Structure of quantum Markov chains and its applications

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1 Introduction

This document is just a brief introduction to my results. My paper hasn't been completed yet. I have written a draft for my results and proofs but it's too long and not suitable for sharing my ideas. I have omitted all of the details and proofs in this document and only keep the most important and understandable concepts and theorems. I hope you could understand my ideas.

Section 2 is the background of my research(In other words, the results in this section don't belong to me). Section 3 to the end of this document are my research results.

I have completed the proof of the results in section 3 and 4 and have checked many times. There are still some details to be filled in the proof of the results in section 5 and 6 but I'm confident with my results.

Briefly speaking, we consider a trace-preserving completely-positive super-operator as a quantum Markov chain. This model is general enough. I studied the period of the irreducible subspaces, limit behavior and so on. Other researchers have proved that the recurrent subspace of a quantum Markov chain could be decomposed to irreducible subspaces two years ago but there were still some problems on the uniqueness of decomposition, which I have also solved. The results that I have got are strong enough(as strong as the theorems on classical Markov chains). The proofs of results on quantum Markov chains are much more complicated than that in classical Markov chains and although the results are similar in some aspects, they have subtle and beautiful differences.

2 Background

This section is the background of my research. Most of the definitions and results in this section are from [1].

In our following discussion we will only consider trace-preserving super-operators.

We use $\mathrm{supp}(\rho)$ to denote the subspace spanned by the eigenvectors of ρ. We use $\mathcal{D}(S)$ to denote the set of density operators that satisfy $\mathrm{supp}(\rho) \subseteq S$.

2.1 Quantum Markov Chains

Definition 2.1. A quantum Markov chain is a pair $\mathcal{G} = (H, \mathcal{E})$, H is a finite-dimensional Hilbert space, $\mathcal{E}$ is a super-operator on H

This definition is general enough for many applications. Our problem is: how to describe the structure of quantum Markov chains? Does it have transient subspace, recurrent subspace, irreducible subspaces and decomposition theorems like classical Markov chains?

2.2 Reachable subspaces and BSCCs

Definition 2.2. For a quantum Markov chain $\mathcal{G} = (H, \mathcal{E})$, define the reachable subspace of ρ as the subspace spanned by $\mathrm{supp}(\rho), \mathrm{supp}(\mathcal{E}(\rho)), \mathrm{supp}(\mathcal{E}^2(\rho)) \cdots$. Denote it as $\mathcal{R}_{\mathcal{G}}(\rho)$.

Definition 2.3. Consider a quantum Markov chain $\mathcal{G} = (H, \mathcal{E})$. We say a subspace S of H is a bottom strongly connected component(BSCC) of $\mathcal{G}$ if it satisfies $\forall \rho \in \mathcal{D}(S), \mathcal{R}_{\mathcal{G}}(\rho) = S$

BSCC is the quantum analog to "irreducible" in classical Markov chains.

I omit the properties of reachable subspaces and BSCCs for simplicity. You could see [1] for details. The definitions of "reachable subspace" and "BSCC" in [1] are different from the definitions here but they are equivalent.

And I will skip the formal definition of "transient subspace" and "recurrent subspace"(See [1] for details) and present the decomposition theorem directly.

Theorem 2.1. *Consider a quantum Markov chain $\mathcal{G} = (H, \mathcal{E})$. H could be decomposed to the direct sum of orthogonal subspaces $B_1, B_2 \cdots B_u, T$, where T is the transient subspace of $\mathcal{G}$, B_is are BSCCs of $\mathcal{G}$.*

Let's see an example, which is the example used in [1]:

Example 2.1. Consider quantum Markov chain $\mathcal{G} = (H, \mathcal{E})$. The state space is $H = \text{span}(|0\rangle, |1\rangle, |2\rangle, |3\rangle, |4\rangle)$, the superoperator is

$$\mathcal{E} = \sum_{i=1}^{5} E_i \cdot E_i^\dagger$$

where

$$E_1 = \frac{1}{\sqrt{2}}(|1\rangle\langle 0+1| + |3\rangle\langle 2+3|)$$
$$E_2 = \frac{1}{\sqrt{2}}(|1\rangle\langle 0-1| + |3\rangle\langle 2-3|)$$
$$E_3 = \frac{1}{\sqrt{2}}(|0\rangle\langle 0+1| + |2\rangle\langle 2+3|)$$
$$E_4 = \frac{1}{\sqrt{2}}(|0\rangle\langle 0-1| + |2\rangle\langle 2-3|)$$
$$E_5 = \frac{1}{10}(|0\rangle\langle 4| + |1\rangle\langle 4| + |2\rangle\langle 4| + 4|3\rangle\langle 4| + 9|4\rangle\langle 4|)$$

where $|i \pm j\rangle = \frac{|i\rangle \pm |j\rangle}{\sqrt{2}}$

Then H has BSCC decomposition $H = B_1 \oplus B_2 \oplus T$, where $T = \text{span}(|4\rangle)$ is the transient subspace, $B_1 = \text{span}(|0\rangle, |1\rangle), B_2 = \text{span}(|2\rangle, |3\rangle)$ are BSCCs of $\mathcal{G}$.

2.3 Uniqueness of BSCC decomposition

We notice that the BSCC decomposition may be not unique. In the example above, we could also decompose H as $D_1 \oplus D_2 \oplus T$, where $D_1 = \text{span}(|0+2\rangle, |1+3\rangle)$, $D_2 = \text{span}(|0-2\rangle, |1-3\rangle)$. Theorem 6 in [1] shows some relations between two different decompositions but the result is not strong enough. This problem has been solved by my results.

2.4 Stationary distribution

The following theorem is also from [1], which is the analog of properties of "stationary distribution" in classical Markov chains:

Theorem 2.2. *For any BSCC S of $\mathcal{G} = (H, \mathcal{E})$, there is a unique density operator $\rho \in \mathcal{D}(S)$ that satisfies $\mathcal{E}(\rho) = (\rho)$.*

But there is no existing results on limit distribution or limit behavior on quantum Markov chains.

3 The period of a BSCC

The first problem is, could we define the "period" of a BSCC? In classical Markov chain, we could decompose the states of an irreducible Markov chain to many classes, where every class is aperiodic(or ergodic) under $\mathcal{G}^p$. I found that there is a similar theorem for quantum Markov chain:

Definition 3.1. We say a BSCC S of quantum Markov chain $(H, \mathcal{E})$ is aperiodic if it satisfies $\forall |\varphi\rangle \in S, \exists N$, $\text{supp}(\mathcal{E}^N(\varphi)) = S$

The following theorem is the period decomposition theorem for BSCCs, which is also the main theorem of this section:

Theorem 3.1 (Period decomposition theorem)**.** *A BSCC S must be one of the following two types:*

1. *Aperiodic BSCC*
2. *There exists a period $p \geq 2$ and a sequence of density operators $\rho_1, \rho_2, \cdots \rho_p$ that satisfy:*
   (a) $\mathrm{span}(\rho_1) \oplus \mathrm{span}(\rho_2) \oplus \cdots \mathrm{span}(\rho_p) = S$ *and subspace sequence* $\{\mathrm{span}(\rho_i)\}_{i=1}^{p}$ *are orthogonal to each other.*
   (b) $\mathcal{E}(\rho_i) = \rho_{i+1 \mod p}$
   (c) $\forall i \in \{1, \cdots p\}$, $\mathrm{span}(\rho_i)$ *is an aperiodic BSCC of* $\mathcal{G}^p$

   For any decomposition that satisfies these conditions, the integer p, the set of $\dim(\mathrm{span}(\rho_i))$ *and the number of occurrence of each element is unique.*

The proof of this theorem is complicated and I have to invent some new tools to prove it. (It occupies 20 pages in my 35-page draft).

I have to omit the proof. I will present some key ideas and tools in my proof.

3.1 Weak fixed point states

Definition 3.2. For a quantum Markov chain $\mathcal{G} = (H, \mathcal{E})$, if the sequence $\{\mathcal{E}^i(\rho)\}_{i=0}^{+\infty}$ has a subsequence that converges to ρ itself, we say ρ is a weak fixed point state of $\mathcal{G}$.

It's easy to see that ρ_1 in the period decomposition theorem is a weak fixed point state. The process of our proof is first finding a weak fixed point state that satisfies some conditions then proving its properties.

3.2 Similarities of subspaces

Definition 3.3. For two subspaces S_1, S_2, define the similarity of them $MS(S_1, S_2)$ as follows:

$$MS(S_1, S_2) = \max_{|\varphi\rangle \in S_1, ||\varphi\rangle|=1} |P_{S_2}(|\varphi\rangle)|$$

This is well-defined because continuous functions in tight set always have the maximum.

It has many useful properties like $MS(\mathcal{E}(S_1), \mathcal{E}(S_2)) \geq MS(S_1, S_2)$. It seems like the fidelity.

4 A stronger version of period decomposition theorem and limit behavior inside a BSCC

After getting the period decomposition theorem of BSCC, the next step is to try to express all of the weak fixed point states in a BSCC. I have proved that $\rho_1, \rho_2 \cdots \rho_p$ are all of the "minimum weak fixed point states" in a BSCC, and weak fixed point state could be written as the linear combination of minimum weak fixed point states(This is nontrivial). This could be seen as a stronger version of period decomposition theorem. I have to omit the details since this is just a brief introduction.

Then we will solve the limit behavior problem when the initial state is in a BSCC. The trick is: for any initial state ρ, consider sequence $\{\mathcal{E}^i(\rho)\}_{i=0}^{+\infty}$. We could prove that all of its convergent subsequences converge to a weak fixed point state. Then we could analyze its limit behavior by analyzing the limit of its convergent subsequences and some other tools.

5 Pseudo-unitary operators and decomposition of recurrent subspace

I won't give the formal definition of pseudo-unitary operator here(actually, I'm still thinking how to give a proper definition such that my proof could be as clear as possible). I will use example to present my ideas.

Let's consider example 2.1 again. Consider the operation of $\mathcal{E}$ on subspace $\mathrm{span}(|0\rangle, |1\rangle, |2\rangle, |3\rangle)$. It's not hard to see all of its BSCCs could be written as $\mathrm{span}(\alpha|0\rangle + \beta|2\rangle, \alpha|1\rangle + \beta|3\rangle)$. Furthermore, the operation of $\mathcal{E}$ on each of these BSCCs are isomorphic to each other. In other words, if we consider the BSCCs as a "monomer", $\mathcal{E}$ could be seen as an identity operator on 2-dimensional Hilbert space.

Then let's consider the following example, which is slightly different from example 2.1:

Example 5.1. Consider quantum Markov chain $\mathcal{G} = (H, \mathcal{E})$. The state space is $H = \text{span}(|0\rangle, |1\rangle, |2\rangle, |3\rangle)$, the superoperator is

$$\mathcal{E} = \sum_{i=1}^{4} E_i \cdot E_i^\dagger$$

where

$$E_1 = \frac{1}{\sqrt{2}}(|1\rangle \langle 0+1| + i\,|3\rangle \langle 2+3|)$$
$$E_2 = \frac{1}{\sqrt{2}}(|1\rangle \langle 0-1| + i\,|3\rangle \langle 2-3|)$$
$$E_3 = \frac{1}{\sqrt{2}}(|0\rangle \langle 0+1| + i\,|2\rangle \langle 2+3|)$$
$$E_4 = \frac{1}{\sqrt{2}}(|0\rangle \langle 0-1| + i\,|2\rangle \langle 2-3|)$$

Then H has unique BSCC decomposition $H = B_1 \oplus B_2$, $B_1 = \text{span}(|0\rangle, |1\rangle)$, $B_2 = \text{span}(|2\rangle, |3\rangle)$ are BSCCs of $\mathcal{G}$.

If we see $|0\rangle, |1\rangle$ as a "monomer", $|2\rangle, |3\rangle$ as another "monomer", $\mathcal{E}$ could be seen as the linear operator $M = \left[\begin{smallmatrix} 1 & \\ & \mathrm{i} \end{smallmatrix}\right]$ on a 2-dimensional space.

This is the idea of "pseudo-unitary operator". It has a constraint: the monomer must be BSCC. And we could also define "pseudo-unitary-evolution subspace": if S is an invariant subspace of $\mathcal{G}$ and the operation of $\mathcal{E}$ restricted on S is pseudo-unitary, we say S is a pseudo-unitary-evolution subspace.

Then we need to introduce "biggest pseudo-unitary-evolution subspace". In example 5.1 B_1 and B_2 are both pseudo-unitary-evolution subspace but the biggest pseudo-unitary-evolution subspace is $B_1 \oplus B_2$.

We omit the formal definition here. Let's see the key theorem directly.

Theorem 5.1. *The recurrent subspace of a quantum Markov chain could be decomposed to the direct sum of orthogonal biggest pseudo-unitary-evolution subspaces uniquely.*

We split the original BSCC decomposition theorem 2.1 to two steps: first decompose the recurrent subspace to pseudo-unitary-evolution subspaces, then decompose pseudo-unitary-evolution subspace to BSCCs. The first step is unique and the second step is easy to study since pseudo-unitary operators have very regular structure. Then we could solve the limit behavior problem with this theorem.

6 Limit behavior of quantum Markov chains

When the initial state is in the recurrent subspace we could know its limit behavior with my research results. When "part" of the initial state falls into the transient subspace there is no simple result, and there is no simple result in classical case either. Since I have written too much I decide to omit it.

7 Applications and summary

I have proved the period decomposition theorem for the BSCCs of quantum Markov chains(in other words, irreducible quantum Markov chains) and solved the limit behavior problem. One possible application of quantum Markov chain is to model the noise in quantum systems. "Pseudo-unitary-evolution subspace" is a generalization of "decoherence-free subspace". We could do quantum computation on "pseudo-unitary-evolution subspace", not necessarily "decoherence-free subspace".

References

[1] Reachability probabilities of quantum Markov chains, Shenggang Ying, CONCUR 2013