We construct a special density operator $\rho_\blacklozenge$ and prove its basic properties. Our aim is to prove that we could see $\rho_\blacklozenge$ as ρ_1 in the period decomposition theorem. In section 3.5 we study subspaces and density operators. We define "similarity" of subspaces and related concepts and prove lots of properties. These results provide important tools for our proof. In section 3.6 we prove a relation between weak fixed point state and similarity. Then in section 3.7, the second part of mainline, we use the tools in section 3.5 and 3.6 and prove more properties of $\rho_\blacklozenge$. In section 3.8 we study "completely-maximally-fidelitous" in depth and in section 3.9 we use these tools to prove a key property of $\rho_\blacklozenge$. The remaining part of proof is easy, which we put at section 3.10.

Dependencies between these subsections are: 3.3, 3.5 don't rely on other subsections; 3.4 relies on 3.3; 3.6 relies on 3.3 and 3.5; 3.7 relies on 3.4, 3.5 and 3.6; 3.8 relies on 3.5; 3.9 relies on 3.7, 3.8; 3.10 relies on 3.9. 3.10 contains the final proof of period decomposition theorem.

3.2 Overview of proof of period decomposition theorem

Then let's see how to prove this theorem. Briefly speaking, we use this technique: Consider a BSCC S that is not aperiodic, if we could find the "initial state" ρ_1 which satisfies:

$$F(\rho, \mathcal{E}^k(\rho)) = \begin{cases} 0 & \text{when } 1 \leq k < K \\ 1 & \text{when } k = K \end{cases}, \text{for some } K \geq 2 \tag{3.1}$$

Which could be divided to three parts:

$$F(\rho, \mathcal{E}^k(\rho)) = 0 \quad \text{when } 1 \leq k < K \tag{3.1a}$$

$$F(\rho, \mathcal{E}^K(\rho)) = 1 \tag{3.1b}$$

$$K \geq 2 \tag{3.1c}$$

we could choose ρ_1, $\mathcal{E}(\rho_1), \mathcal{E}^2(\rho_1) \cdots \mathcal{E}^{K-1}(\rho_1)$ as the sequence of density operators $\rho_1, \rho_2 \cdots \rho_p$ in period decomposition theorem. This density operator sequence satisfies all the requirements in period decomposition theorem except 2.(c): supp(ρ_i) is aperiodic under $\mathcal{G}^K$.

To understand its reason, let's make a comparison with classical Markov chains. The period decomposition theorem is analogous to the case in classical Markov chains. In classical Markov chains, suppose we have a BSCC with period 6, if we drop the third condition it's possible to get a decomposition with "period" 2 or 3. So this is the reason: even if we find a ρ that satisfies Equation (3.1), it's possible that we have only decomposed it partly. But it doesn't matter, as long as we could prove that for any BSCC that is not aperiodic, there is a decomposition that the number of components K is no less than 2, even if some component in this decomposition doesn't satisfy the condition 2.(c), we could use our result again on $\mathcal{G}^K$ and get a bigger decomposition. So it's reasonable to consider the existence of ρ that satisfies Equation (3.1) as our first step to prove our period decomposition theorem. We will construct such a density operator by some techniques in section 3.4 and prove this density operator does satisfy Equation (3.1) in section 3.5 to 3.9. After getting this result, we are very close to proving our original period decomposition theorem 3.1 and the remaining part of proof is in section 3.10.

In more detail, the process of our proof is divided to several parts:

First, in section 3.4, we construct a density operator $\rho_\blacklozenge$ that satisfies some condition. These conditions are not Equation 3.1 because it's not clear whether the density operator that satisfies Equation 3.1 exists. But these conditions are selected carefully, on the one hand, we could prove that such a density operator exists easily(lemma 3.6) and on the other hand these conditions actually imply Equation (3.1). But this is not easy to prove immediately. All of the subsections from section 3.5 to 3.9 are the process of its proof. Our proof of this relation are divided to three steps. In the end of section 3.4 we prove that $\rho_\blacklozenge$ satisfies

$$\forall i \in N, O_i \perp O_{i+k} \quad \text{when } 1 \leq k < K \tag{3.2a}$$

$$\forall i \in N, F(O_i, O_{i+K}) > 0 \quad \text{and is constant} \tag{3.2b}$$

$$K \geq 2 \tag{3.2c}$$

The condition (3.2a) are slightly stronger(actually equivalent) to (3.1a). These conditions are described with the concept of "orbit" that we will introduce in section 3.3. But (3.2b) is much weaker than (3.1b). Although it is very weak, it is the basis of our next step. We will prove $\rho_\blacklozenge$ satisfies the followings in section 3.7:

$$\forall i \in N, O_i \perp O_{i+k} \quad \text{when } 1 \leq k < K \tag{3.3a}$$

$$\forall i \in N, O_i, O_{i+K} \quad \text{is completely-maximally-fidelitous,} \quad F(O_i, O_{i+K}) > 0 \quad \text{and is constant} \tag{3.3b}$$

$$K \geq 2 \tag{3.3c}$$

And our third step is to prove (3.1) in section 3.9.

To explain the meaning of Equation (3.2) and (3.3), let's explain the "orbit" of a initial state, which we will define and study in seciton 3.3. Another keypoint that we observe is: we couldn't just focus on the value $\rho, \mathcal{E}(\rho), \cdots \mathcal{E}^{p-1}(\rho)$. We need to consider the whole sequence $\{\mathcal{E}^i(\rho)\}_{i=0}^{+\infty}$. This is the orbit of ρ, which we will define in definition 3.2. We usually use $\{O_i\}_{i=0}^{+\infty}$ to denote the orbit. It's easy to see that Equation (3.1) is equivalent to Equation (3.2a), (3.2c) together with $\forall i, F(O_i, O_{i+K}) = 1$. Equation (3.2b) is weaker than (3.3b), and Equation (3.3b) is weaker than (3.1b). So what we will do is to prove Equation (3.1) step by step.

In more detail, in section 3.3 we define the orbit of a density operator. We will then define "weak fixed point state" on the basis of orbit. It is very useful and important. Just like the concept of fixed point states play an important role in the proof of BSCC decompositions weak fixed point states play an important role in the proof of period decomposition theorem. It is general enough and easy to construct(lemma 3.3) and also has properties that are strong enough for our proof(lemma 3.2 and 3.14). We will often use these properties in mainline proof.

Section 3.4 is the first part of mainline proof. We will study lots of properties of orbit like orbit dimension and orbit fidelity. We will construct a density operator $\rho_\blacklozenge$ by some special technique on the basis of these functions on orbit and our aim is to prove this density operator satisfies Equation (3.1). We will describe our technique in the beginning of this subsection and at the end of this section we will prove $\rho_\blacklozenge$ satisfies (3.2).

In section 3.5 we will define lots of basic concepts on subpsaces and density operators. This subsection is the basis of lots of proofs and concepts in the following subsections. We will define the similarities MS of two subspaces, maximal similar subspace from one subspace toward another, and we will study some important special case of subspaces and give it a name "completely-maximally-similar". We will also study density operators and use fidelities as a similar tool as similarities. We will define "completely-maximally-fidelitous" for a special case of density operators. Most of the concepts and lemmas in this subsection are the basis of following subsections, but there are two lemmas that will be used directly in the mainline proof: lemma 3.9 and 3.13. These concepts and tools are important and we will use these tools to take full advantage of Equation (3.2b) in section 3.7. Equation (3.2b) seems to be very weak but using the tools in this section and properties of weak fixed point states like lemma 3.2, 3.14(in section 3.6) it's enough to prove interesting result.

Section 3.6 contains only one lemma, 3.14, which shows a property of weak fixed point states about similarities. We will use it in section 3.7.

Section 3.7 is the second part of mainline. We will prove $\rho_\blacklozenge$ satisfies Equation (3.3). We will mainly use Lemma 3.9, 3.13. We will explain our technique in more detail in the beginning of section 3.7.

In section 3.8 we will prove the lemmas that will be used when we prove $\rho_\blacklozenge$ satisfies Equation (3.1). We have proved in section 3.7 that $\rho_\blacklozenge$ satisfies Equation (3.3). We need to find other way to get a density operator that has bigger orbit fidelity. To get deeper results we need to study the property of completely-maximally-fidelitous deeply. We notice that Equation (3.3b) implies the following: O_i, O_{i+K} are completely-maximally-fidelitous, O_{i+1}, O_{i+K+1} are completely-maximally-fidelitous and $F(O_i, O_{i+K}) = F(O_{i+1}, O_{i+K+1}) = C > 0$. So we could see the following problem as a simplification of Equation (3.3b): Given that Given that ρ_1, ρ_2 are completely-maximally-fidelitous, $\mathcal{E}(\rho_1)$, $\mathcal{E}(\rho_2)$ are completely-maximally-fidelitous, $F(\rho_1, \rho_2) = F(\mathcal{E}(\rho_1), \mathcal{E}(\rho_2)) > 0$. What conclusions could we draw? This is the central problem of this subsection and we will use these results in section 3.9.

In section 3.9 we will prove $\rho_\blacklozenge$ satisfies Equation (3.1). We will use the results in section 3.8 to get a density operator ρ that has bigger $F(\rho, \mathcal{E}^K(\rho))$, and violate the requirement of $\rho_\blacklozenge$ that required by its construction. We will explain it in more detail in the beginning of this subsection.

Section 3.10 is the remaining of the proof of period decomposition theorem and some simple corollaries. The result we gets in section 3.9 is very close to theorem 3.1 and the remaining to be done is finding a decomposition that every component is an aperiodic BSCC under some power of the Markov chain and proving the uniqueness. We will also prove corollary 3.24.1, which is a by-product of our proof and will be used in section 4.3.

3.3 Weak fixed point state and its properties

We will first give the formal definitino of orbit in definition 3.2 and then introduce the definition of weak fixed point states in definition 3.3. Proposition 3.2 is a property of weak fixed point state and 3.3 is how to construct a weak fixed point state. These will all be used in section 3.4.

Definition 3.2. For a density operator ρ and a quantum Markov chain $(H, \mathcal{E})$, define the orbit of ρ as the density operator sequence $\{\mathcal{E}^i(\rho)\}_{i=0}^{+\infty}$.

Definition 3.3. For a quantum Markov chain $(H, \mathcal{E})$, if the orbit of a density operator ρ has a subsequence that converges to ρ itself, we say ρ is a weak fixed point state.

Example 3.1. If U is a unitary operator on H, all of the pure states in H is weak fixed point states.
All of the fixed point states are weak fixed point states.
There is no weak fixed point state in transient subspace.

Just as the concept of fixed point state is important in the study of BSCCs, we need the concept of weak fixed point state to prove the period decomposition theorem. From the viewpoint of orbits, fixed point states have good properties because all the terms in their orbits are the same. But this is too special and not enough for our study.

The concept of weak fixed point state is general enough for our study and it has a lot of good properties, for example, some functions on their orbit is a constant, like lemma 3.2 and lemma 3.14.

Proposition 3.2. *If ρ is a weak fixed point state of quantum Markov chain $(H, \mathcal{E})$, the orbit of ρ is density operator sequence $\{O_i\}_{i=0}^{+\infty}$, then $F(O_i, O_{i+k})$ is a constant function of i when k is fixed.*

Proof. Choose a convergent subsequence in $\{\mathcal{E}^i(\rho)\}_\infty$ that converge to ρ. In this convergent subsequence $F(\mathcal{E}^i(\rho), \mathcal{E}^{i+k}(\rho))$ converge to $F(\rho, \mathcal{E}(\rho))$. But the fidelity is non-decreasing. The only possibility is $F(\mathcal{E}^i(\rho), \mathcal{E}^{i+k}(\rho))$ is constant. □

We will see another property of weak fixed point state in section 3.6. We will define "similarity" in section 3.6 and prove that the similarities on two terms with a fixed distance are a constant.

Simultaneously, weak fixed point states are general enough and easy to construct:

Proposition 3.3. *Suppose ρ_0 is an arbitrary density operator. If a subsequence of the orbit of ρ_0 converges to ρ, then ρ is a weak fixed point state.*

Proof. Suppose the orbit of ρ_0 is $\{O_i\}_{i=0}^{+\infty}$. $\forall \delta > 0$, let's construct an i that satisfies $\|\mathcal{E}^i(\rho) - \rho\|_{tr}) < \delta$. Because there is a subsequence converge to ρ in the orbit of $\{O_i\}_{i=0}^{+\infty}$, we could find an O_j that satisfies

$$\|O_j - \rho\|_{tr} < \frac{\delta}{2} \tag{3.4}$$

and an $O_{j+j_2}, j_2 > 0$ that

$$\|O_{j+j_2} - \rho\|_{tr} < \frac{\delta}{2} \tag{3.5}$$

Use the contraction of trace distance on (3.4) we know

$$\|O_{j+j_2} - \mathcal{E}^{j_2}(\rho)\|_{tr} < \frac{\delta}{2} \tag{3.6}$$

then from Equation (3.5) and (3.6), by the triangle inequality of trace distance we know $\|\mathcal{E}^{j_2}(\rho) - \rho\|_{tr} < \delta$. So there is a subsequence of $\{\mathcal{E}^i(\rho)\}_{i=0}^{+\infty}$ that converge to ρ. □

Actually, we will use this proposition to construct a weak fixed point state as the ρ_1 in period decomposition theorem.

3.4 Mainline of proof, part 1: Construction of decomposition

Let's begin the proof of period decomposition theorem. In the mainline of our proof, we will suppose quantum Markov chain $\mathcal{G} = (S, \mathcal{E} = \sum_i E_i \cdot E_i^\dagger)$ contains only one BSCC S, and S is not aperiodic.

Let's make some intuitive thinking to introduce our construction. To prove this theorem, we hope that we could find a density operator $\rho \in S$ that satisfies Equation (3.1). Then $\rho, \mathcal{E}(\rho), \cdots \mathcal{E}^{K-1}(\rho)$ is a decomposition that satisfies the condition (1) and (2) in period decomposition theorem. After finding a decomposition that satisfies (1) and (2), the remaining work will be done in section 3.10.

As we said in section 3.2, Equation (3.1) is almost equivalent to the condition in the period decomposition theorem. But (3.1) is still not easy to handle. Could we find a condition that is almost equivalent to (3.1)?

Let's first try to remove the condition $K \geq 2$. We find that it's possible to get the following initial state: the fixed point state of S. It does no help to the proof of our theorem. One natural idea is to find another condition to replace this requirement. Our idea is to find ρ that has minimum dimension $\dim(\rho)$. And this requirement also has some effects on other conditions in Equation (3.1). When we choose a ρ that has the minimum dimension, we find it's very hard to violate the other two conditions on $F(O_i, O_{i+k})$. Let's give another example to illustrate this.

Example 3.2. Suppose $H = C^3$. Define $\mathcal{E}$ as follows: for any pure state $|\varphi\rangle$, first do a projection measurement on bases $|0\rangle, |1\rangle, |2\rangle$, then make a unitary transformation which satisfies: $U|0\rangle = |1\rangle, U|1\rangle = |2\rangle, U|2\rangle = |0\rangle$. It's easy to see $p = 3$, and the ρ_1 in period decomposition theorem could be $|0\rangle\langle 0|$, $|1\rangle\langle 1|$ or $|2\rangle\langle 2|$.

But the condition now is still not easy to handle: we need to find a density operator ρ with the minimum dimension that satisfies Equation (3.1). It seems that the situation now is even worse and we even don't know whether there exists a density operator that satisfies these strict conditions and has dimension less than $\dim(S)$. The key idea to prove our theorem is to discard the conditions of Equation (3.1) completely and come up with something very different and prove that they are equivalent. Our idea is to use the condition of "minimum dimension" and some other conditions. Since we don't have Equation (3.1) now, it's not enough to consider only $\dim(\rho)$. Our next idea is to consider the behavior of the orbit of ρ as a whole. These two ideas lead us to the definition of orbit dimension function XD, which we will show in definition 3.4. We will consider the density operator ρ where $XD(\rho)$ has the minimum value. Actually, this single condition is "almost enough" for our construction, which means, it's very hard to construct a quantum Markov chain and BSCC S, when we choose ρ as the minimum point of $XD(\rho)$, sequence $\rho, \mathcal{E}(\rho), \mathcal{E}^2(\rho) \cdots$ doesn't satisfy Equation (3.1).

Under this condition ρ is easy to construct but the theorem is hard to prove. We need some other conditions besides XD. We will consider another character of orbit. We will define "orbit fidelity function" in definition 3.4. After we find the minimum point of XD, we will require that the orbit fidelity function value of ρ is as big as possible.

Let's see our construction of the density operator that satisfies Equation (3.1). It has some differences from our intuitive thinking just now and we will explain them. The proof in this subsection is organized as follows. We will first define two functions in definition 3.4. One is orbit dimension function XD, which operates on a density operator ρ. Another is orbit fidelity function LF_k, which operates on an integer k and a density operator. Then we will define a strict total order in definition 3.5 and prove the existence of the maximal element of this order relation in lemma 3.5. (lemma 3.4 is the properties of two functions). Then come the differences: after finding the "maximal" density operator ρ in lemma 3.5, we will consider its orbit and choose a convergent subsequence. We denote it as $\rho_{\blacklozenge}$ in definition 3.6. We will prove in lemma 3.6 that $\rho_{\blacklozenge}$ is a weak fixed point state and is still maximal in our order relation. Which means, we have proved the existence of the density operator that satisfies (1)it is a weak fixed point state (2) it is maximal under our order relation. These two conditions will provide lots of tools and properties to us and our task in the following subsections until section 3.9 is to prove $\rho_{\blacklozenge}$ satisfies Equation (3.1).

So let's introduce two functions, XD and LF_k.

Definition 3.4. Suppose the orbit of ρ is $\{O_i\}_{i=0}^{+\infty}$. Define orbit dimension function XD and orbit fidelity function LF_k on all the possible density operators $\rho \in S$ as follows:

$$XD(\rho) = \max_{i \in N} \dim(O_i) \tag{3.7}$$

$$LF_k(\rho) = \lim_{i \to +\infty} F(O_i, O_{i+k}) \tag{3.8}$$

Since $F(O_i, O_{i+k})$ is nondecreasing with i when k is fixed, LF_k is well-defined. XD is well-defined obviously.

Then let's begin the first step of construction. We need to define a strict total order on all the density operators in S and prove the existence of maximum point.

Definition 3.5. Define a strict total order on all the density operators in S with the following pseudocode comparison function:

Listing 1: Order structure

```
def compare(ρ1,ρ2):
    if XD(ρ1) < XD(ρ2):#return value is opposite to comparison
        return '>'
    if XD(ρ1) > XD(ρ2):
        return '<'
    if XD(ρ1) = XD(ρ2):
        for k in 1..n:
            if LF_k(ρ1) < LF_k(ρ2):
                return '<'
            if LF_k(ρ1) > LF_k(ρ2):
                return '>'
    return '=' #they are 'equal' if it comes here
```

We use $\rho_1 \prec \rho_2, \rho_1 \sim \rho_2, \rho_1 \succ \rho_2$ to represent the cases where $compare(\rho_1, \rho_2)$ returns $<, =, >$. Then define $\precsim, \succsim$ like $\leq, \geq$. Listing 1 a strict total order obviously so this is well-defined.

Lemma 3.4.

$$\max_{i \in N} XD(\rho_i) \geq XD(\lim_{i \to +\infty} \rho_i) \tag{3.9}$$

$$\lim_{i \to +\infty} LF_k(\rho_i) = LF_k(\lim_{i \to +\infty} \rho_i) \tag{3.10}$$

Proof. Assume $\lim_{i \to \infty} \rho_i = \rho$

The inequality (3.9):

$$\max_{i \in N} XD(\rho_i) = \max_{i \in N} \max_{j \in N} \dim(\mathcal{E}^j(\rho_i)) = \max_{j \in N} \max_{i \in N} \dim(\mathcal{E}^j(\rho_i))$$

Suppose $XD(\rho) = \dim(spn(\mathcal{E}^j(\rho)))$, then $\lim_{i \to +\infty} \mathcal{E}^j(\rho_i) = \mathcal{E}^j(\rho)$. So $\exists i, \dim(\mathcal{E}^j(\rho_i)) \geq \mathcal{E}^j(\rho)$, which completes our proof.

The Equation (3.10):

We know $A(\rho_1, \rho_2) = \arccos F(\rho_1, \rho_2)$ are a distance measure on density operators. So by triangle inequality we know

$$|A(\mathcal{E}^j(\rho_i), \mathcal{E}^{j+k}(\rho_i)) - A(\mathcal{E}^j(\rho), \mathcal{E}^{j+k}(\rho))| \leq A(\mathcal{E}^j(\rho_i), \mathcal{E}^j(\rho)) + A(\mathcal{E}^{j+k}(\rho_i), \mathcal{E}^{j+k}(\rho)) \leq 2A(\rho_i, \rho)$$

By Lagrange's mean value theorem we have

$$|F(\mathcal{E}^j(\rho_i), \mathcal{E}^{j+k}(\rho_i)) - F(\mathcal{E}^j(\rho), \mathcal{E}^{j+k}(\rho))| \leq 2A(\rho_i, \rho)$$

This bound holds for all j so

$$|LF_k(\rho_i) - LF_k(\rho)| \leq 2A(\rho_i, \rho)$$

Take limit on i we know $\lim_{i \to +\infty} LF_k(\rho_i) = LF_k(\lim_{i \to +\infty} \rho_i)$. □

Lemma 3.5. *There exists a "maximal" density operator ρ, which means for all density operators ρ' we have $\rho' \prec \rho$ or $\rho' \sim \rho$.*

Proof. First, we could find a set in which every element has smallest XD. Then by the inequality (3.9) we know this is a compact set. Then we find a set with biggest LF_1. Then by equation (3.10) it is also a compact set. Repeat this process and we could find such a density operator ρ that is "maximal". □

But this is just the first step of our construction. We have found the maximum point ρ in our order relation, we may expect $F(\rho, \mathcal{E}^i(\rho))$ could have some properties. But since LF_k is defined by the limit, it's possible that $F(\rho, \mathcal{E}^k(\rho))$ is not equal to $LF_k(\rho)$. So we need the second step of our construction: Choose a convergent subsequence in its orbit and consider its limit. We will see it doesn't affect the property of being maximum and it does make $F(O_i, O_{i+k})$ constant due to the property of weak fixed point states.

Definition 3.6. Define $\rho_\blacklozenge$ as follows: Suppose ρ is one of the maximum point in our order relation. Choose a convergent subsequence in the orbit of ρ, denote the limit of this subsequence as $\rho_\blacklozenge$. Choose any one if there are multiple choices.

We will prove that this is an initial state that satisfies Equation (3.1). We will finally prove this in section 3.9.

Obviously, we have found a weak fixed point state by choosing convergent subsequence. And $\rho_\blacklozenge$ is also a maximum point in our order relation:

Proposition 3.6. *$\rho_\blacklozenge$ is a weak fixed point state. And $\rho_\blacklozenge$ is also a maximum point in our order relation, in other words, for any ρ' it holds that $\rho' \prec \rho$ or $\rho' \sim \rho$.*

Proof. $\rho_\blacklozenge$ is the limit of a convergent subsequence in the orbit of some density operator ρ. By proposition 3.3 we know $\rho_\blacklozenge$ is a weak fixed point state.

Then by the definition of XD and LF_k every term in this convergent subsequence has equal or smaller orbit dimension and equal orbit fidelty. Then by lemma 3.9 and 3.10 we know $\rho_\blacklozenge \succsim \rho$ in this order structure. But ρ is maximum under the order structure so $\rho_\blacklozenge$ must also be maximum. □

Corollary 3.6.1. *Suppose* $\{O_i\}_{i=0}^{+\infty}$ *is the orbit of* $\rho_\blacklozenge$, *then* $F(O_i, O_{i+k})$ *is constant for all* i *when* k *is fixed.* $F(O_i, O_{i+k}) = LF_k(\rho)$.

Suppose the orbit of $\rho_\blacklozenge$ *is* $\{O_i\}_{i=0}^{+\infty}$, *then Equation* (3.2) *holds.*

To prove $\rho_\blacklozenge$ satisfies Equation (3.1) we need more properties of $\rho_\blacklozenge$. We will study the basic properties of subspaces and density operators in the next subsection. We will see in section 3.6 that $\rho_\blacklozenge$ has not only constant fidelity but also constant "similarity".

3.5 Subspaces and similarities, density operators and fidelities

This subsection is the basis of the following subsections. We have seen in section 3.4 that O_i, O_{i+K} are not orthogonal, and our aim is to prove $O_i = O_{i+K}$. There seems to be a big gap between them, but due to $\rho_\blacklozenge$ is a maximum point in the order relation it's possible to prove it. The first problem is how to use the property of being non-orthogonal. We will define "maximal similar subspace", and use this concept and lemmas to prove a property of O_i and O_{i+K}.

Let's first introduce a trick. Fidelity has the following property: $F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \geq F(\rho, \sigma)$. Suppose $S_1 = \text{supp}(\rho)$, $S_2 = \text{supp}(\sigma)$. If we don't consider the fidelity of ρ and σ, but choose $|\varphi\rangle \in S_1$, $|\phi\rangle \in S_2$ such that $F(\varphi, \phi) = |\langle\varphi|\phi\rangle|$ is maximal, we will get $F(\mathcal{E}(\varphi), \mathcal{E}(\phi)) \geq F(\varphi, \phi)$, which is a very tight lower bound. On the other hand, we have lots of methods to get an upper bound on $F(\mathcal{E}(\varphi), \mathcal{E}(\phi))$. For example, lemma 2.1 and proposition 3.11 followed. We will see lots of cases where these two bounds get the same value and inequality becomes equality. This condition will show lots of properties on structure of $\mathcal{E}$. The first example of this trick is lemma 3.9. So the first question is how to find $|\varphi\rangle, |\phi\rangle$ such that $F(\varphi, \phi)$ is maximal. To solve this problem we introduce the concept of similarity and maximal similarity subspace. Then we introduce lots of other concepts and lemmas which are related to this trick and are also the basis of the following subsections.

This subsection is organized as follows. First we introduce the similarity of two subspaces $MS(S_1, S_2)$, and prove one of its properties: lemma 3.8. Then we introduce the concept of maximal similarity subspace $MSPOS$ in one subspace toward another subspace. We present its properties in lemma 3.7 and then introduce the concept of completely-maximal-similar based on properties of $MSPOS$. Two subspaces that are completely-maximal-similar could be expressed with some form of bases, which we will show in lemma 3.10. Then we introduce lemma 3.9 which is an important lemma and will be used in the mainline proof in section 3.7. Then we turn our attention to density operators and fidelities. We will introduce an upper bound on fidelities in proposition 3.11 and introduce the concept of completely-maximally-fidelitous based on this inequality. Then we list the basic properties of completely-maximally-fidelitous density operators in fact 3.12. Then we will prove lemma 3.13, which is another important lemma which will be used directly in mainline. We will use it in section 3.7.

Let's first study properties of subspaces. Let's introduce two notations MS, $MSPOS$ which operate on subspaces.

Definition 3.7. For two subspaces S_1, S_2, define the similarity of them $MS(S_1, S_2)$ as follows:

$$MS(S_1, S_2) = \max_{|\varphi\rangle \in S_1, ||\varphi\rangle|=1} |P_{S_2}(|\varphi\rangle)|$$

This is well-defined because continuous functions in tight set always have the maximum.

Suppose S_1, S_2 are two subspaces. Define the maximal similarity subspace in S_1 toward S_2 as the set of pure states in S_1 that satisfy $|P_{S_2}(|\varphi\rangle)| = MS(S_1, S_2)| |\varphi\rangle |$. Denote it as $MSPOS_{S_1}(S_2)$.

Example 3.3. $S_1 = \text{span}(|0\rangle, |1\rangle)$, $S_2 = \text{span}(\frac{|0\rangle+|2\rangle}{\sqrt{2}}, \frac{|1\rangle+|3\rangle}{\sqrt{2}})$, $S_3 = \text{span}(\frac{|0\rangle+|2\rangle}{\sqrt{2}}, \frac{|1\rangle+|2\rangle}{\sqrt{2}})$, $S_4 = \text{span}(|2\rangle, |3\rangle)$, then $MS(S_1, S_2) = MS(S_2, S_4) = \frac{1}{\sqrt{2}}$, $MS(S_1, S_3) = |\frac{\langle 0|-\langle 1|}{\sqrt{2}} \frac{|0\rangle-|1\rangle}{\sqrt{2}}| = 1$, $MS(S_2, S_3) = |\frac{\langle 0|+\langle 2|}{\sqrt{2}} \frac{|0\rangle+|2\rangle}{\sqrt{2}}| = 1$.

$MSPOS_{S_1}(S_2) = S_1$, $MSPOS_{S_2}(S_1) = S_2$, $MSPOS_{S_1}(S_3) = MSPOS_{S_3}(S_1) = \text{span}(\frac{|0\rangle-|1\rangle}{\sqrt{2}})$. Furthermore, if $S_5 = \text{span}(\frac{|0\rangle+|1\rangle+|2\rangle+|3\rangle}{2})$, then we have

$$MSPOS_{S_1}(S_5) = \text{span}(\frac{|0\rangle + |1\rangle}{\sqrt{2}})$$
$$MSPOS_{S_5}(S_1) = \text{span}(\frac{|0\rangle + |1\rangle + |2\rangle + |3\rangle}{2})$$

It means similarity MS is a mutual property of two subspaces. Actually, similarity MS is the maximum of $|\langle\varphi|\phi\rangle|$, where $|\varphi\rangle \in S_1$, $|\phi\rangle \in S_2$. Its meaning is how close these two subspaces are to intersection. If two subspaces intersect with each other, their similarity is 1. If they are orthogonal, their similarity is 0.

Lemma 3.7. *1.* $MS(S_1, S_2) = MS(S_2, S_1)$

2. $MSPOS_{S_1}(S_2)$ *is a subspace in* S_1*.*

3. Suppose $S_3 = MSPOS_{S_1}(S_2)$, $S_4 = MSPOS_{S_2}(S_1)$, *we have*

(a) $\dim(S_3) = \dim(S_4)$, $MS(S_1, S_2) = MS(S_3, S_4)$

(b) $MSPOS_{S_3}(S_4) = S_3$, $MSPOS_{S_4}(S_3) = S_4$

4. $MS(S, \mathrm{supp}(\lim_{i\to+\infty} \rho_i)) \le \max_{i\in N} MS(S, \mathrm{supp}(\rho_i))$

Another reason for introducing the concept of similarity is because it has analogous property as fidelity:

Lemma 3.8 (Monotonicity of the similarity)**.**

$$MS(\mathcal{E}(S_1), \mathcal{E}(S_2)) \ge MS(S_1, S_2) \tag{3.11}$$

Proof. Choose $|\varphi\rangle \in S_1$, $|\phi\rangle \in S_2$ such that $MS(S_1, S_2) = |\langle\varphi|\phi\rangle|$. Then

$$\begin{aligned} MS(S_1, S_2) &= F(\varphi, \phi) \\ &\le F(\mathcal{E}(\varphi), \mathcal{E}(\phi)) \\ &\le \sqrt{tr(P_{\mathrm{span}(\mathcal{E}(\varphi))}\mathcal{E}(\phi))} \\ &\le \sqrt{tr(P_{\mathcal{E}(S_1)}\mathcal{E}(\phi))} \\ &\le \max_{|\phi'\rangle\in\mathcal{E}(S_2)} \sqrt{tr(P_{\mathcal{E}(S_1)}\phi')} \\ &= MS(\mathcal{E}(S_1), \mathcal{E}(S_2)) \end{aligned}$$

□

We will use this lemma in section 3.6 to prove that weak fixed point states have constant similarities between the span of two terms with a fixed distance. Which means, due to properties of weak fixed point states, we need to study what will happen when the equality holds. This case is important in our proof but we have no idea about its properties currently.

We have said that we are interested in the case $MS(S_1, S_2) = MS(\mathcal{E}(S_1), \mathcal{E}(S_2))$, and our result is the following lemma, which will be used in mainline proof.

Lemma 3.9. *Suppose quantum Markov chain* $\mathcal{G} = (H, \mathcal{E})$*. Subspaces* $S_1, S_2 \in H$ *satisfy* $MS(S_1, S_2) = MS(\mathcal{E}(S_1), \mathcal{E}(S_2)) = C$*. Suppose the maximal similarity subspaces between them are*

$$B_1 = MSPOS_{S_1}(S_2) \tag{3.12a}$$
$$C_1 = MSPOS_{S_2}(S_1) \tag{3.12b}$$
$$B_2 = MSPOS_{\mathcal{E}(S_1)}(\mathcal{E}(S_2)) \tag{3.12c}$$
$$C_2 = MSPOS_{\mathcal{E}(S_2)}(\mathcal{E}(S_1)) \tag{3.12d}$$

Then we have $\mathcal{E}(B_1) \subseteq B_2, \mathcal{E}(C_1) \subseteq C_2$*.*

Proof. For any $|\varphi\rangle \in S_1$, we could find a $|\phi\rangle \in S_2$ that satisfy $F(\varphi, \phi) = |\langle\varphi|\phi\rangle| = C$. Then we have $F(\mathcal{E}(\varphi), \mathcal{E}(\phi)) \ge C$. But $MS(\mathcal{E}(S_1), \mathcal{E}(S_2)) = C$, so $MS(\mathcal{E}(\varphi), \mathcal{E}(\phi)) = C$ and they are completely-maximally-similarity, so $\mathcal{E}(\varphi) \in B_2, \mathcal{E}(\phi) \in C_2$ □

We will introduce concepts of maximal similarity subspace, completely-maximally-similar and study the properties of them. Then we will prove lemma 3.9, which will be used in the proof of section 3.7, part 2 of mainline proof.

Definition 3.8. If S_1, S_2 satisfy $S_1 = MSPOS_{S_1}(S_2)$, $S_2 = MSPOS_{S_2}(S_1)$, we say S_1, S_2 are completely-maximally-similar.

The intuitive meaning of "maximal similar subspace" from S_1 toward S_2 is the set of states that .

Before we present the properties of spaces that are completely-maximally-similar, let's first define similarity on sequences.

Definition 3.9. If a sequence $\mathfrak{s}_1 = \{\varphi_i\}_{i=1}^n$ satisfies $\forall i, j, i \neq j, |\varphi_i\rangle \perp |\varphi_j\rangle, ||\varphi_i\rangle| = ||\varphi_j\rangle|$, we say $\mathfrak{s}_1$ is a orthogonal consistent sequence.

If two sequences of orthogonal pure states with the same length $\mathfrak{s}_1 = \{\varphi_i\}_n$, $\mathfrak{s}_2 = \{\phi_i\}_n$ satisfy $\forall i, \langle\varphi_i|\phi_i\rangle = C$, $\forall i, j, i \neq j, |\varphi_i\rangle \perp |\phi_j\rangle, ||\varphi_i\rangle| = ||\varphi_j\rangle|, ||\phi_i\rangle| = ||\phi_j\rangle|$, we say $\{\varphi_i\}_n$, $\{\phi_i\}_n$ are completely-maximally-similar, their similarity is C.

We will use lowercase Gothic font characters to denote sequences, for example, $\mathfrak{s}_1, \mathfrak{s}_2$.

Two subspaces that are completely-maximally-similar have the following properties, which enable us to express two subspaces with bases:

Lemma 3.10. *These conditions are equivalent:*

1. *S_1, S_2 are completely-maximally-similar and $MS(S_1, S_2) = C > 0$*
2. *There exist orthogonal bases $\mathfrak{s}_1$ of S_1, $\mathfrak{s}_2$ in S_2 that are completely-maximally-similar, their similarity is C*
3. *$\dim(S_1) = \dim(S_2) = n$, and if we denote the orthogonal complement of S_1 in $S_1 \oplus S_2$ as $S_1^\perp$, then $= \dim(S_1^\perp) = n$. And we could choose orthogonal bases $\mathfrak{s}_1$ in S_1 , orthogonal bases $\mathfrak{s}_2$ in $S_1^\perp$ such that $S_2 = \mathrm{span}(C\mathfrak{s}_1 \boxplus \sqrt{1 - C^2}\mathfrak{s}_2)$.*
4. *Suppose two orthogonal subspaces S_0, $S_0^\perp$ satisfy $\dim(S_0) = \dim(S_0^\perp) = n$, $\mathfrak{s}_1$, $\mathfrak{s}_2$ are orthogonal bases of $S_1, S_1^\perp$ respectively, vector $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in C^2$ satisfy $|(\alpha_1, \beta_1)| = |(\alpha_2, \beta_2)| = 1$ and they are not orthogonal, S_1, S_2 are two subspaces expressed as follows: $S_1 = \mathrm{span}(\alpha_1\mathfrak{s}_1 \boxplus \beta_1\mathfrak{s}_2)$, $S_2 = \mathrm{span}(\alpha_2\mathfrak{s}_1 \boxplus \beta_2\mathfrak{s}_2)$. $C = |\alpha_1^+\alpha_2 + \beta_1^\dagger\beta_2|$.*

With this lemma, we could express two subspaces that are completely-maximally-similar with bases. This is useful in the study of some problems and will mainly be used in section 3.8.

The concept of "completely-maximally-similar" will be used in section 3.7. It will also be used in our discussion of "completely-maximally-fidelitous", and we will see their relation sooner. On the one hand, "completely-maximally-similar" is the necessary condition of "completely-maximally-fidelitous"(fact 3.12) and we could get the property of "completely-maximally-fidelitous" with "completely-maximally-similar" under some condition(lemma 3.13).

Then let's study properties of density operators and fidelities, especially density operators in two subspaces that are completely-maximally-similar. This theorem is the basis:

Proposition 3.11. *Suppose $S_1 = \mathrm{supp}(\rho_1), S_2 = \mathrm{supp}(\rho_2)$, then we have $F(\rho_1, \rho_2) \leq MS(S_1, S_2)$. The equality holds if and only if one of the following holds:*

1. *$\rho_1 \perp \rho_2$*
2. *$MS(S_1, S_2) > 0$, then there must be $\dim(S_1) = \dim(S_2) = n$(denote them as n) and there exist orthogonal bases of S_1 $\mathfrak{s}_1$, orthogonal bases of S_2 $\mathfrak{s}_2$, coefficients $\{c_i\}_n$ such that*
   (a) *$\rho_1 = \sum_i c_i\varphi_i$, $\rho_2 = \sum_i c_i\phi_i$*
   (b) *$\mathfrak{s}_1, \mathfrak{s}_2$ are completely-maximally-similar, similarity is $MS(S_1, S_2)$*

This proposition is intuitive. It's actually weaker than lemma 2.1. But we still put it here because we are interested in the equality condition and it is the basis of definition 3.10. Let's first give some example.

Example 3.4. $\rho_1 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$, $\rho_2 = \frac{1}{4}(|0\rangle + |2\rangle)(\langle 0| + \langle 2|) + \frac{1}{4}(|1\rangle + |3\rangle)(\langle 1| + \langle 3|)$ satisfy the equality, while ρ_1, $\rho_3 = \frac{1}{8}(|0\rangle + |2\rangle)(\langle 0| + \langle 2|) + \frac{3}{8}(|1\rangle + |3\rangle)(\langle 1| + \langle 3|)$ only satisfy the inequality.

Definition 3.10. If density operators ρ_1, ρ_2 satisfies $F(\rho_1, \rho_2) = MS(\mathrm{supp}(\rho_1), \mathrm{supp}(\rho_2))$, we say ρ_1, ρ_2 are completely-maximally-fidelitous.

Example 3.5. $\rho_1 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$, $\rho_2 = \frac{1}{4}(|0\rangle+|2\rangle)(\langle 0|+\langle 2|)+\frac{1}{4}(|1\rangle+|3\rangle)(\langle 1|+\langle 3|)$ are completely-maximally-fidelitous, fidelity is $\frac{1}{\sqrt{2}}$ And this is also the similarity of $\mathrm{supp}(\rho_1), \mathrm{supp}(\rho_2)$. And ρ_1, $\rho_3 = \frac{1}{8}(|0\rangle+|2\rangle)(\langle 0|+\langle 2|)+\frac{3}{8}(|1\rangle+|3\rangle)(\langle 1|+\langle 3|)$ aren't completely-maximally-fidelitous, but $\mathrm{supp}(\rho_1)$, $\mathrm{supp}(\rho_2)$ are completely-maximally-similarity, their similarity is $\frac{1}{\sqrt{2}}$.

The following facts are obvious from proposition 3.11 and definition 3.10:

Fact 3.12.

1. *ρ_1, ρ_2 are completely-maximally-fidelitous, their fidelity is $C \Rightarrow \mathrm{supp}(\rho_1)$, $\mathrm{supp}(\rho_2)$ are completely-maximally-similarity, similarity is C.*

2. *$F(\rho_1, \rho_2) = MS(\mathrm{supp}(\rho_1), \mathrm{supp}(\rho_2)) \Leftrightarrow \rho_1 \perp \rho_2$ or ρ_1, ρ_2 are completely-maximally-fidelitous and $F(\rho_1, \rho_2) > 0$.*

3. *ρ_1, ρ_2 are completely-maximally-fidelitous, their fidelity is $C > 0 \Leftrightarrow$ There exist spectral decomposition $\rho_1 = \{c_i\}_{i=1}^n \boxed{DenM}\, \mathfrak{s}_1$, $\rho_2 = \{c_i\}_{i=1}^n \boxed{DenM}\, \mathfrak{s}_2$ that satisfies $\mathfrak{s}_1$, $\mathfrak{s}_2$ are completely-maximally-similar, their similarity is C.*

We could come up with an idea from the first fact.

Lemma 3.13. *S_1, S_2 are two subspaces that are completely-maximally-similar, their similarity is $C > 0$, $\mathrm{supp}(\mathcal{E}(S_1)) \subseteq S_2$, then there exist two density operators $\rho_1 \in S_1$, $\rho_2 \in S_2$ that satisfy*

1. *ρ_1, ρ_2 are completely-maximally-fidelitous, their fidelity is C*

2. *$\mathcal{E}(\rho_1) = \rho_2$*

Proof. Suppose $|\varphi_i\rangle$ are orthogonal bases of S_1 and $|\phi_i\rangle$ are orthogonal bases of S_2, $\langle\varphi_i|\phi_i\rangle = MS(S_1, S_2)$. Since S_1 is bottom invariant in $U\mathcal{E}U^+$, where $U = \sum_i(|\varphi_i\rangle\langle\phi_i| + |\varphi_i^\perp\rangle\langle\phi_i^\perp|)$, $|\varphi^\perp\rangle, |\phi_i^\perp\rangle \in \mathrm{span}(|\varphi_i\rangle, |\phi_i\rangle)$, which maps S_2 to S_1. Choose a ρ that $U\mathcal{E}(\rho)U^+ = \rho$ and we have $F(\rho, \mathcal{E}(\rho)) = MS(S_1, S_2)$. So $\rho, \mathcal{E}(\rho)$ is completely-maximally-fidelitous, fidelity is C. □

Conclusions and applications

We use similar approach when studying subspaces and density operators. First we introduce the concepts: "similarity" for subspaces and the existing concept of fidelity for density operators. Then lemma 3.8 for similarities is analogous to the monotonicity of fidelities. And we study the case where the equalities hold in these inequalities: we get lemma 3.9 for similarities, and for fidelity, we are interested the case where $F(\rho, \sigma) = F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) > 0$ with the properties of being completely-maximal-fidelitous, which we will study in section 3.8.

Most of lemmas and definitions in this subsections are the basis of section 3.6 and 3.8, but lemma 3.9 and 3.13 will be used directly in section 3.7. We will use these lemmas in the following way(See example 3.6 for details):

In mainline proof, we need lemmas that could construct a density operator that has smaller $\max(\dim(\rho), \dim(\mathcal{E}(\rho)))$ or bigger $F(\rho, \mathcal{E}(\rho))$. We will use these lemmas on the orbit of $\rho_\blacklozenge$ and try to construct a new density operator that has smaller orbit dimension or bigger orbit fidelity(with equal orbit dimension). This violates the property of $\rho_\blacklozenge$ of being maximum under the order structure and we will prove some properties of the orbit of $\rho_\blacklozenge$ with such proof by contradiction.

In more detail, we will use lemma 3.9 and lemma 3.13 in section 3.7. Our aim in section 3.7 is to prove $\rho_\blacklozenge$ satisfies (3.2). We try to use lemma 3.9 to get a density operator with smaller $\max(\dim(\rho), \dim(\mathcal{E}(\rho)))$, and violate the property of $\rho_\blacklozenge$ that it has the smallest orbit dimension. The result of this proof by contradiction is that $\forall i, O_i, O_{i+K}$ are completely-maximally-similar. Then we use lemma 3.13 to try to violate the property of $\rho_\blacklozenge$ that it has the biggest orbit fidelity when the orbit dimension is smallest. The result is that we get O_i, O_{i+K} are completely-maximally-fidelitous.

We will study how to handle the case that O_i, O_{i+K} are completely-maximally-fidelitous in section 3.8.

3.6 A proposition about weak fixed point states and similarities

Proposition 3.14. *If ρ is a weak fixed point state of $\mathcal{G} = (H, \mathcal{E})$, the orbit of ρ is the density operator sequence $\{O_i\}_{i=0}^{+\infty}$. Suppose $S_i = \mathrm{supp}(O_i)$, then we have $MS(S_i, S_{i+k})$ is a constant of i when k is fixed.*

Proof. First, it's easy to know that for two pure states $|\varphi\rangle, |\phi\rangle$, $|\langle\varphi|\phi\rangle| = C \Rightarrow \|\varphi - \phi\|_{tr} = 2\sqrt{1-C^2}$.

Define a function $T(x) = 2\sqrt{1-x^2}$, which maps inner product of pure states to their trace distance times 2.

By lemma 3.8 we know $MS(S_i, S_{i+k})$ is non-decreasing with i. Assume it increases at some position, let's derive a contradiction. Without loss of generality, we could assume $MS(S_1, S_{k+1}) > MS(S_0, S_k)$. So there exists some $\delta > 0$, $T(MS(S_1, S_{k+1})) = T(MS(S_0, S_k)) - \delta$

Assume $|\varphi\rangle \in S_1, |\phi\rangle \in S_{k+1}$ satisfy $|\langle\varphi|\phi\rangle| = MS(S_1, S_{1+k})$. Choose δ' such that $\delta'\varphi < O_1, \delta'\phi < O_{1+k}$ hold. Such δ' always exists.

Suppose the orbits of φ and ϕ are $\{Q_{i+1}\}_{i=0}^{+\infty}$ and $\{R_{i+k+1}\}_{i=0}^{+\infty}$. The indexes of Q start from 1. The indexes of R start from $k+1$.

By monotonicity of trace distance we have

$$\forall i \geq 0, \|R_{i+k+1} - Q_{i+1}\|_{tr} = \|\mathcal{E}^i(\varphi) - \mathcal{E}^i(\phi)\|_{tr} \leq \|\varphi - \phi\|_{tr} = T(MS(S_1, S_{k+1})) \leq T(MS(S_0, S_k)) - \delta \tag{3.13}$$

And we have the conditions: $\forall i \geq 0$,

$$\delta' Q_{i+1} \leq O_{i+1} \tag{3.14a}$$

$$\delta' R_{i+k+1} \leq O_{i+k+1} \tag{3.14b}$$

Assume that $\{O_i\}_{i=0}^{+\infty}$ converges to $O_0 = \rho$ in subsequence of indexes $\{i_j\}_{j=1}^{+\infty}$. So $\{O_{i_j+k}\}_{j=1}^{+\infty}$ converges to O_k.

Consider $P_{O_0} Q_i P_{O_0}, P_{O_k} R_i P_{O_k}$. Due to convergence of $\{O_{i_j}\}_{j=1}^{+\infty}$ and $\{O_{i_j+k}\}_{j=1}^{+\infty}$ we know

$$\lim_{j\to+\infty} \|O_{i_j} - P_{O_0} O_{i_j} P_{O_0}\|_{tr} = 0 \tag{3.15a}$$

$$\lim_{j\to+\infty} \|O_{i_j+k} - P_{O_k} O_{i_j+k} P_{O_k}\|_{tr} = 0 \tag{3.15b}$$

From (3.15) we know

$$\lim_{j\to+\infty} tr(P_{O_0^\perp}(O_{i_j})) = 0 \tag{3.16a}$$

$$\lim_{j\to+\infty} tr(P_{O_k^\perp}(O_{i_j+k})) = 0 \tag{3.16b}$$

From (3.16) and (3.14) we know

$$\lim_{j\to+\infty} tr(P_{O_0^\perp}(\delta_0 Q_{i_j})) = 0 \tag{3.17a}$$

$$\lim_{j\to+\infty} tr(P_{O_k^\perp}(\delta_0 R_{i_j+k})) = 0 \tag{3.17b}$$

From the Equation (3.17) we know

$$\lim_{j\to+\infty} \|\delta_0(Q_{i_j} - P_{O_0} Q_{i_j} P_{O_0})\|_{tr} = 0 \tag{3.18a}$$

$$\lim_{j\to+\infty} \|\delta_0(R_{i_j+k} - P_{O_k} R_{i_j+k} P_{O_k})\|_{tr} = 0 \tag{3.18b}$$

Use triangle inequality of trace distance on Equation (3.18), we know

$$\lim_{j\to+\infty} \|\delta_0(R_{i_j+k} - Q_{i_j})\|_{tr} - \|P_{O_k}(\delta_0 R_{i_j+k}) P_{O_0} - P_{O_0}(\delta_0 Q_{i_j}) P_{O_0}\|_{tr} = 0 \tag{3.19}$$

From Equation (3.17) we could also know

$$\lim_{j\to+\infty} tr(P_{O_0}(\delta_0 Q_{i_j})) = \delta_0 \tag{3.20a}$$

$$\lim_{j\to+\infty} tr(P_{O_k}(\delta_0 R_{i_j+k})) = \delta_0 \tag{3.20b}$$

Which imply

$$\lim_{j\to+\infty} \|\frac{P_{O_0}(\delta_0 Q_{i_j}) P_{O_0}}{tr(P_{O_0}(\delta_0 Q_{i_j}))} - P_{O_0}(Q_{i_j}) P_{O_0}\|_{tr} = 0 \tag{3.21a}$$

$$\lim_{j\to+\infty} \|\frac{P_{O_k}(\delta_0 R_{i_j}) P_{O_k}}{tr(P_{O_k}(\delta_0 R_{i_j+k}))} - P_{O_k}(R_{i_j+k}) P_{O_k}\|_{tr} = 0 \tag{3.21b}$$

And because

$$\forall \rho_1 \in \text{supp}(O_0), \rho_2 \in \text{supp}(O_k), \|\rho_1 - \rho_2\|_{tr} \geq T(MS(S_0, S_k))$$

we know

$$\|\frac{P_{O_0}(\delta_0 Q_{i_j})P_{O_0}}{tr(P_{O_0}(\delta_0 Q_{i_j}))} - \frac{P_{O_k}(\delta_0 R_{i_j+k})P_{O_k}}{tr(P_{O_k}(\delta_0 R_{i_j+k}))}\|_{tr} \geq T(MS(S_0, S_k)) \tag{3.22}$$

Use formulas (3.21) and (3.22) we know

$$\liminf_{j\to+\infty} \|P_{O_k}(\delta_0 Q_{i_j})P_{O_0} - P_{O_k}(\delta_0 R_{i_j+k})P_{O_k}\|_{tr} \geq \delta_0 T(MS(S_0, S_k)) \tag{3.23}$$

From Equation (3.23) and (3.19) we know

$$\liminf_{j\to+\infty} \|\delta_0(R_{i_j+k} - Q_{i_j})\|_{tr} \geq \delta_0 T(MS(S_0, S_k))$$

But by (3.13) we have

$$\sup_{j\geq 1} \|\delta_0(R_{i_j+k} - Q_{i_j})\|_{tr} \leq \delta_0(T(MS(S_0, S_k)) - \delta)$$

Which is a contradiction. □

Weak fixed point states have the same property on fidelities and similarities because both fidelity and similarity satisfy monotonicity like Lemma 3.8, and both of them have some obvious or hidden continuity that could be used in our proof.

3.7 Mainline of proof, part 2: Corollary by properties of similarity and fidelity

Our aim in this section is to prove $\rho_\blacklozenge$ satisfies Equation (3.2). What we have known is Equation (3.1), property of $\rho_\blacklozenge$ of being maximum under the order structure, properties of similarities, fidelities and weak fixed point states.

Lemma 3.15. *Suppose the orbit of $\rho_\blacklozenge$ is $\{O_i\}_{i=0}^{+\infty}$. We have:*

$$\begin{aligned}
&\forall i \in N, O_i \perp O_{i+k} \quad \textit{when } 1 \leq k < K \\
&\forall i \in N, F(O_i, O_{i+K}) > 0 \quad \textit{and is constant} \\
&K \geq 2
\end{aligned} \tag{(3.2) revisited}$$

Let's raise some examples to show the proof of this lemma.

Example 3.6. 1. Use the example that we have raised before. Assume $H = C^3 \mathcal{E}$ is such an operator: for a pure state $|\varphi\rangle$, the operation of $\mathcal{E}$ first does a projection on bases $|0\rangle, |1\rangle, |2\rangle$, then does a unitary transformation U: $U|0\rangle = |1\rangle, U|1\rangle = |2\rangle, U|2\rangle = |0\rangle$. It's easy to see $p = 3$, and all the weak fixed point states are $c_0|0\rangle\langle 0| + c_1|1\rangle\langle 1| + c_2|2\rangle\langle 2|$, where $c_1 + c_2 + c_3 = 1$. Imagine that we use the construction in section 3.4 on this quantum Markov chain. We will get $\rho_\blacklozenge$, its orbit, value of orbit dimension function XD, orbit fidelity function LF_k and so on. As we have said in section 3.4, we will try to construct a initial state ρ' that has smaller orbit dimension or bigger orbit fidelity, which is a contradiction, and get some properties of $\rho_\blacklozenge$. Let's consider the features of the orbits of density operators that are not maximum under the order structure in this example, and see how to handle these cases.

$\rho = \frac{1}{3}|0\rangle\langle 0| + \frac{2}{3}|1\rangle\langle 1|$. Suppose its orbit is $\{O_i\}_{i=0}^{\infty}$, $\text{supp}(O_i) = S_i$. By calculation we know

$$O_1 = \mathcal{E}(\rho) = \frac{1}{3}|1\rangle\langle 1| + \frac{2}{3}|2\rangle\langle 2|$$
$$O_2 = \mathcal{E}^2(\rho) = \frac{1}{3}|2\rangle\langle 2| + \frac{2}{3}|0\rangle\langle 0|$$

We could see S_0, S_1 are completely-maximal-similar, similarity is 1. $S_1 = \mathcal{E}(S_0), S_2 = \mathcal{E}(S_1)$ are completely-maximal-similar, similarity is 1. Use Lemma 3.9, take maximal similary subspaces on S_0 and S_1 and we will get an initial state with smaller orbit dimension

$$MSPOS_{S_1}(S_0) = \text{span}(|1\rangle)$$

Then $\rho' = |1\rangle\langle 1|$ satisfies $XD(\rho') < XD(\rho)$

2. Suppose $H = C^4$. $\mathcal{E}$ is a superoperator that operates as follows: first make projection $|\varphi\rangle$ on bases $|0\rangle, |1\rangle, |2\rangle, |3\rangle$, and take unitary transformation: $U|0\rangle = |1\rangle, U|1\rangle = |2\rangle, U|2\rangle = |3\rangle, U|3\rangle = |0\rangle$. We could see $p = 4$, and all the weak fixed point states are $c_0|0\rangle\langle 0| + c_1|1\rangle\langle 1| + c_2|2\rangle\langle 2| + c_3|3\rangle\langle 3|$. Analogous to the example before, let's consider how to handle the following density operator ρ:

   $\rho = \frac{1}{3}|0\rangle\langle 0| + \frac{2}{3}|2\rangle\langle 2|$. Suppose its orbit is $\{O_i\}_{i=0}^{\infty}$, $\mathrm{supp}(O_i) = S_i$. By calculation we know

$$O_1 = \mathcal{E}(\rho) = \frac{1}{3}|1\rangle\langle 1| + \frac{2}{3}|3\rangle\langle 3|$$
$$O_2 = \mathcal{E}^2(\rho) = \frac{1}{3}|2\rangle\langle 2| + \frac{2}{3}|0\rangle\langle 0|$$
$$O_3 = \mathcal{E}^3(\rho) = \frac{1}{3}|3\rangle\langle 3| + \frac{2}{3}|1\rangle\langle 1|$$

   $S_0 \perp S_1.S_1 = \mathcal{E}(S_0) \perp S_2 = \mathcal{E}(S_1)$, we can't use Lemma 3.9 on these pairs of subspaces. And we have $S_0 = S_2, S_1 = S_3$, the result of using Lemma 3.9 is trivial. But we could use Lemma 3.13 under this case:

   Because $S_0, \mathcal{E}^2(S_0)$ are completely-maximal-similarity, their similarity is 1, based on Lemma 3.13 we could find a density operator $\rho' \in S_0$ such that $F(\rho', \mathcal{E}^2(\rho')) = MS(S_0, S_2) = 1 > F(O_0, O_2)$. Then ρ' has equal orbit dimension and bigger orbit fidelity LF_2.($LF_1(\rho) = 0$ and can't be smaller).

Proof. Let $S_i = \mathrm{supp}(O_i)$.

Because $\rho_\blacklozenge$ is a weak fixed point state, from proposition 3.2, 3.14, $F(O_i, O_{i+K})$, $MS(S_i, S_{i+K})$ are constant with i when K is fixed.

Choose the minimum K such that $F(O_i, O_{i+K}) > 0$, that is to say, we have $LF_k = 0$ when $0 < k < K$, and $LF_K > 0$. First, obviously we have $K \leq \dim(S)$.

If $F(O_i, O_{i+K}) = 1$, the lemma holds automatically.

If $F(O_i, O_{i+K}) = MS(S_i, S_{i+K})$, which means they are completely-maximally-fidelitous and $F(O_i, O_{i+K}) > 0$(fact 3.12(2)), the lemma holds automatically.

From proposition 3.11 we know $F(O_i, O_{i+K}) \leq MS(S_i, S_{i+K})$. So the remaining case is $F(O_i, O_{i+K}) < MS(S_i, S_{i+K})$. For this case, we will use the property of $\rho_\blacklozenge$ of being maximum under the order structure to get a contradiction.

Construct two sequences of subspaces $\{B_i\}_{n=0}^{+\infty}$ $\{C_i\}_{n=0}^{+\infty}$ as follows:

$$B_i = MSPOS_{S_i}(S_{i+K})$$

$$C_i = MSPOS_{S_{i+K}}(S_i)$$

From Lemma 3.9 we know $\mathcal{E}(B_i) \subseteq B_{i+1}$,$\mathcal{E}(C_i) \subseteq C_{i+1}$.

Then from the definition of C_i we know $\dim(C_i) \leq \dim(S_{i+K})$. Then we discuss whether there is an i that makes these two value equal:

1. If $\forall i, \dim(C_i) < \dim(S_{i+K})$, then choose any $\rho' \in C_0$, we have $\mathcal{E}^i(\rho') \in C_i \subset S_{i+K}$, $\mathrm{sodim}(\mathcal{E}^i(\rho')) < \dim(S_{i+K})$, so $XD(\rho') < XD(\rho)$, which violates the property of $\rho_\blacklozenge$ of being maximum under the order structure.

2. If $\exists i, \dim(C_i) = \dim(S_{i+K})$, then $\mathcal{E}^K(B_i) \in S_{i+K} = \mathrm{supp}(C_i)$. Because B_i, C_i are completely-maximally-similar (From the construction above, Lemma 3.7(2) and Definition 3.8). So by Lemma 3.13 we could find a density operator ρ' which satisfies $\rho' \in B_i$, and

$$F(\rho', \mathcal{E}^K(\rho')) = MS(S_i, S_{i+K}) > F(O_i, O_{i+K})$$

   Then we get a density operator ρ' which has bigger orbit fidelity function value $LF_K(\rho)$ and equal orbit dimension function value $XD(\rho)$. When $1 \leq k < K$ $LF_k(\rho_\blacklozenge) = 0$ and couldn't be smaller, which violates the property of $\rho_\blacklozenge$.

So the only possibility is $\forall i, O_i$ and O_{i+K} are completely-maximally-fidelitous, and $F(O_i, O_{i+K}) > 0$. And we have showed these value are constant with i.

□

3.8 Properties of completely-maximally-fidelitous density operators

We have known O_i, O_{i+K} are completely-maximally-fidelitous, which implies O_{i+1}, O_{i+1+K} are also completely-maximally-fidelitous. So to get deeper results, we could consider the following problem, which is a simplification of the problem that we are facing:

Given that ρ_1, ρ_2 are completely-maximally-fidelitous, $\mathcal{E}(\rho_1)$, $\mathcal{E}(\rho_2)$ are completely-maximally-fidelitous, $F(\rho_1, \rho_2) = F(\mathcal{E}(\rho_1), \mathcal{E}(\rho_2)) > 0$. What conclusions could we draw?

We could imagine we have known the value of $\mathcal{E}(\rho_1)$, $\mathcal{E}(\rho_2)$. We hope we could find other ρ besides ρ_1, ρ_2 that $\mathcal{E}(\rho))$ could be determined uniquely. Our technique is to use the monotonicity of the fidelity.

Consider the density operator ρ that make $(F(\rho_1, \rho))^2 + (F(\rho_2, \rho))^2$ get its maximum. We could first imagine what will happen if ρ_1, ρ_2 are both pure states, and imagine the case where ρ_1 and ρ_2 are completely-maximally-fidelitous and $F(\rho_1, \rho_2) > 0$. We will introduce a notation MID to represent this density operator and use $\bullet \boxed{DenM} (\bullet \boxplus \bullet)$ form to express it. Since $(F(\mathcal{E}(\rho_1), \mathcal{E}(\rho)))^2 + (F(\mathcal{E}(\rho_2), \mathcal{E}(\rho)))^2 \geq (F(\rho_1, \rho))^2 + (F(\rho_2, \rho))^2$ $\mathcal{E}(\rho)$ could be uniquely determined. This is what we do in corollary 3.19.1.

But corollary 3.19.1 is only a special case of a general proposition, proposition 3.19. This theorem tells us for some density operators, what is the result $\mathcal{E}(\rho)$. This proposition is general enough but not very convenient for usage. We will introduce two special cases of this proposition, corollary 3.19.1 and 3.19.2. The Corollary 3.19.1 is both a lemma for the proof of Proposition 3.19 and a special case of it. Since we prove this corollary by other method there is no circular reasoning here.

Then we will prove some lemmas that will be used in section 3.9. Although we have found some density operators and the value of $\mathcal{E}(\rho)$, we need to consider how to get bigger value of $F(\rho, \mathcal{E}(\rho))$ so that we could use these results in the mainline proof to get bigger orbit fidelity. For most of the cases we could use the MID operator. Lemma 3.21 is the basis of it. But the equality in Lemma 3.21 could hold in a special case, which we call "fidelity C cyclic". For this case we will use the $AXLE$ operator and use Lemma 3.22 to prove our new density operator has bigger orbit fidelity.

Definition 3.11. Define two operators MID, $AXLE$ on two pure states $|\varphi_1\rangle$, $|\varphi_2\rangle$ as follows:

1. If $|\varphi_1\rangle = e^{i\theta} |\varphi_2\rangle$, $MID(|\varphi_1\rangle, |\varphi_2\rangle) = \text{span}(|\varphi_1\rangle)$, $AXLE(|\varphi_1\rangle, |\varphi_2\rangle)$ is meaningless.
2. If $|\varphi_1\rangle \perp |\varphi_2\rangle$, both $MID(|\varphi_1\rangle, |\varphi_2\rangle)$ and $AXLE(|\varphi_1\rangle, |\varphi_2\rangle)$ is meaningless.
3. If $|\langle\varphi_1|\varphi_2\rangle| \in (0, 1)$: it's easy to see that $|\varphi_2\rangle$ could be uniquely expressed as $|\varphi_2\rangle = \cos\theta_1 e^{i\theta_2} |\varphi_1\rangle + \sin\theta_1 |\varphi_1^\perp\rangle$, where $\theta_1 \in (0, \frac{\pi}{2})$, $\theta_2 \in [0, 2\pi)$, $|\varphi_1^\perp\rangle$ is a state in $\text{span}(|\varphi_1\rangle, |\varphi_2\rangle)$ that is orthogonal to $|\varphi_1\rangle$. Let's define MID and $AXLE$ as:

$$MID(|\varphi_1\rangle, |\varphi_2\rangle) = \text{span}(\frac{1}{\sqrt{2}} |\varphi_1\rangle + \frac{1}{\sqrt{2}} |\varphi_1^\perp\rangle)$$
$$AXLE(|\varphi_1\rangle, |\varphi_2\rangle) = \text{span}(\cos\frac{\theta_1}{2} e^{i\theta_2} |\varphi_1\rangle + \sin\frac{\theta_1}{2} |\varphi_1^\perp\rangle)$$

Fact 3.16. *Both $MID(|\varphi_1\rangle, |\varphi_2\rangle)$ and $AXLE(|\varphi_1\rangle, |\varphi_2\rangle)$ are one-dimensional subspace in $\text{span}(|\varphi_1\rangle) \oplus \text{span}(|\varphi_2\rangle)$. $MID(|\varphi_1\rangle, |\varphi_2\rangle)$ is the set of states that maximize $|\langle\varphi_1|\varphi_3\rangle|^2 + |\langle\varphi_2|\varphi_3\rangle|^2$.*

From Figure 1 it's easy to see that in Bloch sphere MID is the "middle state" of $|\varphi_1\rangle$ and $|\varphi_2\rangle$, and $AXLE$ is the "axis" of "rotation" from $|\varphi_1\rangle$ to $|\varphi_2\rangle$. Furthermore, the global phase of $|\varphi_1\rangle$ and $|\varphi_2\rangle$ have no effect on the results of MID and $AXLE$.

Example 3.7.

$$MID(|0\rangle, \frac{|0\rangle + |1\rangle}{\sqrt{2}}) = \text{span}(\cos\frac{\pi}{8} |0\rangle + \sin\frac{\pi}{8} |1\rangle)$$
$$AXLE(|0\rangle, \frac{|0\rangle + |1\rangle}{\sqrt{2}}) = \text{span}(\frac{|0\rangle + i|1\rangle}{\sqrt{2}})$$

Recall Fact 3.12: if ρ_1, ρ_2 are completely-maximally-fidelitous, they have spectral decompositions that satisfy $|\langle\varphi_i|\phi_i\rangle| = C > 0$ and when $i \neq j$, $|\langle\varphi_i|\phi_i\rangle| = 0$. So it's natural to think about generalizing these two operators to density operators that are completely-maximally-fidelitous: we could take MID on the components of their spectral decomposition and get the "middle state" $MID(\rho_1, \rho_2)$ and the "axis of rotation" $AXLE(\rho_1, \rho_2)$.

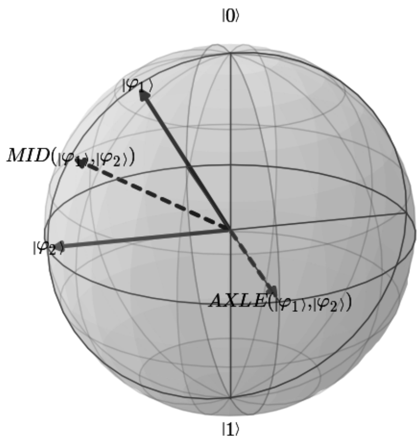

Figure 1: MID and $AXLE$ operators

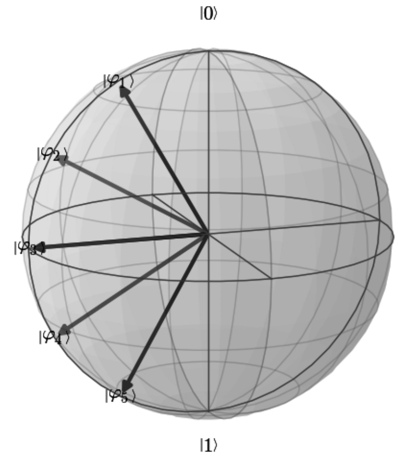

Figure 2: fidelity C cycling

Definition 3.12. Suppose density operators ρ_1, ρ_2 are completely-maximally-fidelitous, $F(\rho_1, \rho_2) > 0$, suppose one of their spectral decompositions $\rho_1 = \sum_i c_i \varphi_i$, $\rho_2 = \sum_i c_i \phi_i$ satisfy $\langle \varphi_i | \phi_i \rangle = C$, $| \langle \varphi_i | \phi_j \rangle | = 0 (i \neq j)$. Then

1. define $MID(\rho_1, \rho_2) = \sum_i c_i \psi_i$, where for all i, ψ_i is the density operator of a pure state in $MID(|\varphi_i\rangle, |\phi_i\rangle)$.
2. when $F(\rho_1, \rho_2) \neq 1$, define $AXLE(\rho_1, \rho_2) = \sum_i c_i \psi_i$, where for all i, ψ_i is the density operator of a pure state in $AXLE(|\varphi_i\rangle, |\phi_i\rangle)$.

The well-definedness of these two operators is not very obvious: it's possible that ρ_1, ρ_2 have multiple possible decompositions that satisfy the required conditions. But we could prove that both $MID(\rho_1, \rho_2)$ and $AXLE(\rho_1, \rho_2)$ are uniquely determined using the basic properties that we have proved in section 3.5.

Fact 3.17. *$MID(\rho_1, \rho_2)$ and $AXLE(\rho_1, \rho_2)$ are uniquely determined when ρ_1, ρ_2 are completely-maximally-fidelitous and $F(\rho_1, \rho_2) > 0$(for AXLE, $\in (0, 1)$). In other words, definition 3.12 is well-defined.*

Since when we define $MID(\rho_1, \rho_2)$, $AXLE(\rho_1, \rho_2)$ we require ρ_1 and ρ_2 are completely-maximally-fidelitous, and we have known from section 3.5 that density operators that are completely-maximally-fidelitous could be expressed by $\bullet \boxed{DenM} (\bullet \boxplus \bullet)$ form, it's reasonable to expect that $MID(\rho_1, \rho_2)$, $AXLE(\rho_1, \rho_2)$ could also be expressed by this form. This is the following fact.

Fact 3.18. *Suppose S_1, S_2 are two orthogonal subspaces of H, $\dim(S_1) = \dim(S_2) = n$, $\{|\varphi_i\rangle\}_n$, $\{|\chi_i\rangle\}_n$ are orthogonal bases of S_1 and S_2 respectively, density operators ρ_1, ρ_2 could be expressed as*

$$\rho_1 = \{c_i\}_n \boxed{DenM} \alpha_1 \{|\varphi_i\rangle\}_n \boxplus \beta_1 \{|\chi_i\rangle\}_n)$$
$$\rho_2 = \{c_i\}_n \boxed{DenM} \alpha_2 \{|\varphi_i\rangle\}_n \boxplus \beta_2 \{|\chi_i\rangle\}_n)$$

(Where the only differences are α, β), and the norms of both (α_1, β_1) and (α_2, β_2) are 1 and they are not orthogonal. Then we have

1. $$MID(\rho_1, \rho_2) = \{c_i\}_n \boxed{DenM} (\alpha_{mid} \{|\varphi_i\rangle\}_n \boxplus \beta_{mid} \{|\chi_i\rangle\}_n)$$
   Where $\alpha_{mid}, \beta_{mid}$ satisfy $\alpha_{mid} |0\rangle + \beta_{mid} |1\rangle \in MID(\alpha_1 |0\rangle + \beta_1 1, \alpha_2 |0\rangle + \beta_2 |1\rangle)$
2. *If $\rho_1 \neq \rho_2$, then*
   $$AXLE(\rho_1, \rho_2) = \{c_i\}_n \boxed{DenM} (\alpha_{axle} \{|\varphi_i\rangle\}_n \boxplus \beta_{axle} \{|\chi_i\rangle\}_n)$$
   Where $\alpha_{axle}, \beta_{axle}$ satisfy $\alpha_{axle} |0\rangle + \beta_{axle} |1\rangle \in AXLE(\alpha_1 |0\rangle + \beta_1 1, \alpha_2 |0\rangle + \beta_2 |1\rangle)$

Example 3.8. Suppose

$$\{|\varphi_i\rangle\}_2 = \{|0\rangle, |2\rangle\}_2$$
$$\{|\chi_i\rangle\}_2 = \{|1\rangle, |3\rangle\}_2$$
$$\{|\phi\rangle\}_2 = \{\frac{|0\rangle + |1\rangle}{\sqrt{2}}, \frac{|2\rangle + |3\rangle}{\sqrt{2}}\}_2$$
$$\rho_1 = \{\frac{1}{2}, \frac{1}{2}\}_2 \boxed{DenM} \{|\varphi_i\rangle\}_2$$
$$\rho_2 = \{\frac{1}{2}, \frac{1}{2}\}_2 \boxed{DenM} \{|\phi_i\rangle\}_2$$

then we have:

$$MID(\rho_1, \rho_2) = \{\frac{1}{2}, \frac{1}{2}\}_2 \boxed{DenM} \{\cos\frac{\pi}{8}|0\rangle + \sin\frac{\pi}{8}|1\rangle, \cos\frac{\pi}{8}|2\rangle + \sin\frac{\pi}{8}|3\rangle\}_2$$
$$AXLE(\rho_1, \rho_2) = \{\frac{1}{2}, \frac{1}{2}\}_2 \boxed{DenM} \{\frac{|0\rangle + i|1\rangle}{\sqrt{2}}, \frac{|2\rangle + i|3\rangle}{\sqrt{2}}\}_2$$

The following proposition is the main result in this subsection. Although it won't be used directly in our proof, it has many corollaries and is the basis of our following discussions.

Proposition 3.19. *Suppose quantum Markov chain $\mathcal{G} = (H, \mathcal{E})$. Suppose $\mathfrak{s}_1 \cup \mathfrak{s}_2$ are orthogonal bases of S_1, $\rho_1, \rho_2 \in S_1$, $\mathfrak{s}'_1 \cup \mathfrak{s}'_2$ are orthogonal bases of S_2, $\mathfrak{s}_1 \perp \mathfrak{s}_2$, $\mathfrak{s}'_1 \perp \mathfrak{s}'_2$, $\mathcal{E}(\rho_1), \mathcal{E}(\rho_2) \in S_2$, the norm of (α_1, β_1) and (α_2, β_2) are both 1 and they are not orthogonal, and ρ_1, ρ_2 satisfy*

$$\rho_1 = \{c_i\}_n \boxed{DenM} (\alpha_1\mathfrak{s}_1 \boxplus \beta_1\mathfrak{s}_2)$$
$$\rho_2 = \{c_i\}_n \boxed{DenM} (\alpha_2\mathfrak{s}_1 \boxplus \beta_2\mathfrak{s}_2)$$
$$\mathcal{E}(\rho_1) = \{c'_i\}_m \boxed{DenM} (\alpha_1\mathfrak{s}'_1 \boxplus \beta_1\mathfrak{s}'_2)$$
$$\mathcal{E}(\rho_2) = \{c'_i\}_m \boxed{DenM} (\alpha_2\mathfrak{s}'_1 \boxplus \beta_2\mathfrak{s}'_2)$$

Then $\forall \alpha, \beta$, $|\alpha|^2 + |\beta|^2 = 1$, we have

$$\mathcal{E}(\{c_i\}_n \boxed{DenM} (\alpha\mathfrak{s}_1 \boxplus \beta\mathfrak{s}_2)) = \{c'_i\}_m \boxed{DenM} (\alpha\mathfrak{s}'_1 \boxplus \beta\mathfrak{s}'_2) \tag{3.24}$$

We will first give two corollaries of this proposition and then prove the proposition itself. These corollaries are just special cases of this proposition. But to prove this proposition, we will first prove one of its special cases in other way(not relying on this proposition), and use this special case to prove this proposition itself. Let's first give some explanations to this proposition.

Example 3.9. 1. If

$$\mathcal{E}(|0\rangle\langle 0|) = |2\rangle\langle 2|$$
$$\mathcal{E}(\frac{1}{2}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|)) = \frac{1}{2}(|2\rangle + |3\rangle)(\langle 2| + \langle 3|)$$

then $\forall \alpha, \beta$, $|\alpha|^2 + |\beta|^2 = 1$, we have

$$\mathcal{E}((\alpha|0\rangle + \beta|1\rangle)(\alpha^\dagger\langle 0| + \beta^\dagger\langle 1|)) = (\alpha|2\rangle + \beta|3\rangle)(\alpha^\dagger\langle 2| + \beta^\dagger\langle 3|)$$

2. If(we align these equations in this way to make them more readable)

$$\rho_1 = \frac{1}{2}|0+2\rangle\langle 0+2| \qquad +\frac{1}{2}|1\rangle\langle 1|$$
$$\rho_2 = \frac{1}{2}|0\rangle\langle 0| \qquad +\frac{1}{2}|1+3\rangle\langle 1+3|$$
$$\mathcal{E}(\rho_1) = \rho_2 = \frac{1}{2}|0\rangle\langle 0| \qquad +\frac{1}{2}|1+3\rangle\langle 1+3|$$
$$\mathcal{E}(\rho_2) = \frac{1}{2}\frac{|0\rangle + i|2\rangle}{\sqrt{2}}\frac{\langle 0| - i\langle 2|}{\sqrt{2}} \qquad +\frac{1}{2}|3\rangle\langle 3|$$

then we have $\forall\alpha,\beta,\ |\alpha|^2+|\beta|^2=1$,

$$\begin{aligned}&\mathcal{E}(\frac{1}{2}(\alpha\,|0+2\rangle+\beta\,|0-2\rangle)(\alpha^\dagger\,\langle 0+2|+\beta^\dagger\,\langle 0-2|)+\frac{1}{2}(\alpha\,|1\rangle+\beta\,|3\rangle)(\alpha^\dagger\,\langle 1|+\beta^\dagger\,\langle 3|))\\ =&\frac{1}{2}(\alpha\,|0\rangle+i\beta\,|2\rangle)(\alpha^\dagger\,\langle 0|-i\beta^\dagger\,\langle 2|)+\frac{1}{2}(\alpha\,|1+3\rangle+\beta\,|1-3\rangle)(\alpha^\dagger\,\langle 1+3|+\beta^\dagger\,\langle 1-3|)\end{aligned}$$

where $|i+j\rangle=\frac{|i\rangle+|j\rangle}{\sqrt{2}}, |i-j\rangle=\frac{|i\rangle-|j\rangle}{\sqrt{2}}$

From fact 3.18 we know both MID and $AXLE$ could be expressed as $\bullet\ \boxed{DenM}\,(\bullet\boxplus\bullet)$ form, so the followings are direct corollaries of this proposition:

Corollary 3.19.1. *If ρ_1,ρ_2 are completely-maximally-fidelitous, $\mathcal{E}(\rho_1),\mathcal{E}(\rho_2)$ are completely-maximally-fidelitous, $F(\rho_1,\rho_2)=F(\mathcal{E}(\rho_1,\rho_2))>0$, then $\mathcal{E}(MID(\rho_1,\rho_2))=MID(\mathcal{E}(\rho_1),\mathcal{E}(\rho_2))$.*

Corollary 3.19.2. *If ρ_1,ρ_2 are completely-maximally-fidelitous, $\mathcal{E}(\rho_1),\mathcal{E}(\rho_2)$ are completely-maximally-fidelitous, $F(\rho_1,\rho_2)=F(\mathcal{E}(\rho_1,\rho_2))\in(0,1)$, then $\mathcal{E}(AXLE(\rho_1,\rho_2))=AXLE(\mathcal{E}(\rho_1),\mathcal{E}(\rho_2))$.*

We will prove proposition 3.19 and its corollaries with the following order: first prove corollary 3.19.1 with other methods, then prove proposition 3.19 with corollary 3.19.1, then shows 3.19.1 and 3.19.2 are indeed special cases of proposition 3.19.

Proof of corollary 3.19.1. Suppose $\rho_3=MID(\rho_1,\rho_2)$. Suppose $S_1=\mathrm{supp}(\mathcal{E}(\rho_1)), S_2=\mathrm{supp}(\mathcal{E}(\rho_2))$. Suppose $F(\rho_1,\rho_2)=C$.

$$\begin{aligned}tr(P_{S_1}\mathcal{E}(\rho_3))&\geq F(\mathcal{E}(\rho_1),\mathcal{E}(\rho_3))^2\quad\text{(Lemma 2.1)}\\ &\geq F(\rho_1,\rho_3)^2\\ &=\frac{1+C}{2}\end{aligned}\tag{3.25a}$$

Similarly,

$$tr(P_{S_2}\mathcal{E}(\rho_3))\geq\frac{1+C}{2}\tag{3.25b}$$

Suppose we have the spectral decomposition $\mathcal{E}(\rho_3)=\sum_j c_j\,|\psi_j\rangle\,\langle\psi_j|$. So

$$\begin{aligned}1+C\leq tr((P_{S_1}+P_{S_2})\mathcal{E}(\chi))&=\sum_j c_j tr((P_{S_1}+P_{S_2})(\psi_j))\\ &\leq\sum_j c_j(1+C)\\ &=1+C\end{aligned}\tag{3.26}$$

So the equality must holds, so we have $\forall j,|\psi_j\rangle\in\mathrm{supp}(MID(\mathcal{E}(\rho_1),\mathcal{E}(\rho_2)))$, which means $\mathrm{supp}(\mathcal{E}(MID(\rho_1,\rho_2)))\subseteq\mathrm{supp}(MID(\mathcal{E}(\rho_1),\mathcal{E}(\rho_2)))$. Next we prove $\mathcal{E}(MID(\rho_1,\rho_2))=MID(\mathcal{E}(\rho_1),\mathcal{E}(\rho_2))$.

Since the similarity of $\mathrm{supp}(MID(\mathcal{E}(\rho_1),\mathcal{E}(\rho_2)))$ with both $\mathrm{supp}(\mathcal{E}(\rho_1))$ and $\mathrm{supp}(\mathcal{E}(\rho_2))$ is $\frac{1+C}{2}$, to ensure $F(\mathcal{E}(\rho_1),\mathcal{E}(\rho_3))^2\geq\frac{1+C}{2}$ there must be $MID(\mathcal{E}(\rho_1),\mathcal{E}(\rho_2))$ is completely-maximally-fidelitous with both $\mathcal{E}(\rho_1)$ $\mathcal{E}(\rho_2)$, fidelity is $\frac{1+C}{2}$. The only possibility is $\mathcal{E}(MID(\rho_1,\rho_2))=MID(\mathcal{E}(\rho_1),\mathcal{E}(\rho_2))$. ☐

The following lemma is a temporary lemma that will be used in the proof of Proposition 3.19.

Lemma 3.20. *Suppose three sequences of orthogonal states $\{|\varphi_i\rangle\}_n$, $\{|\phi_i\rangle\}_n$, $\{|\psi_i\rangle\}_n$ satisfies:*

$$\forall i,\langle\varphi_i|\phi_i\rangle=C>0,\forall i,j,i\neq j,\langle\varphi_i|\phi_i\rangle=0.\forall i,|\psi_i\rangle\in MID(|\varphi_i\rangle,|\phi_i\rangle)\tag{3.27}$$

If $|\eta_1\rangle\in\mathrm{span}(\{|\varphi_i\rangle\}_n),|\eta_2\rangle\in\mathrm{span}(\{|\phi_i\rangle\}_n)$ satisfies $|\eta_1\rangle+|\eta_2\rangle\in\mathrm{span}(\{|\psi_i\rangle\}_n)$, then there exists a sequence of coefficients $\{c_i\}_n$ that

$$|\eta_1\rangle=\sum_{i=1}^n c_i\,|\varphi_i\rangle\,,|\eta_2\rangle=\sum_{i=1}^n c_i\,|\phi_i\rangle\tag{3.28}$$

Proof. Suppose $|\eta_1\rangle = \sum_{i=1}^n c_i |\varphi_i\rangle, |\eta_2\rangle = \sum_{i=1}^n c_i' |\phi_i\rangle$. Then

$$|\eta_1\rangle + |\eta_2\rangle = \sum_{i=1}^n c_i |\varphi_i\rangle + \sum_{i=1}^n c_i' |\phi_i\rangle \in \mathrm{span}(\{|\psi_i\rangle\}_n)$$

Which means

$$\forall i, c_i |\varphi_i\rangle + c_i' |\phi_i\rangle \in \mathrm{span}(|\psi_i\rangle) = MID(|\varphi_i\rangle, |\phi_i\rangle)$$

Because $\langle\varphi_i|\phi_i\rangle = C > 0$ we know $c_i' = c_i$, which completes our proof. □

Proof of proposition 3.19. First suppose

$$\{|\phi_i\rangle\}_n = \alpha_1\{|\varphi_i\rangle\}_n \boxplus \beta_1\{|\chi_i\rangle\}_n$$
$$\{|\psi_i\rangle\}_n = \alpha_2\{|\varphi_i\rangle\}_n \boxplus \beta_2\{|\chi_i\rangle\}_n$$
$$\{|\phi_i'\rangle\}_m = \alpha_1\{|\varphi_i'\rangle\}_m \boxplus \beta_1\{|\chi_i'\rangle\}_m$$
$$\{|\psi_i'\rangle\}_m = \alpha_2\{|\varphi_i'\rangle\}_m \boxplus \beta_2\{|\chi_i'\rangle\}_m$$

It's easy to see $\forall i, \langle\psi_i|\phi_i\rangle = \langle\psi_i'|\phi_i'\rangle = \alpha_1^\dagger\alpha_2 + \beta_1^\dagger\beta_2$. This is a complex number. Without loss of generality, we could make a rotation on α_2, β_2 and move the phase into $\{|\psi_i\rangle\}_n$ and $\{|\psi_i'\rangle\}_m$. So we only need to consider the cases where $C = \alpha_1^\dagger\alpha_2 + \beta_1^\dagger\beta_2$ is non-negative real number. By condition of the proposition we know $C > 0$.

By Corollary 3.19.1 we know that for $(\alpha_{mid}, \beta_{mid})$ Equation (3.24) holds, and our aim is to prove this equation holds for all α, β.

Consider $|\varphi\rangle \in \mathrm{supp}(\rho_1), |\phi\rangle \in \mathrm{supp}(\rho_2)$ that satisfies $\langle\varphi|\phi\rangle = C > 0$. Then for any operator E in the operator-sum representation of $\mathcal{E}$, , $E|\varphi\rangle \in \mathrm{supp}(\mathcal{E}(\rho_1)), E|\phi\rangle \in \mathrm{supp}(\mathcal{E}(\rho_2))$. Since $\langle\varphi|\phi\rangle = C > 0$ it's not hard to see the coefficients of $|\varphi\rangle$ on bases $\{|\phi_i\rangle\}_n$ are the same as the coefficients of $|\phi\rangle$ on the bases $\{|\psi_i\rangle\}_n$. Then we know $|\varphi\rangle + |\phi\rangle \in \mathrm{supp}(MID(\rho_1, \rho_2))$ So we know $E(|\varphi\rangle + |\phi\rangle) \in \mathrm{span}(\{|\psi_i'\rangle\}_m \boxplus \{|\phi_i'\rangle\}_m)))$.

So by Lemma 3.20 we know $E|\varphi\rangle = \sum_{i=1}^m c_i |\varphi_i'\rangle, E|\phi\rangle = \sum_{i=1}^m c_i |\phi_i'\rangle$. On the other hand the coefficients of $|\varphi\rangle$ on bases $\{|\phi_i\rangle\}_n$ are also the same as the coefficients of $|\phi\rangle$ on the bases $\{|\psi_i\rangle\}_n$. If we consider the state in $\mathrm{span}(|\varphi\rangle, |\phi\rangle)$ we will know:

For all α, β, if a pure states $|\eta\rangle \in \alpha\{|\varphi_i\rangle\}_n \boxplus \beta\{|\phi_i\rangle\}_n$ and the coefficients of $|\eta\rangle$ under bases $\alpha\{|\varphi_i\rangle\}_n \boxplus \beta\{|\phi_i\rangle\}_n$ are fixed(for different α, β), then the coefficients of $E|\eta\rangle$ under bases $\alpha\{|\varphi_i'\rangle\}_m \boxplus \beta\{|\phi_i'\rangle\}_m$ are fixed.

Since this is true for all the Es in the operator-sum representation of $\mathcal{E}$, we know if the representation of ρ_1 and ρ_2 have the same coefficients under their corresponding bases, $\mathcal{E}(\rho_1)$ and $\mathcal{E}(\rho_2)$ has the same coefficients under their corresponding bases respectively. Which means the map

$$\mathcal{E}: S_1 \to S_2, S_1 = \mathrm{span}(\alpha\{|\varphi_i\rangle\}_n \boxplus \beta\{|\chi_i\rangle\}_n), S_2 = \mathrm{span}(\alpha\{|\varphi_i'\rangle\}_m \boxplus \beta\{|\chi_i'\rangle\}_m)$$

are the isomorphic to each other for different α and β, and the isomorphism maps every state in one S_1, S_2 to the state in S_1 or S_2 that has the same coefficients with the original state. Since (3.24) holds for α_1, β_1, we could generalize this equation to all of the α, β as long as we consider the density operators that have the same coefficients under their bases $\alpha\{|\varphi_i\rangle\}_n \boxplus \beta\{|\chi_i\rangle\}_n$ and $\alpha\{|\varphi_i'\rangle\}_m \boxplus \beta\{|\chi_i'\rangle\}_m$, which means Equation (3.24) holds for all α, β. □

Return to the problem that we concern about. Suppose we know ρ_1, ρ_2 are completely-maximally-fidelitous, $\mathcal{E}(\rho_1), \mathcal{E}(\rho_2)$ are completely-maximally-fidelitous, we could use Corollary 3.19.1 and 3.19.2 to get some corollaries. But what we want is to get some results that could be used in our proof of period decomposition theorem, and we need to consider how to find a density operator that has bigger $F(\rho, \mathcal{E}(\rho))$. We have found the cases where $\mathcal{E}(\rho)$ could be determined, and let's consider the value of $F(\rho, \mathcal{E}(\rho))$. To simplify the description of a special case let's introduce a new concept.

Definition 3.13. If sequence $\{|\varphi_i\rangle\}_n$ satisfies:

1. $\forall i, |\varphi_i\rangle \in \mathrm{span}(|\varphi_1\rangle, |\varphi_2\rangle)$
2. $\forall i = 1, 2 \cdots n-2, |\varphi_{i+1}\rangle \in MID(|\varphi_i\rangle, |\varphi_{i+2}\rangle)$

We say $\{|\varphi_i\rangle\}_n$ are fidelity C cyclic, where $C = |\langle\varphi_1|\varphi_2\rangle| \in (0, 1]$

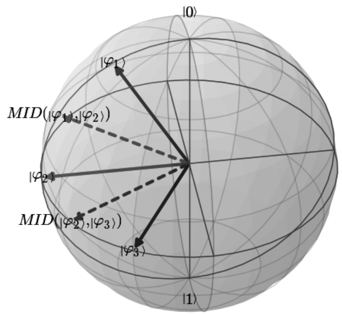

Figure 3: Figure for Lemma 3.21

We could see Figure 2 to get an intuitive understanding of fidelity $\cos\theta$ cyclic. If $\{|\varphi_i\rangle\}_n$ is fidelity C cyclic, then these states are cyclic on a big circle in Bloch sphere, and the angle of rotation is 2θ. Furthermore, it's obvious that $\forall i \in \{1, 2 \cdots n-2\}, AXLE(|\varphi_i\rangle, |\varphi_{i+1}\rangle) = AXLE(|\varphi_{i+1}\rangle, |\varphi_{i+2}\rangle)$.

Example 3.10. (1)$\{\cos\frac{n\pi}{6}|0\rangle + \sin\frac{n\pi}{6}|1\rangle\}_{n=1}^{10}$ is fidelity $\frac{\sqrt{3}}{2}$ cyclic. (2)If $\rho_1 = \rho_2 = \rho_3$, then $\{\rho_i\}_3$ is fidelity 1 cyclic.

Analogously, we could also define "fidelity C cyclic" on density operators:

Definition 3.14. We say $\{\rho_i\}_n$ is fidelity C cyclic if it satisfies: there exists sequence of coefficients $\{c_j\}_{j=1}^m$, such that each term in $\{\rho_i\}_n$ could be written as $\rho_i = \{c_j\}_{j=1}^m \boxed{DenM} \{|\varphi_{ij}\rangle\}_{j=1}^m$, and $\forall j, \{|\varphi_{ij}\rangle\}_{i=1}^n$ is fidelity C cyclic.

Example 3.11. Suppose $\rho_1 = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$, $\rho_2 = \frac{1}{2}(|0+2\rangle\langle 0+2| + |1+3\rangle\langle 1+3|)$, $\rho_3 = \frac{1}{2}(|2\rangle\langle 2| + |3\rangle\langle 3|)$, then $\{\rho_i\}_3$ is fidelity $\frac{1}{\sqrt{2}}$ cyclic.

Lemma 3.21. *If ρ_1, ρ_2 are completely-maximally-fidelitous, ρ_2, ρ_3 are completely-maximally-fidelitous, $F(\rho_1, \rho_2) = F(\rho_2, \rho_3) = C > 0$, then*

$$F(MID(\rho_1, \rho_2), MID(\rho_2, \rho_3)) \geq C$$

And the equality holds if and only if sequence ρ_1, ρ_2, ρ_3 is fidelity C cyclic.

You could see Figure 3.

Proof. First consider the case where ρ_1, ρ_2, ρ_3 are all pure states. Let $|\varphi_4\rangle \in MID(|\varphi_1\rangle, |\varphi_2\rangle)$, $|\varphi_5\rangle \in MID(|\varphi_3\rangle, |\varphi_2\rangle)$. Then $|\langle\varphi_4|\varphi_2\rangle| = |\langle\varphi_5|\varphi_2\rangle| = \sqrt{\frac{1+C}{2}}$. From Lemma 3.16 we know $1+C = |\langle\varphi_4|\varphi_2\rangle|^2 + |\langle\varphi_5|\varphi_2\rangle|^2 \leq 1 + |\langle\varphi_4|\varphi_5\rangle|$. So the equality holds only when $|\varphi_2\rangle \in MID(|\varphi_4\rangle, |\varphi_5\rangle)$ and the lemma is true.

For density operators that are completely-maximally-fidelitous,

$$\begin{aligned}
&F(MID(\rho_1, \rho_2), MID(\rho_2, \rho_3)) \\
=&F(\sum_i c_i MID(\varphi_i, \phi_i), \sum_i c_i MID(\phi_i, \chi_i)) \\
\geq&\sum_i c_i F(MID(\varphi_i, \phi_i), MID(\phi_i, \chi_i)) \\
\geq&\sum_i c_i C = C
\end{aligned}$$

This completes the proof. ∎

The following lemma is very obvious from the Figure 2, and it's useful in our proof.

Lemma 3.22. *If $\{\rho_i\}_n$ are fidelity C cyclic, $C \in (0,1)$, then $\forall i, AXLE(\rho_i, \rho_{i+1}) = AXLE(\rho_{i+1}, \rho_{i+2})$.*

We will use Corollary 3.19.1, 3.19.2 and Lemma 3.21, 3.22 in section 3.9. We will use Lemma 3.21 and 3.22 to try to construct a density operator that has bigger orbit fidelity and use corollaries 3.19.1, 3.19.2 to prove that this density operator has equal orbit dimension with $\rho_\blacklozenge$. We will prove by cases: for the case that "fidelty C cyclic" doesn't hold, we use corollary 3.19.1 and 3.21 to get a contradiction; for the case that "fidelty C cyclic, $C \in (0,1)$" holds, we will use Corollary 3.19.2 and Lemma 3.22 to get a contradiction.

3.9 Mainline of proof, part 3: Corollary by properties of completely-maximally-fidelitous density operators

Let's introduce our idea. In section 3.7 we have proved formulas (3.2), and our problem is how to construct a density operator with (possible) bigger orbit fidelity. Since when $k < K$ the orbit fidelity is 0 and couldn't be smaller, we could only consider LF_K. We could use corollary 3.19.1 and 3.19.2 to construct new density operators that $\mathcal{E}^i(\rho)$ could be determined uniquely and it's easy to see these density operators have the same orbit dimension as $\rho_\blacklozenge$. The problem is how to choose the density operators that has bigger orbit fidelity. We notice that when we construct the density operator with the value of ρ, $\mathcal{E}^K(\rho)$ and the value of $\mathcal{E}^K(\rho), \mathcal{E}^{2K}(\rho)$, these density operator satisfies the conditions in Lemma 3.21. So we could use Lemma 3.21 to get a contradiction. The result is that we know ρ, $\mathcal{E}^K(\rho)$ are fidelity $C > 0$ cyclic, and then we could use Lemma 3.22 to get another contradiction. This contradiction will imply $C = 1$, which completes our proof.

Lemma 3.23. *$\rho_\blacklozenge$ satisfies formulas (3.1):*

$$F(\rho, \mathcal{E}^k(\rho)) = \begin{cases} 0 & \text{when } 1 \le k < K \\ 1 & \text{when } k = K \end{cases}, \text{for some } K \ge 2 \tag{(3.1) revisited}$$

Proof. Suppose the orbit of $\rho_\blacklozenge$ is $\{O_i\}_{i=0}^{+\infty}$.

We have proved in corollary 3.6.1 that $\rho_\blacklozenge$ is a weak fixed point state and satisfies Equation (3.2):

$$\begin{aligned} &\forall i \in N, O_i \perp O_{i+k} \quad \text{when } 1 \le k < K \\ &\forall i \in N, F(O_i, O_{i+K}) > 0 \quad \text{and is constant} \\ &K \ge 2 \end{aligned} \tag{(3.2) revisited}$$

First consider $MID(O_i, O_{i+K})$. Let's use it to prove that $\forall i_0 \in \{0, 1 \cdots K-1\}$, the orbit of O_{i_0} under $\mathcal{E}^K$, in other words, $\{O_{Ki+i_0}\}_{i=0}^{+\infty}$, must be fidelity C cyclic, where $C = F(O_i, O_{i+K})$

Otherwise, suppose for some i_0, the orbit of O_{i_0} under $\mathcal{E}^K$, in other words, $\{O_{Ki+i_0}\}_{i=0}^{+\infty}$ isn't fidelity C cyclic. By Lemma 3.21 we know there is some i such that $\rho' = MID(O_{Ki+i_0}, O_{Ki+K+i_0})$, $\rho'' = MID(O_{Ki+K+i_0}, O_{Ki+2K+i_0})$ satisfy $F(\rho', \rho'') > C$, which means $LF_K(\rho') > LF_K(\rho_\blacklozenge)$.

Then let's prove $C = 1$. $XD(\rho') = XD(\rho_\blacklozenge)$(corollary 3.19.1), and when $k < K$ $LF_k(\rho') \ge 0 = LF_k(\rho_\blacklozenge)$, which are contradictory to the property that $\rho_\blacklozenge$ is maximum under the order relation in listing 1. So $\forall i_0 = 0, 1 \cdots K-1$, the orbit of O_{i_0} under $\mathcal{E}^K$ must be fidelity C cyclic.

Furthermore, if $C < 1$, let $\rho' = AXLE(O_1, O_{K+1})$. By corollary 3.19.2 we know:

$$\begin{aligned} \mathcal{E}(\rho') &= AXLE(O_2, O_{K+2}) \\ \mathcal{E}^2(\rho') &= AXLE(O_3, O_{K+3}) \\ &\cdots \\ \mathcal{E}^K(\rho') &= AXLE(O_{K+1}, O_{2K+1}) = AXLE(O_1, O_{K+1}) = \rho' (\text{lLemma 3.22}) \end{aligned}$$

Which is contradictory to the property that $\rho_\blacklozenge$ is maximum under order relation. So we have $C = 1$.

Finally, because S is not aperiodic, we have $K > 1$(Otherwise there will be $XD(\rho_\blacklozenge) = \dim(S)$). This completes our proof. □

Corollary 3.23.1. *A BSCC S must be one of the following two types:*

1. *Aperiodic BSCC*

2. *There exist a period $K \ge 2$ and a sequence of density operators $\rho_1, \rho_2, \cdots \rho_K$ that satisfy:*

(a) $\mathrm{supp}(\rho_1) \oplus \mathrm{supp}(\rho_2) \oplus \cdots \mathrm{supp}(\rho_K) = S$ *and subspace sequence* $\{\mathrm{supp}(\rho_i)\}_{i=1}^{K}$ *are orthogonal to each other.*

(b) $\mathcal{E}(\rho_i) = \rho_{i+1 \mod K}$

Proof. This corollary just expresses Lemma 3.23 in another way. The proof of Lemma 3.23 is under the condition that S is not aperiodic. So we know when S is not aperiodic, by Lemma 3.23 the density operator $\rho_\blacklozenge$ that we construct in definition 3.6 satisfies Equation (3.1). And if we choose $\rho_\blacklozenge, \mathcal{E}(\rho_\blacklozenge), \cdots \mathcal{E}^{K-1}(\rho_\blacklozenge)$ as $\rho_1, \rho_2 \cdots \rho_p$ we will see $\rho_i \perp \rho_j$ when $i \neq j$ because $F(\mathcal{E}^i(\rho_\blacklozenge), \mathcal{E}^j(\rho_\blacklozenge)) = 0$, $\mathcal{E}(\rho_i) = \rho_{i+1}$ and $\mathcal{E}(\rho_p) = \rho_1$ because $F(\rho_1, \mathcal{E}(\rho_p)) = F(\rho_\blacklozenge, \mathcal{E}^K(\rho_\blacklozenge)) = 1$. So all the conditions in (2) are satisfied. □

3.10 Mainline of proof, part 4: Finding "maximal decomposition" and uniqueness

Proposition 3.24. *If S is an aperiodic BSCC of $\mathcal{G}$, we have*

1. $\exists N, \forall |\varphi\rangle \in S, \forall n > N, \mathrm{supp}(\mathcal{E}^n(\varphi)) = S$
2. $\forall i$, *S is an aperiodic BSCC of* $\mathcal{G}^i$

The first one exchanges the symbols $\forall$ and $\exists$ and seems to be stronger than the original definition of aperiodic BSCCs. This will be used in the following section. The second one is useful for our proof of main theorem.

Proof of theorem 3.1. Compare theorem 3.1 with corollary 3.23.1, we don't have the property of being aperiodic under $\mathcal{G}^p$, and haven't proved the uniqueness. Our remaining work is to find the "maximal decomposition" that satisfies the aperiodic property and prove its uniqueness.

To get stronger results we will use Lemma 3.23 instead of corollary 3.23.1 as our basis.

Suppose BSCC S is not aperiodic. Based on Lemma 3.23, $\{\mathcal{E}^i(\rho_\blacklozenge)\}_{i=0}^{K-1}$ satisfies condition 1,2 in theorem 3.1. Let's prove that they also satisfy condition 3. Without loss of generality, suppose ρ_1 is not aperiodic under $\mathcal{E}^K$. Consider $\mathcal{E}^K$, and use Lemma 3.23 to get a decomposition of S_1. Suppose they are $\rho'_{11}, \rho'_2 \cdots \rho'_m$. From $F(\mathcal{E}^K(\rho'_i), \mathcal{E}^K(\rho'_j)) = 0$ we know $\forall k, F(\mathcal{E}^k(\rho_{1i}), \mathcal{E}^k(\rho_{1j})) = 0$. Then we know $\forall k, 1 \leq k \leq K, \{\mathcal{E}^k(\rho_{1i})\}_{i=1}^{m}$ is a decomposition of S_k under $\mathcal{E}^K$, which means we could decompose S to mK subspaces and ρ'_1 has smaller orbit dimension function value, which is a contradiction. So $\{\mathcal{E}^i(\rho_\blacklozenge)\}_{i=0}^{K-1}$ satisfies condition 3 and is a period decomposition that we want to find.

To prove the uniqueness: suppose there are two period decompositions of S, first assume they have different period p. Denote them as p_1, p_2. Consider the BSCC decomposition of $\mathcal{G}^{p_1 p_2}$. Based on proposition 3.24-2 we get two BSCC decompositions with different numbers of BSCCs, which is a contradiction. So period $p_1 = p_2$. Then the uniqueness of dimension is easy to get by considering the BSCC decomposition of $\mathcal{G}^{p_1}$. □

The following corollary is a by-product of our proof. It is useful in section 4.3.

Corollary 3.24.1. *Suppose $\rho_1, \rho_2 \cdots \rho_p$ is a period decomposition of S, then $\forall \rho \in S, \forall i, \rho \precsim \rho_i$.*

Proof. We know from the proof of theorem 3.1 that $\rho_\blacklozenge$, $\mathcal{E}(\rho_\blacklozenge)$, $\cdots$, $\mathcal{E}^K(\rho_\blacklozenge)$ are a period decomposition of S.Then by uniqueness of dimension we know $\rho_i \sim \rho_\blacklozenge$. So $\forall \rho \in S, \rho \precsim \rho_\blacklozenge \sim \rho_i$. □

Since p is uniquely determined we could define the period for a BSCC:

Definition 3.15. We call the number of subspaces p in the period decomposition of a BSCC as the period of this BSCC.

4 Strong uniqueness of period decomposition theorem and limit behavior inside BSCC

4.1 Introduction

Although we have proved a similar theorem as in classical case the uniqueness in this theorem is not satisfying. In this section we will prove the strong uniqueness of period decomposition theorem with the language of weak fixed point states. We will find all of the weak fixed point states in a BSCC and will study the limit behavior inside a BSCC with the tool of weak fixed point states. We will see weak fixed point states are very powerful and important: we could get more good properties with the help of period decomposition theorem 3.1, and we could use it to solve the problem of limit behavior of quantum Markov chains when the initial state is in a BSCC. We could consider the orbit of some initial state and all of its convergent subsequence converges to a weak fixed point state. Since the strong uniqueness theorem has told us all of the "minimum weak fixed point state" in a BSCC the limit behavior is not hard to handle.

In more detail, in section 4.2 we study the limit behavior of aperiodic BSCCs. In section 4.3 we study the further properties of weak fixed point states. We introduce the concept of "self-converge synchronously", "minimum weak fixed point states". In section 4.4 we introduce the concept of "similar sequence space" and develop lots of symbol to describe it. In section 4.5 we study further properties of "completely-maximally-fidelitous" and in section 4.6 we combine our results and solve the limit behavior problem inside a BSCC.

4.2 Limit behavior of aperiodic BSCCs

As the basis, we will first study the limit behavior of aperiodic BSCCs.

Lemma 4.1. *If S is an aperiodic BSCC, then $|\varphi\rangle \in S$, $\lim_{i\to+\infty}\mathcal{E}^i(\varphi)=\rho$, where ρ is the fixed point state of S.*

Proof. By Lemma 3.24 we know

$$\exists N, \forall |\varphi\rangle \in S, \mathrm{supp}(\mathcal{E}^N(\varphi)) = S$$

Which implies

$$\exists N, \delta > 0, \forall \sigma \in S, \mathcal{E}^N(\sigma) = \delta\rho + (1-\delta)\sigma'$$

So $\forall |\varphi\rangle \in S$,

$$\begin{aligned}
\lim_{i\to+\infty}\mathcal{E}^i(\varphi) &= \lim_{i\to+\infty}\mathcal{E}^r(\mathcal{E}^{pN}(\varphi)) \quad (t=\lfloor\frac{i}{N}\rfloor, i=tN+r)\\
&= \lim_{i\to+\infty}\mathcal{E}^r((\delta+\delta(1-\delta)+\cdots+\delta(1-\delta)^{t-1}(\rho)+(1-\delta)^t\sigma) \quad (t=\lfloor\frac{i}{N}\rfloor, i=tN+r)\\
&= \lim_{i\to+\infty}((1-(1-\delta)^t)\rho+(1-\delta)^t\sigma') \quad (t=\lfloor\frac{i}{N}\rfloor, i=tN+r)\\
&= \rho
\end{aligned}$$

□

Corollary 4.1.1. *There is only one weak fixed point state in an aperiodic BSCC, which is its fixed point state.*

But although we have get the period decomposition theorem it's not enough to handle the general case of BSCCs. One reason is that $|\varphi\rangle$ in some BSCC may not be in any component in the period decomposition. To handle the general case we need more tools.

4.3 Further properties of weak fixed point states

Our aim is to find all of the weak fixed point states in a BSCC. In this subsection we will see more properties of weak fixed point states, which is the basis of our following discussion. With the help of the period decomposition theorem we could know a lot more about it.

First, Weak fixed point states are an important concept in our discussions, but what we know about it is still very limited, for example, we even don't know whether it is additive. The reason is that for two different weak fixed point states, we don't know whether there exists a sequence of indexes that both density operators converge to themselves. This leads to the following definition.

Definition 4.1. Suppose ρ, σ are two weak fixed point states, their orbits are $\{O_i\}_{i=0}^{+\infty}$, $\{Q_i\}_{i=1}^{+\infty}$ respectively. If there exists a sequence of integers $\{i_j\}_{j=1}^{+\infty}$ such that $\{O_{i_j}\}_{j=1}^{+\infty}$, $\{Q_{i_j}\}_{j=1}^{+\infty}$ converge to ρ, σ respectively, we say ρ, σ are two weak fixed point states that self-converge synchronously.

And with the help of period decomposition theorem it's not hard to get the property of self-converging synchronously:

Lemma 4.2. *The followings hold for weak fixed point states:*

1. *If ρ, σ are two weak fixed point states that self-converge synchronously, their orbits are $\{O_i\}_{i=0}^{+\infty}$, $\{Q_i\}_{i=1}^{+\infty}$ respectively, then*

(a) $F(O_i, Q_i)$ *are constant with* i.

(b) $MS(\text{supp}(O_i), \text{supp}(Q_i))$ *are constant with* i.

2. *If* ρ *is a weak fixed point state of* $\mathcal{G}$*, then for all* $i > 0$*, it is a weak fixed point state of* $\mathcal{G}^i$.

3. *If* ρ *is a weak fixed point state, then* $tr(P_T(\rho)) = 0$*,* T *is the transient subspace.*

4. *Suppose* $\rho_1, \rho_2 \cdots \rho_p$ *are the density operators in a period decomposition of a BSCC,* σ *is a weak fixed point state, then* $\forall i$*,* ρ_i *and* σ *self-converge synchronously.*

Proof. For the first lemma, Proofs are similar to proposition 3.2 and proposition 3.14. We omit the proofs here.

Without loss of generality, suppose we are considering ρ_1 and σ. Denote the orbit of σ as $\{O_i\}_{i=0}^{+\infty}$. Let's construct a sequence of indexes $\{i_j\}_{j=1}^{+\infty}$ that ρ_1 and σ self-converge synchronously.

$\forall \epsilon_j > 0$, we could find a term in the orbit of σ that satisfies

$$\|O_j - \sigma\|_{tr} < \frac{\epsilon_j}{p} \tag{4.1}$$

So

$$\|O_{pj} - \sigma\|_{tr} \leq \|O_{pj} - O_{(p-1)j}\|_{tr} + \|O_{(p-1)j} - O_{(p-2)j}\|_{tr} + \cdots + \|O_j - \sigma\|_{tr} < \epsilon_j \tag{4.2}$$

Since $p|pj$ we know $\mathcal{E}^{pj}(\rho_1) = \rho_1$ so when $\epsilon_j \to 0$, we could construct an increasing sequence of j with Equation (4.1) and ρ_1 and σ self-converge synchronously on indexes $\{pj_k\}_{k=1}^{+\infty}$. □

As we have said, our aim is to find all of the weak fixed point states in a BSCC. One natural idea is to find a set of "smaller" weak fixed point states and express the weak fixed point states as the sum of them. As the basis, we have the following lemma:

Lemma 4.3. *For a quantum Markov chain* $\mathcal{G} = (H, \mathcal{E})$*, suppose* S *is an aperiodic BSCC,* ρ_0 *is the fixed point state of* S*. If* σ *is a weak fixed point state,* ρ_0 *and* σ *are not orthogonal, then there exists a weak fixed point state* $\rho_2 \in \text{supp}(\sigma)$ *that is completely-maximally-fidelitous with* ρ_0*. Furthermore,* $\exists c_1 > 0$ *that* $\sigma = c_1\rho_2 + (1 - c_1)\rho_3$*, where* ρ_3 *is a weak fixed point state that satisfies* $MS(\text{supp}(\rho_3), \text{supp}(\rho_0)) < MS(\text{supp}(\sigma), \text{supp}(\rho_0))$*, and* ρ_2*,* ρ_3 *self-converge synchronously.*

Proof. Suppose the orbit of σ is $\{O_i\}_{i=0}^{+\infty}$, $S_i = \text{supp}(O_i)$.

First by Lemma 4.2 $MS(S_i, S)$ are constant.

Then suppose $O_i = c_iO_{i1} + (1 - c_i)O_{i2}$. $\text{supp}(O_{i1})$ is completely-maximally-similar with some subspace of S and $MS(\text{supp}(O_{i2}), S) < MS(S_i, S)$. It's not hard to see such decomposition always exists. Suppose O_{i1} are completely-maximally-fidelitous with $\sigma_i \in S$.

Because $MS(S_i, S)$ are constant and σ_i satisfies $F(\sigma_i, O_{i1}) = MS(S_i, S)$, we know $\mathcal{E}(O_{i1}) \in \text{supp}(O_{(i+1)1})$. So c_i is non-decreasing.

Suppose O_i converges to σ at indexes $\{i_j\}_{j=1}^{+\infty}$.

We also have $\sigma = O_0 = c_0O_{01} + (1 - c_0)O_{02}$. On the one hand c_i is non-decreasing and on the other hand there is a convergent subsequence that converge to σ itself, we know c_i is constant.

Then we know $\mathcal{E}(\sigma_i) = \sigma_{i+1}$. Because $\{\sigma_i\}$ is the orbit of $\sigma_0 \in S$, we know $\lim_{i\to+\infty} \sigma_i = \rho_0$.

Then suppose O_{i1} converge to O_r on $\{i_j\}_{j=1}^{+\infty}$(If they don't converge we could choose a convergent subsequence and replace $\{i_j\}_{j=1}^{+\infty}$ with it). Since σ_i converge to ρ_0 we know $F(O_r, \rho_0) = MS(S_i, S)$. It's easy to see O_r is completely-maximally-fidelitous with ρ_0 and since c_i is constant we know $O_r = O_0$, so O_{01} is a weak fixed point state that self-converges synchronously with σ. So O_{i2} is also a weak fixed point state and they self-converge synchronously. □

Now it's time to introduce the concept of minimum weak fixed point states.

Definition 4.2. If a weak fixed point state ρ satisfies: there is no weak fixed point state σ that satisfies $\text{supp}(\sigma) \subset \text{supp}(\rho)$, we say ρ is a minimum weak fixed point state.

Although in Lemma 4.3 we only consider aperiodic BSCCs, we could still use it to study general BSCC by choose a period decomposition and consider the power of quantum Markov chains. Together with Lemma 4.2 we could get the following corollaries on minimum weak fixed point states:

Lemma 4.4. *The followings hold for minimum weak fixed point states:*

1. *Suppose $\rho_1, \rho_2 \cdots \rho_p$ is a period decomposition of some BSCC, ρ is a minimum weak fixed point state, then for all i, ρ is completely-maximally-fidelitous with ρ_i.*

2. *For a quantum Markov chain $\mathcal{G} = (H, \mathcal{E})$, σ is a weak fixed point state in its recurrent subspace, then σ could be written as $\sigma = c_1\sigma_1 + c_2\sigma_2 + \cdots + c_m\sigma_m$, where every σ_i is a minimum weak fixed point state.*

So to find all the weak fixed point states in a BSCC we only need to consider the minimum weak fixed point states. We could use corollary 4.4 to study their properties. Actually, we will see there is no minimum weak fixed point states other than the density operators in the period decomposition. First we will use corollary 3.24.1 in section 3.10 to prove the following property:

Lemma 4.5. *Suppose S is a BSCC of $\mathcal{G} = (H, \mathcal{E})$, one of period decompositions of S is $\rho_1, \rho_2 \cdots \rho_p$. Suppose $\sigma \in S$ is a minimum weak fixed point, then $XD(\sigma) = XD(\rho_1)$, $LF_k(\sigma) = 0 (k < p)$, $LF_p(\sigma) = 1$.*

Proof. Without loss of generality, suppose ρ_1 and σ are not orthogonal. Since σ is a minimum weak fixed point state we know ρ_1 and σ are completely-maximally-fidelitous.

Then let's prove $\mathcal{E}^i(\rho_1)$ is completely-maximally-fidelitous with $\mathcal{E}^i(\sigma)$. This is because by lemma 4.2 we know $MS(\text{supp}(\mathcal{E}^i(\rho_1)), \text{supp}(\mathcal{E}^i(\sigma)))$ is a constant. So $F(\mathcal{E}^i(\rho_1), \mathcal{E}^i(\sigma)) \geq F(\rho_1, \sigma) = MS(\text{supp}(\rho_1), \text{supp}(\sigma)) = MS(\text{supp}(\mathcal{E}^i(\rho_1)), \text{supp}(\mathcal{E}^i(\sigma)))$ so $\mathcal{E}^i(\rho_1)$ is completely-maximally-fidelitous with $\mathcal{E}^i(\sigma)$.

This implies $\dim(\mathcal{E}^i(\rho_1)) = \dim(\mathcal{E}^i(\sigma))$. So $XD(\sigma) = XD(\rho_1)$.

By corollary 3.24.1 we know $\sigma \precsim \rho_1$. But $LF_k(\rho_1) = 0 (k < p)$, there must be $LF_k(\sigma) = 0 (k < p)$, which implies $\sigma, \mathcal{E}(\sigma), \cdots \mathcal{E}^{p-1}(\sigma)$ are orthogonal to each other, and $\mathcal{E}(\sigma), \mathcal{E}^2(\sigma), \cdots \mathcal{E}^p(\sigma)$.

So the only thing remaining to be proved is $LF_p(\sigma) = 1$, or equivalently, $\sigma = \mathcal{E}^p(\sigma)$. We have known both σ and $\mathcal{E}^p(\sigma)$ are orthogonal to $\mathcal{E}(\sigma), \mathcal{E}^2(\sigma), \cdots \mathcal{E}^{p-1}(\sigma)$. There must be $\text{supp}(\sigma) = \text{supp}(\mathcal{E}^p(\sigma))$. But the density operator in $\text{supp}(\sigma)$ that is completely-maximally-fidelitous with ρ_1 is unique, so $\sigma = \mathcal{E}^p(\sigma)$. □

Corollary 4.5.1. *Suppose S is a BSCC of $\mathcal{G} = (H, \mathcal{E})$, suppose its period is p, if $\sigma \in S$ is a minimum weak fixed point state, then $\sigma, \mathcal{E}(\sigma), \cdots \mathcal{E}^{p-1}(\sigma)$ is a period decomposition of S.*

So our next task is to prove it's impossible to have two "different" period decomposition. We quote "different" here because if $\sigma = \mathcal{E}^i(\rho)$ holds for some i, we see these two decomposition as the same decomposition.

4.4 Similar sequence space

To study the properties of completely-maximally-fidelitous density operators we need to create some symbols that are more concise.

Definition 4.3. We say a linear space $\mathfrak{H}$ is a similar sequence space if $\mathfrak{H}$ satisfies:

1. The field of coefficients is the complex field.
2. The elements in it are sequences with a fixed length, every element in it is orthogonal consistent sequence and every two elements in the space are completely-maximally-similar.
3. The addition and multiplication are defined as $\alpha\mathfrak{s}_1 \boxplus \beta\mathfrak{s}_2$

And we define the inner product of two elements in this space as their similarity. So a similar sequence space is a unitary space.

Example 4.1. The set of sequence $\{\alpha |0\rangle + \beta |1\rangle, \alpha |2\rangle + \beta |3\rangle\}_2$ is a similar sequence space, its dimension is 2.

Since similar sequence spaces are unitary space, we could use the common concepts in linear spaces like orthogonal, length and normalizing on it. It's obvious that the similar sequence space is isomorphism to C^m. We could also express the elements with orthogonal bases like $\mathfrak{s} = \alpha_1\mathfrak{s}_1 \boxplus \alpha_2\mathfrak{s}_2 \cdots \alpha_m\mathfrak{s}_m$.

Definition 4.4. Denote the similar sequence space that spanned by $\mathfrak{s}_1, \mathfrak{s}_2 \cdots \mathfrak{s}_n$ as $\mathfrak{span}(\mathfrak{s}_1, \mathfrak{s}_2, \cdots \mathfrak{s}_n)$. Suppose H is the normal linear space spanned by all the terms in sequences $\mathfrak{s}_1, \mathfrak{s}_2 \cdots \mathfrak{s}_n$. We call H as the parent space of $\mathfrak{span}(\mathfrak{s}_1, \mathfrak{s}_2, \cdots \mathfrak{s}_n)$.

It's easy to see the concept of parent space is well-defined because the different choices of bases don't affect the result of parent space.

Example 4.2. The parent space of $\mathfrak{H} = \mathfrak{span}(\{|0\rangle, |2\rangle\}_2, \{|1\rangle, |3\rangle\}_2)$ is $\text{span}(|0\rangle, |1\rangle, |2\rangle, |3\rangle)$.

We could also study unitary transformation, unitary operator and its eigenvalue on similar sequence space. We call the eigenstate of a unitary operator on similar sequence spaces as eigensequence. Following are some symbols on unitary transformation and unitary operator.

Definition 4.5.
1. Consider the unitary transformation between two similar sequence space $\mathfrak{H}_1$ and $\mathfrak{H}_2$, we write $\mathfrak{H}_1 \overset{\mathfrak{U}}{\mapsto} \mathfrak{H}_2$ if the unitary transform $\mathfrak{U}$ is a bijection between $\mathfrak{H}_1$ and $\mathfrak{H}_2$.
2. Consider the unitary operator $\mathfrak{U}$ on a similar sequence space $\mathfrak{H}$. $\dim(\mathfrak{H}) = n$. Suppose $\mathfrak{U}$ has orthogonal normalized eigensequences $\mathfrak{s}_1, \mathfrak{s}_2 \cdots \mathfrak{s}_n$, we write $\mathfrak{U} = \boxplus_{i=1}^{k} \mathrm{e}^{\mathrm{i}\theta_i} \mathfrak{s}_i \mathfrak{s}_i^\dagger$

Following is a convenient symbol, which will be used consequently:

Example 4.3. Use the $\mathfrak{H}$ in example 4.2. Suppose $\mathfrak{U} : \mathfrak{H} \mapsto \mathfrak{H}$ is defined as follows: $\mathfrak{U}\{|0\rangle, |2\rangle\}_2 = \{|0\rangle, |2\rangle\}_2$, $\mathfrak{U}\{|1\rangle, |3\rangle\}_2 = i\{|1\rangle, |3\rangle\}_2$, then the eigenvalues of $\mathfrak{U}$ are $1, i$, the eigensequences are $\{|0\rangle, |2\rangle\}_2, \{|1\rangle, |3\rangle\}_2$ respectively.

Definition 4.6. Suppose $\mathfrak{s} = \{|\varphi_i\rangle\}_n$, $\mathfrak{s}' = \{|\varphi_i'\rangle\}_n$, we say $|\phi\rangle \xleftrightarrow{\mathfrak{s} \leftrightarrow \mathfrak{s}'} |\phi'\rangle$ if there exists $\{a_i\}_n$ such that $|\phi\rangle = \sum_{i=1}^n a_i |\varphi_i\rangle, |\phi'\rangle = \sum_{i=1}^n a_i |\varphi_i'\rangle$.

Example 4.4. Suppose $\mathfrak{s}_1 = \{|0\rangle, |2\rangle\}_2, \mathfrak{s}_2 = \{|1\rangle, |3\rangle\}_2$, then $\frac{1}{\sqrt{2}}(|0\rangle + |2\rangle) \xleftrightarrow{\mathfrak{s}_1 \leftrightarrow \mathfrak{s}_2} \frac{1}{\sqrt{2}}(|1\rangle + |3\rangle)$.

One usage of "similar sequence space" is to describe a special kind of superoperator. To do this we need to introduce the following concept, "coexist":

Definition 4.7. $\mathfrak{U}$ is a unitary transformation from similar sequence space $\mathfrak{H}_1$ to $\mathfrak{H}_2$. Their parent spaces are H_1, H_2 respectively. If a linear map $E : H_1 \mapsto H_2$ satisfies:

1. $\forall \mathfrak{s} \in \mathfrak{H}_1,, |\varphi\rangle \in \text{span}(\mathfrak{s})$, $E|\varphi\rangle \in \text{span}(\mathfrak{U}\mathfrak{s})$
2. $\forall \mathfrak{s}_1, \mathfrak{s}_2 \in \mathfrak{H}_1$, $|\varphi_1\rangle \in \text{span}(\mathfrak{s}_1), |\varphi_2\rangle \in \text{span}(\mathfrak{s}_2)$, $|\varphi_1\rangle \xleftrightarrow{\mathfrak{s}_1 \leftrightarrow \mathfrak{s}_2} |\varphi_2\rangle$ implies $E|\varphi_1\rangle \xleftrightarrow{\mathfrak{U}\mathfrak{s}_1 \leftrightarrow \mathfrak{U}\mathfrak{s}_2} E|\varphi_2\rangle$

we say E coexists with U.

Example 4.5. Suppose $\mathcal{E} = \sum_{i=1}^4 E_i \cdot E_i^\dagger$ is defined as follows:

$$
\begin{aligned}
E_1 &= |0\rangle\langle 2| + i|1\rangle\langle 3| \\
E_2 &= |0\rangle\langle 0| + i|1\rangle\langle 1| \\
E_3 &= |2\rangle\langle 0| + i|3\rangle\langle 1| \\
E_4 &= |2\rangle\langle 2| + i|3\rangle\langle 3|
\end{aligned}
$$

U and $\mathfrak{H}$ are defined as example 4.3, then E_1, E_2, E_3, E_4 all coexist with U.

The following lemma describes the properties of coexisting with bases:

Lemma 4.6. *Suppose unitary transformation $\mathfrak{U} : \mathfrak{H}_1 \to \mathfrak{H}_2$, $\{\mathfrak{s}_i\}_n$ are orthogonal bases of $\mathfrak{H}_1$, $\{\mathfrak{U}\mathfrak{s}_i\}$ are orthogonal bases of $\mathfrak{H}_2$, then E coexists with $\mathfrak{U}$ if and only if $\forall i, j, i \neq j, \forall |\varphi\rangle \in \text{span}(\mathfrak{s}_i), \forall |\phi\rangle \in \text{span}(\mathfrak{s}_j)$, $|\varphi\rangle \xleftrightarrow{\mathfrak{s}_i \leftrightarrow \mathfrak{s}_j} |\phi\rangle$, we have $E|\varphi\rangle \xleftrightarrow{\mathfrak{U}\mathfrak{s}_i \leftrightarrow \mathfrak{U}\mathfrak{s}_j} E|\phi\rangle$.*

Corollary 4.6.1.
1. *If E_1, E_2 coexist with $\mathfrak{U}$, then $aE_1 + bE_2$ coexist with $\mathfrak{U}$.*
2. *If E_1 coexists with $\mathfrak{U}_1 : \mathfrak{H}_1 \mapsto \mathfrak{H}_2$, E_2 coexists with $\mathfrak{U}_2 : \mathfrak{H}_2 \mapsto \mathfrak{H}_3$, then E_2E_1 coexists with $\mathfrak{U}_2 \circ \mathfrak{U}_1$*

$$
\begin{array}{ccc}
|\varphi\rangle & \xleftrightarrow[\mathfrak{s}_1 \leftrightarrow \mathfrak{s}_2]{} & |\phi\rangle \\
\Big\downarrow E & \Big\downarrow \mathfrak{U} & E \Big\downarrow \\
E\,|\varphi\rangle & \xleftrightarrow{\mathfrak{U}\mathfrak{s}_1 \leftrightarrow \mathfrak{U}\mathfrak{s}_2} & E\,|\phi\rangle
\end{array}
\tag{4.3}
$$

Figure 4: A figure about coexisting and similar sequence space

4.5 Further properties of completely-maximally-fidelitous density operators

Our aim in this subsection is to develop some useful tools and prove that the period decomposition of a BSCC is unique.

We have seen in section 3.8 that when ρ_1, ρ_2 are completely-maximally-fidelitous, $\mathcal{E}(\rho_1), \mathcal{E}(\rho_2)$ are completely-maximally-fidelitous and their fidelities are equal we could get a lot of interesting results. But the symbols are very complex and not easy to generalize to handle the condition where we need to consider ρ_1, ρ_2, $\mathcal{E}(\rho_1), \mathcal{E}(\rho_2)$, $\mathcal{E}^2(\rho_1), \mathcal{E}^2(\rho_2) \cdots$. But this is what we are facing when we consider the uniqueness of period decomposition. So let's first introduce a lemma, which is similar but deeper than Lemma 3.19:

Lemma 4.7. *Consider a quantum Markov chain $\mathcal{G} = (H, \mathcal{E} = \sum_{i=1}^{k} E_i \cdot E_i^\dagger)$. Suppose*

$$\rho_1 = \{c_i\}_n \boxed{DenM}\, \mathfrak{s}_1 \tag{4.4}$$

$$\rho_2 = \{c_i\}_n \boxed{DenM}\, \mathfrak{s}_2 \tag{4.5}$$

$$\mathcal{E}(\rho_1) = \{d_i\}_n \boxed{DenM}\, \mathfrak{s}_1' \tag{4.6}$$

$$\mathcal{E}(\rho_2) = \{d_i\}_n \boxed{DenM}\, \mathfrak{s}_2' \tag{4.7}$$

$\mathfrak{s}_1, \mathfrak{s}_2$ are completely-maximally-similar, $\mathfrak{s}_1', \mathfrak{s}_2'$ are completely-maximally-similar, their similarities are equal, suppose $\mathfrak{span}(\mathfrak{s}_1, \mathfrak{s}_2) = \mathfrak{H}_1$, $\mathfrak{span}(\mathfrak{s}_1', \mathfrak{s}_2') = \mathfrak{H}_2$, $\mathfrak{U} : \mathfrak{H}_1 \mapsto \mathfrak{H}_2$ satisfies $\mathfrak{U}\mathfrak{s}_1 = \mathfrak{s}_1', \mathfrak{U}\mathfrak{s}_2 = \mathfrak{s}_2'$, then we have:

1. *$\forall \mathfrak{s} \in \mathfrak{H}_1, \mathcal{E}(\{c_i\}_n \boxed{DenM}\, \mathfrak{s}) = \{c_i\}_n \boxed{DenM}\, \mathfrak{U}\mathfrak{s}$.*
2. *$\forall i$, E_i coexists with $\mathfrak{U}$.*

The first one is the same as Lemma 3.19. The second one builds the relation between the superoperator $\mathcal{E}$ that satisfies (4.4) and the similar sequence space it generates.

Proof. Use Lemma 3.19, which means $\forall \alpha, \beta$, $\mathcal{E}(\{c_i\}_n \boxed{DenM} (\alpha\mathfrak{s}_1 \boxplus \beta\mathfrak{s}_2)) = \{d_i\}_n \boxed{DenM} (\alpha\mathfrak{s}_1' \boxplus \beta\mathfrak{s}_2'$ So 1 is true obviously.

For 2, choose an orthogonal bases $\mathfrak{s}_1, \mathfrak{s}_2 \in \mathcal{H}_1$. Suppose $|\varphi\rangle \xleftrightarrow{\mathfrak{s}_1 \leftrightarrow \mathfrak{s}_2} |\chi\rangle$. We have $E\,|\varphi\rangle \in \text{span}(\mathfrak{U}\mathfrak{s}_1), E\,|\chi\rangle \in \text{span}(\mathfrak{U}\mathfrak{s}_2)$, and $E(|\varphi\rangle + |\chi\rangle) \in \text{span}(\mathfrak{U}\mathfrak{s}_1 + \mathfrak{U}\mathfrak{s}_2)$. It's easy to see $E\,|\varphi\rangle \xleftrightarrow{\mathfrak{U}\mathfrak{s}_1 \leftrightarrow \mathfrak{U}\mathfrak{s}_2} E\,|\chi\rangle$ so by Lemma 4.6 E coexists with $\mathfrak{U}$. □

Lemma 4.8. *Consider a quantum Markov chain $\mathcal{G} = (H, \mathcal{E} = \sum_i E_i \cdot E_i^\dagger)$. If a unitary matrix $U \neq \mathrm{e}^{\mathrm{i}\theta} I : H \to H$ satisfies $[U, E_i] = 0$ for all i, H is not a BSCC.*

Proof. Since $U \neq \mathrm{e}^{\mathrm{i}\theta} I$ we know the eigenvalues of U are not all the same. Then consider the eigenspace of some eigenvalue $\mathrm{e}^{\mathrm{i}\theta_1}$. Denote this eigenspace as S_1. Then $\forall i, \forall\,|\varphi\rangle \in S_1, [U, E_i]\,|\varphi\rangle = UE_i\,|\varphi\rangle - \mathrm{e}^{\mathrm{i}\theta_1} E_i\,|\varphi\rangle = 0$, which means $E_i\,|\varphi\rangle$ is also a eigenstate with eigenvalue $\mathrm{e}^{\mathrm{i}\theta}$, so $\forall i, E_i\,|\varphi\rangle \in S_1$. So H is not a BSCC. □

And our aim is to construct a U on S under the condition that S has two different period decompositions.

Lemma 4.9. *If $\rho_1, \rho_2 \cdots \rho_p$, $\sigma_1, \sigma_2 \cdots \sigma_p$ are two different period decompositions of S, ρ_i, σ_i are positive completely-maximally-fidelitous, then we could decompose them as $\rho_i = \{c_j\}_i \boxed{DenM}\, \mathfrak{s}_i$, $\sigma_i = \{c_j\}_i \boxed{DenM}\, \mathfrak{s}_i'$ such that $\forall i$, $\mathfrak{s}_i$ and $\mathfrak{s}_i'$ are completely-maximally-similar and similarities are a fixed number $C > 0$ and $\bigcup_i \mathfrak{s}_i$, $\bigcup_i \mathfrak{s}_i'$ are two different bases of S.*

This lemma constructs two sets of bases on S. On the one hand, we could construct a unitary operator that maps one set of bases to another; on the other hand, the condition of being completely-maximally-similar gives us the tools to prove $[U, E_i] = 0$.

Lemma 4.10. *If there are two different decompositions, there exists a unitary matrix $U \neq e^{i\theta} I$ on S that satisfies $\forall i, [U, E_i] = 0$*

Proof. Choose bases $\{\mathfrak{s}_i\}_p$, $\{\mathfrak{s}'_i\}_p$ based on Lemma 4.9. Then consider the unitary transformation U that maps $\bigcup_i \mathfrak{s}_i$ to $\bigcup_i \mathfrak{s}'_i$. Then we need to prove that U satisfies $\forall i, \forall |\varphi_j\rangle \in \mathfrak{s}_i, [U, E_i] |\varphi_j\rangle = 0$.

Without loss of generality, consider $UE_i |\varphi_1\rangle - E_i U |\varphi_1\rangle$, where $|\varphi_1\rangle$ is the first element of $\mathfrak{s}_1$. Then $U |\varphi_1\rangle$ is the first element of $\mathfrak{s}'_1$. Then we have $|\varphi_1\rangle \xleftrightarrow{\mathfrak{s}_1 \leftrightarrow \mathfrak{s}'_1} U |\varphi_1\rangle$.

Consider $\mathfrak{U} : \mathfrak{span}(\mathfrak{s}_1, \mathfrak{s}'_1) \to \mathfrak{span}(\mathfrak{s}_2, \mathfrak{s}'_2)$ that satisfies $\mathfrak{U}\mathfrak{s}_1 = \mathfrak{s}_2, \mathfrak{U}\mathfrak{s}'_1 = \mathfrak{s}'_2$. By Lemma 4.7 we know E_i coexists with $\mathfrak{U}$.

Because E_i coexists with $\mathfrak{U}$ and $|\varphi_1\rangle \xleftrightarrow{\mathfrak{s}_1 \leftrightarrow \mathfrak{s}'_1} U |\varphi_1\rangle$ we know $E_i |\varphi_1\rangle \xleftrightarrow{\mathfrak{s}_2 \leftrightarrow \mathfrak{s}'_2} E_i U |\varphi_1\rangle$. By the construction of U we know $UE_i |\varphi_1\rangle = E_i U |\varphi_1\rangle$, which means $[U, E_i] |\varphi_1\rangle = 0$.

So $[U, E_i] = 0$ for all i. Obviously $U \neq e^{i\theta} I$. □

Then we could prove the uniqueness of period decomposition:

Theorem 4.11. *There are at most 1 possible period decomposition for a BSCC except for the circular shift of this period decomposition.*

Proof. By Lemma 4.8 and 4.10. If there are two different period decompositions for a BSCC, we could construct a unitary operator $U \neq e^{i\theta} I$ based on Lemma 4.10 that satisfies $\forall i, [U, E_i] = 0$. Then by Lemma 4.8 we know this is not a BSCC, which is a contradiction. □

4.6 Strong period decomposition theorem and limit behavior in BSCC

But the uniqueness of period decomposition is still not strong enough. What we want is to find all of the weak fixed point states in a BSCC. So we need to combine our results in section 4.3 and 4.5. Then we could solve the problem of limit behavior when the initial state is in a BSCC.

The following theorem gives us all of the weak fixed point states:

Theorem 4.12. *Suppose a BSCC has a period decomposition where the density operators in this decomposition are $\rho_1, \rho_2 \cdots \rho_p$, then $\rho_1, \rho_2, \cdots \rho_p$ are all of the minimum weak fixed point states in this BSCC. Furthermore, all of the weak fixed point states in this BSCC could be written as $c_1\rho_1 + c_2\rho_2 + \cdots + c_p\rho_p$, where $\sum_{i=1}^{p} c_i = 1$.*

Proof. Suppose $\rho_1, \rho_2 \cdots \rho_p$ are a period decomposition of a BSCC. Assume there is some other minimum weak fixed point state σ by corollary 4.5.1 we know $\sigma, \mathcal{E}(\sigma), \cdots \mathcal{E}^{p-1}(\sigma)$ is another period decomposition of this BSCC. But by 4.11 it's impossible. So $\rho_1, \rho_2 \cdots \rho_p$ are all of the minimum weak fixed point states. □

The following theorem solves the limit behavior of a quantum Markov chain when the initial state is in a BSCC:

Theorem 4.13. *Suppose S is a BSCC, $S_1, S_2 \cdots S_p$ are its period decomposition and the density operators in this decomposition are $\rho_1, \rho_2 \cdots \rho_p$, for a initial state $\sigma \in S$, denote $tr(P_{S_i}(\sigma))$ as p_i, then $\lim_{k\to+\infty} \mathcal{E}^{kp+i_0}(\sigma) = \sum_{i=1}^{k} p_i \rho_{i+i_0 \mod p}$*

Proof. By Lemma 2.1 and monotonicity of fidelity we know $tr(P_{S_{j+i_0}} \mathcal{E}^{kp+i_0}(\sigma)) \geq \sqrt{p_j}$. If we fix i_0, consider all of the convergent subsequences in sequence $\{\mathcal{E}^{kp+i_0}\}_{k=0}^{+\infty}$, their limit must be a weak fixed point state that satisfies $tr(P_{S_{i_0+1}} \sigma') \geq \sqrt{p_1}, tr(P_{S_{i_0+2}} \sigma') \geq \sqrt{p_2} \cdots$, and the only weak fixed point state that satisfies these conditions is $\sum_{i=1}^{p} p_i \rho_{i+i_0 \mod p}$. So all of the convergent subsequences converge to this density operator so the sequence itself converges to it. □

Corollary 4.13.1. $\lim_{k\to+\infty} \mathcal{E}^{kp+i_0} =$

5 Structure of superoperators

5.1 Introduction

5.2 Pseudo-unitary operators

We have studied the unitary transformation generated from $\mathcal{E}$, we will now study how to construct a $\mathcal{E}$ from a unitary operator on sequence space.

Definition 5.1.

$$\rho \xrightarrow{\mathfrak{s}_1 \to \mathfrak{s}_2} \rho'$$
$$\mathcal{E} \xrightarrow{\mathfrak{s}_1 \to \mathfrak{s}_2} \mathcal{E}'$$

Lemma 5.1. *If* $\forall i, E_i$ *coexists with* $U : \mathcal{H} \longmapsto \mathcal{H}$, $U = \boxplus_{i=1}^{k} \mathrm{e}^{\mathrm{i}\theta_i} \mathfrak{s}_i \mathfrak{s}_i^{\dagger}$, *then*

1. $\forall i, \mathrm{span}(\mathfrak{s}_i)$ *is an invariant subspace of* $\mathcal{E}$.
2. $\forall \mathfrak{s}, \mathcal{E}(\mathrm{span}(\mathfrak{s})) \subseteq \mathrm{span}(U\mathfrak{s})$
3. $\rho_1 \xleftrightarrow{\mathfrak{s}_1 \leftrightarrow \mathfrak{s}_2} \rho_2 \Rightarrow \mathcal{E}(\rho_1) \xleftrightarrow{U\mathfrak{s}_1 \leftrightarrow U\mathfrak{s}_2} \mathcal{E}(\rho_2)$
4. $\mathcal{E}|\,\mathrm{span}(\mathfrak{s}_i) \xleftrightarrow{\mathfrak{s}_i \leftrightarrow \mathfrak{s}_j} \mathcal{E}|\,\mathrm{span}(\mathfrak{s}_j)$

Definition 5.2. Suppose $\mathcal{G} = (H, \mathcal{E})$, we say $\mathcal{E}$ is a pseudo-unitary operator on H, if there exists a superoperator $\mathcal{E}_{monomer}$ on space S and S is a BSCC of $\mathcal{E}_{monomer}$, a sequence of orthogonal bases $\mathfrak{s}$ of S, and a similar sequence subspace $\mathfrak{H}$ whose parent space is H, a unitary operator U on $\mathfrak{H}$ and $\mathcal{E}_{monomer}$ satisfies $\forall \mathfrak{s} \in \mathfrak{H}$, $\mathcal{E}(\rho) = \rho'$, $\mathcal{E}_{monomer}(\sigma) = \sigma'$, $\sigma \xrightarrow{\mathfrak{s} \to \mathfrak{s}_1} \rho$, $\sigma' \xrightarrow{\mathfrak{s} \to U\mathfrak{s}_1} \rho'$, Denote it as

$$\mathcal{E}_{monomer} \overset{\mathfrak{s} \to \mathfrak{s}_2, \mathfrak{H}}{\boxtimes} \mathfrak{U} = \mathcal{E}$$

U is called as architecture operator. $\mathfrak{H}$ is called as architecture space. $\mathcal{E}_{monomer}$ is called as monomer operator.

Definition 5.3. if $\mathcal{E}_{monomer} \boxtimes^{\mathfrak{s} \to \mathfrak{H}} \mathfrak{U} = \mathcal{E}$, $\mathrm{span}(\mathfrak{s})$ is a BSCC of $\mathcal{E}_{monomer}$, we say $\mathcal{E}$ is a pseudo-unitary operator, and $\mathfrak{H}$ is its architecture space, $\mathfrak{U}$ is its architecture operator.

Example 5.1.

Lemma 5.2. *Suppose* S_1 *is a pseudo-unitary-evolution subspace of* $\mathcal{E}$, $\mathcal{E}|S_1 = \mathcal{E}_{monomer} \boxtimes^{\mathfrak{s} \to \mathfrak{H}} \mathfrak{U}$, $S_1 \oplus S_2$ *is also pseudo-unitary-evolution subspace, then we could express* $\mathcal{E}|S_1 \oplus S_2$ *as xxx*

Lemma 5.3. *Since* $E_1 E_2$ *coexists with* $\mathfrak{U}^2$ *we know iteration of pseudo-unitary operator is also a pseudo-unitary operator. Since the monomer operator converges to xx we know* $\lim_{i \to +\infty} \mathcal{E}_{monomer}^{i} \boxtimes^{\mathfrak{s} \to \mathfrak{H}} \mathfrak{U}^i = ..$

Definition 5.4. If S is an invariant subspace of $\mathcal{E}$ and $\mathcal{E}|S$ is a pseudo-unitary operator on S, we say S is a pseudo-unitary-evolution subspace of $\mathcal{E}$. If S is a pseudo-unitary subspace of $\mathcal{E}$ and $S' \supseteq S \Rightarrow S' = S$, we say S is a biggest pseudo-unitary-evolution subspace of $\mathcal{E}$.

Fact 5.4.

1. *Span of eigensequences decomposition of a pseudo-unitary-evolution subspace are orthogonal BSCCs.*
2. *If* S_1, S_2 *are two pseudo-unitary-evolution subspaces, then* $S_1 \cap S_2$ *is a pseudo-unitary-evolution subspace.*

Lemma 5.5. *If pseudo-unitary-operator has two different architecture spaces, they must satisfies, and under this map we have the architecture operator satisfies: monomer operator and monomer operator base sequence satisfies:*

Lemma 5.6. *If* $S_1 \subseteq S_2$, S_1, S_2 *are pseudo-unitary-evolution subspaces, the orthogonal complement of* S_1 *in* S_2 *are pseudo-unitary-evolution subspace.*

Lemma 5.7. *If* S_1,S_2 *are two pseudo-unitary-evolution subspaces,* $\dim(S_1 \cap S_2) > 0$, *then* $S_1 \oplus S_2$ *is a pseudo-unitary-evolution subspace with monomer operator uniquely determined by* S_1 *or* S_2.

Proof. Consider $S_1 \oplus S_2$ and this is a pseudo-unitary-evolution subspace. Consider $\mathcal{E}|S_1 \oplus S_2$. This is a pseudo-unitary operator. With the uniqueness in representation of pseudo-unitary operator we could write $\mathcal{E}|S_1$ and $\mathcal{E}|S_2$ as and they have the same representation on $\mathcal{E}|S_1 \oplus S_2$, then xxx

Old proof:

$$\mathcal{E}|S_1 \oplus S_2 = \mathcal{E}_{monomer} \overset{\mathfrak{s} \to \mathcal{H}_1 \oplus \mathcal{H}_2}{\boxtimes} U_1 \oplus U_2$$
$$\mathcal{E}|S_2 \oplus S_3 = \mathcal{E}_{monomer} \overset{\mathfrak{s} \to \mathcal{H}_2 \oplus \mathcal{H}_3}{\boxtimes} U_2 \oplus U_3$$

Then

$$\mathcal{E}|(S_1 \oplus S_2 \oplus S_3) = \mathcal{E}_{monomer} \overset{\mathfrak{s} \to \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3}{\boxtimes} U_1 \oplus U_2 \oplus U_3$$

□

Lemma 5.8. *properties of psusubspace, biggest...*

Then let's introduce our main theorem in this section. xxx

5.3 Further properties of weak fixed point states

Lemma 5.9. *Suppose ρ is a minimum weak fixed point state, S is a BSCC, its period decomposition is $S_1, S_2 \cdots S_p$, $tr(P_{S_1}\rho) > 0$, the weak fixed point state in S_i is ρ_i, then*

1. $\dim(\mathrm{span}(\mathcal{E}^{kp+i}(\rho))) = \dim(\rho_i)$.
2. $\forall i$, $\mathcal{E}^i(\rho)$ *is a minimum weak fixed point state.*

Proof. The first one is the same as corollary

Since $MS(\mathcal{E}^j(\rho), \mathcal{E}^j(\rho_1))$ and $F(\mathcal{E}^j(\rho), \mathcal{E}^j(\rho_1))$ are constant, we know $\forall i, \mathcal{E}^{kp+i}(\rho)$ and $\mathcal{E}^{kp+i}(\rho_1)$ are completely-maximally-fidelitous.

Suppose $\mathcal{E}^i(\rho)$ is not a minimum weak fixed point state, we could find a minimum weak fixed point state in $\mathrm{supp}(\mathcal{E}^i(\rho))$ whose dimension is less than $\mathrm{supp}(\mathcal{E}^{kp+i}(\rho))$, which contradicts the one we have proved. □

Lemma 5.10. *If ρ is a minimum weak fixed point state, $\forall \sigma \in \mathrm{supp}(\rho)$, $\lim_{i\to+\infty} \|\mathcal{E}^i(\sigma) - \mathcal{E}^i(\rho)\|_{tr} = 0$*

Lemma 5.11. *If ρ_1, ρ_2 are two minimum weak fixed point states that self-converge synchronously, then they are completely-maximally-fidelitous.*

Proof. Suppose they are not completely-maximally-fidelitous, choose $\rho_1' \in \mathrm{supp}(\rho_1)$, $\rho_2' \in \mathrm{supp}(\rho_2)$, use lemma 5.10 and we know $F(\rho_1, \rho_2) = MS(\rho_1, \rho_2)$ □

Lemma 5.12. *If $\mathcal{E}$ is pseudo-unitary on S, all of its minimum weak fixed point states satisfies $\rho \in \mathrm{span}(\mathfrak{s})$ for some $\mathfrak{s} \in \mathfrak{H}$.*

Proof. □

Lemma 5.13. *If $\mathcal{E}$ is pseudo-unitary on S, its monomer operator is, its architecture operator is, the weak fixed point states of the monomer operator is , then all of the weak fixed point states of $\mathcal{E}$ on S is xx.*

Proof. Suppose $\mathfrak{U} = \sum_{i=1}^{k} \mathrm{e}^{\mathrm{i}\theta_i} \mathfrak{s}_i \mathfrak{s}_i^\dagger$. Suppose σ is a minimum weak fixed point state, then suppose $\rho_1, \rho_2 \cdots \rho_k$ is the state in S_1 that are completely-maximally-fidelitous with σ, then $\rho_1, \rho_2, \cdots \rho_k$ are minimum weak fixed point states. Then □

5.4 The most simple nontrivial case: the space is the direct sum of two aperiodic BSCCs

Lemma 5.14. *Suppose $\mathcal{G} = (S_1 \oplus S_2, \mathcal{E})$ is a quantum Markov chain, S_1, S_2 are two aperiodic BSCCs of $\mathcal{G}$, their fixed point states are ρ_1, ρ_2 respectively, σ is a minimum weak fixed point state that is different from ρ_1, ρ_2, S_1, S_2 satisfy $\forall |\varphi\rangle \in S_i, span(\mathcal{E}(\varphi)) = S_i$, suppose ρ_1, ρ_2, σ could be written as*

$$\begin{aligned}
\rho_1 &= \{c_i\}_{i=1}^n \boxed{DenM}\, \mathfrak{s}_1 \\
\rho_2 &= \{c_i\}_{i=1}^n \boxed{DenM}\, \mathfrak{s}_2 \\
\sigma &= \{c_i\}_{i=1}^n \boxed{DenM}\, \alpha\mathfrak{s}_1 \boxplus \beta\mathfrak{s}_2
\end{aligned}$$

Then $\mathcal{E}(\sigma)$ could be written as

$$\mathcal{E}(\sigma) = \{c_i\}_{i=1}^n \boxed{DenM}\, \alpha'\mathfrak{s}_1 \boxplus \beta'\mathfrak{s}_2$$

Proof. σ, $\mathcal{E}(\sigma)$ self-converge synchronously, so they are completely-maximally-fidelitous, so $\mathrm{supp}(\sigma)$, $\mathrm{supp}(\mathcal{E}(\sigma))$ are completely-maximally-similar. And we have

$$\mathrm{span}(\alpha\{\varphi_i\}_n \boxplus \beta U\{\chi_i\}_n) = \mathrm{span}(\alpha\{\varphi_i\}_n \boxplus \beta U_0^\dagger D U_0\{\chi_i\}_n) = \mathrm{span}(\alpha U_0\{|\varphi_i\rangle\}_n \boxplus \beta U_0\{\mathrm{e}^{\mathrm{i}\theta_i}|\chi_i\rangle\}_n)$$

So we could consider

$$\mathrm{supp}(\sigma) = \mathrm{span}(\alpha\{\varphi_i\}_n \boxplus \beta\{\chi_i\}_n), \mathrm{supp}(\mathcal{E}(\sigma)) = \mathrm{span}(\alpha'\{\varphi_i\}_n \boxplus \beta' D\{\chi_i\}_n)$$

And we need to prove $D = \mathrm{e}^{\mathrm{i}\theta} I$

Choose E such that $E|\varphi_1\rangle = |\varphi_2\rangle$. Since E coexists with $\mathfrak{U}$ we know E coexists with $\mathfrak{U}' : U_0\mathfrak{H} \mapsto U_0\mathfrak{H}$. So we know

$$E|\varphi_1\rangle = |\varphi_2\rangle$$
$$E(\alpha|\varphi_1\rangle + \beta|\chi_1\rangle) \in \mathrm{span}(\alpha'\{\varphi_i\}_n \boxplus \beta' D\{\chi_i\}_n)$$
$$E|\chi_1\rangle \in \mathrm{span}(\{|\chi_i\rangle\}_n)$$

So $E|\chi_1\rangle = \mathrm{e}^{\mathrm{i}\theta_2}|\chi_2\rangle$. Similarly, we have:

If $E|\varphi_1\rangle = |\varphi_1\rangle$, then $E|\chi_1\rangle = \mathrm{e}^{\mathrm{i}\theta_1}|\chi_1\rangle$

If $E|\varphi_2\rangle = |\varphi_1\rangle$, then $E|\chi_2\rangle = \mathrm{e}^{\mathrm{i}\theta_1}|\chi_1\rangle$

Then consider $\mathcal{E}(\mathcal{E}(\sigma))$. We have $E(\alpha'|\varphi_1\rangle + \beta'\mathrm{e}^{\mathrm{i}\theta_1}|\chi_1\rangle) \in \mathcal{E}(\mathcal{E}(\sigma))$, $E(\alpha'|\varphi_2\rangle + \beta'\mathrm{e}^{\mathrm{i}\theta_2}|\chi_2\rangle)) \in \mathcal{E}(\mathcal{E}(\sigma))$. So $\theta_1^2 = \theta_2\theta_1$, and similarly, $\theta_2^2 = \theta_2\theta_1$, so $\theta_1 = \theta_2$. □

Lemma 5.15. *Suppose $\mathcal{G} = (S_1 \oplus S_2, \mathcal{E})$ is a quantum Markov chain, S_1, S_2 are two aperiodic BSCCs of $\mathcal{G}$, their fixed point states are ρ_1, ρ_2 respectively, σ is a minimum weak fixed point state that is different from ρ_1, ρ_2, suppose ρ_1, ρ_2, σ could be written as*

$$\rho_1 = \{c_i\}_{i=1}^n \boxed{DenM}\, \mathfrak{s}_1$$
$$\rho_2 = \{c_i\}_{i=1}^n \boxed{DenM}\, \mathfrak{s}_2$$
$$\sigma = \{c_i\}_{i=1}^n \boxed{DenM}\, \alpha\mathfrak{s}_1 \boxplus \beta\mathfrak{s}_2$$

Then $\mathcal{E}(\sigma)$ could be written as

$$\mathcal{E}(\sigma) = \{c_i\}_{i=1}^n \boxed{DenM}\, \alpha'\mathfrak{s}_1 \boxplus \beta'\mathfrak{s}_2$$

Proof. All of the minimum weak fixed point states are weak fixed point states of $\mathcal{G}^i$. Choose i such that $\mathcal{G}^i$ satisfies the conditions in lemma 5.14. □

Lemma 5.16. *If there is a minimum weak fixed point state that is not ρ_1 or ρ_2, then $\mathcal{E}$ is a pseudo-unitary operator*

Proof. Suppose the minimum weak fixed point state is σ. Consider σ, $\mathcal{E}(\sigma)$. We have

$$\rho_1 = \{c_i\}_{i=1}^n \boxed{DenM}\, \mathfrak{s}_1$$
$$\rho_2 = \{c_i\}_{i=1}^n \boxed{DenM}\, \mathfrak{s}_2$$
$$\sigma = \{c_i\}_{i=1}^n \boxed{DenM}\, \alpha\mathfrak{s}_1 \boxplus \beta\mathfrak{s}_2$$
$$\mathcal{E}(\sigma) = \{c_i\}_{i=1}^n \boxed{DenM}\, \alpha'\mathfrak{s}_1 \boxplus \beta'\mathfrak{s}_2$$

□

5.5 A more complicated case: two BSCCs

Lemma 5.17. *Suppose S_1, S_2 are two BSCCs if $\mathcal{G}$, then one of the following holds:*

1. *For all minimum weak fixed point states $\rho \in S_1 \oplus S_2$, $\rho \in S_1$ or $\rho \in S_2$*
2. *We could find orthogonal sequences $\mathfrak{s}_1, \mathfrak{s}_2 \cdots \mathfrak{s}_p$ in S_1, $\mathfrak{s}'_1, \mathfrak{s}'_2 \cdots \mathfrak{s}'_p$ in S_2, coefficients $\mathfrak{c}_1, \mathfrak{c}_2 \cdots \mathfrak{c}_p$, and express one of the minimum weak fixed point states as $\mathfrak{c}_i \boxed{DenM}\, \alpha\mathfrak{s}_i \boxplus \beta\mathfrak{s}'_i$,then this is a pseudo-unitary operator with architecture operator xx and monomer operator xx.*

Proof. Suppose the it's of type (2)

$\mathfrak{U}_1 = \mathfrak{s}_2\mathfrak{s}_1^\dagger + \mathrm{e}^{\mathrm{i}\theta_1}\mathfrak{s}'_2\mathfrak{s}_1^\dagger$, $\mathfrak{U}_2 =, \cdots$

$\forall E$, E coexists with $\mathfrak{U}_1, \mathfrak{U}_2 \cdots \mathfrak{U}_p$

Consider $\mathfrak{H}$: the concatanation of $\mathfrak{s}_1, \mathfrak{s}_2 \cdots$, the concatanaiton of $\mathfrak{s}'_1 \cdots$, $\mathfrak{U}$ is defined as follows:

Then $\forall E$, E coexists with $\mathfrak{U}$ xx... □

5.6 Pseudo-unitary operators and weak fixed point states

Lemma 5.18. *Suppose ρ is a minimum weak fixed point state in $S_1 \oplus S_2 \cdots S_{m-1}$, S_i are BSCCs, $tr(P_{S_i}\rho) > 0$ for all i, if $S_1 \oplus S_2 \cdots S_m$ is a pseudo-unitary-evolution subspace of $\mathcal{E}$, then $S_1 \oplus S_2 \cdots S_m$ is a pseudo-unitary operator of $\mathcal{E}$.*

Proof. Suppose ρ is positive-completely-maximally-fidelitous to $\sigma_1 \in S_m$ and $\sigma_2 \in \oplus_{i=1}^{m-1} S_i$, $\mathcal{E}(\rho)$ is positive-completely-maximally-fidelitous to $\sigma_1' \in S_m$ and $\sigma_2' \in \oplus_{i=1}^{m-1} S_i$. Then by the property of pseudo-unitary operator we know σ_2, σ_2' are completely-maximally-fidelitous. Suppose $\sigma_2 = \{c_i\} \boxed{DenM} \mathfrak{s}_2$, $\sigma_2' = \{c_i\} \boxed{DenM} \mathfrak{s}_2'$, by a similar technique as the proof of lemma 5.14 we know $\forall i, E_i$ coexists with . ☐

Lemma 5.19. *If S_1, S_2 are two biggest pseudo-unitary-evolution subspaces, all the minimum weak fixed point state ρ in $S_1 \oplus S_2$ satisfies $\rho \in S_1$ or $\rho \in S_2$.*

Lemma 5.20. *Suppose ρ is a minimum weak fixed point state in $S_1 \oplus S_2 \cdots S_m$, S_i are BSCCs, $tr(P_{S_i}\rho) > 0$ for all i, then $S_1 \oplus S_2 \cdots S_m$ is a pseudo-unitary operator of $\mathcal{E}$.*

Proof. Use mathematical induction, when $m = 2$ the lemma is true. Suppose the lemma is true for $2, 3 \cdots m - 1$.

Consider the similar sequence space spanned by ρ and some ρ_i in S_1 where ρ_i is not perpendicular to ρ. Then there is a density operator in $S_2 \oplus S_3 \cdots S_m$ that is a minimum weak fixed point state. So $S_2 \oplus S_3 \cdots S_m$ is a pseudo-unitary evolution subspaces of $\mathcal{E}$ by the induction hypothesis. Then we could also know $S_1 \oplus S_2 \cdots S_{m-1}$ is a pseudo-unitary evolution subspaces. Then by lemma 5.18 we know $S_1 \oplus S_2 \cdots S_m$ is a pseudo-unitary operator of $\mathcal{E}$. ☐

Proposition 5.21. *The recurrent subspace of any superoperator could be uniquely decomposed to orthogonal biggest pseudo-unitary subspaces, and all of the minimum weak fixed point states falls into one of these subspaces.*

Proof. Consider the set of weak fixed point states. Because non-orthogonal minimum fixed point states span a pseudo-unitary subspaces. So R is .. of orthogonal pseudo-unitary subspaces. ☐

Theorem 5.22. *Consider quantum Markov chain $\mathcal{G} = (H, \mathcal{E})$.*

1. *First, H could be uniquely decomposed to transient subspace T and recurrent subspace R*
2. *Secondly, the recurrent subspace R could be uniquely decomposed to orthogonal biggest pseudo-unitary subspaces, and all of the minimum weak fixed point states falls into one of these subspaces.*
3. *Then, every pseudo-unitary subspace S satisfies $\mathcal{E}|S = \mathcal{E}_{monomer} \boxplus^{\mathfrak{s}\to\mathfrak{H}} \mathfrak{U}$, where $\text{span}(\mathfrak{s})$ is a BSCC of $\mathcal{E}_{monomer}$*
4. *Finally, every BSCC has a period, and it could be uniquely decomposed to aperiodic subspace under $\mathcal{G}^p$.*

6 Limiting behavior of quantum Markov chains

Lemma 6.1. *When S is a BSCC, its period decomposition is ., $|\varphi\rangle \in S$, the limit behavior of $|\varphi\rangle$*

Lemma 6.2. *Suppose S is a pseudo-unitary subspace, $|\varphi\rangle \in S$, the limit behavior of $|\varphi\rangle$*

Proposition 6.3. *Suppose the recurrent subspace R of quantum Markov chain $(H, \mathcal{E})$ could be decomposed to $E_1 \oplus E_2 \oplus \cdots E_u$, where E_is are pseudo-unitary-evolution subspaces, $|\varphi\rangle$ could be expressed as $|\varphi\rangle = \sqrt{c_1}\,|\varphi_1\rangle + \sqrt{c_2}\,|\varphi_2\rangle + \cdots \sqrt{c_u}\,|\varphi_u\rangle$, where $|\varphi_i\rangle \in E_i$, then $\lim_{i\to+\infty} \|\mathcal{E}^i(\varphi) - \sum_{j=1}^{u} c_j \mathcal{E}^i(\varphi_j)\|_{tr} = 0$*

7 Related algorithms

8 Other applications

8.1 Generalization of decoherence free subspaces

The subspace of pseudo-unitary operators are generalization of decoherence free subspaces.

8.2 Quantum graph

TBA.

9 Conclusions and outlooks

Acknowledgements

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