Primal-Dual Algorithms for Clustering and Feature Allocation

Nathan Cordner

Boston University

1 October 2018

Introduction

Clustering Problem

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Feature Allocation Problem

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(Metric) Uncapacitated Facility Location

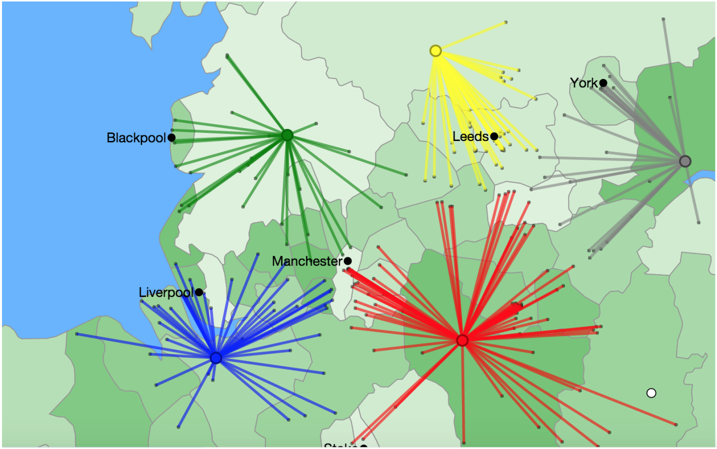

Figure: http://examples.gurobi.com/facility-location

(Metric) Uncapacitated Facility Location

Given a set of facilities F, and a set of clients C

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- c_{ij} = distance between facility i and client j
- f_i = cost of opening facility i
- x_{ij} = indicator for whether client j connects to facility i
- y_i = indicator for whether facility i is open

(Metric) Uncapacitated Facility Location

Primal IP:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i \in F, j \in C} c_{ij} x_{ij} + \sum_{i \in F} f_i y_i \\
\text{subject to} \quad & \forall j \in C : \sum_{i \in F} x_{ij} \geq 1, \\
& \forall i \in F, j \in C : y_i - x_{ij} \geq 0, \\
& \forall i \in F, j \in C : x_{ij} \in \{0, 1\}, \\
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\end{aligned}
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c_{ij} = distance, f_i = facility cost,
x_{ij} = client connection, y_i = facility open

(Metric) Uncapacitated Facility Location

Primal LP Relaxation:

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i \in F, j \in C} c_{ij} x_{ij} + \sum_{i \in F} f_i y_i \\ \text{subject to} & \forall j \in C : \displaystyle\sum_{i \in F} x_{ij} \geq 1, \\ & \forall i \in F, j \in C : y_i - x_{ij} \geq 0, \\ & \forall i \in F, j \in C : x_{ij} \geq 0, \\ & \forall i \in F : y_i \geq 0. \end{array}$$

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(Metric) Uncapacitated Facility Location

Dual LP:

$$
\begin{aligned}
\text{maximize} \quad & \sum_{j \in C} \alpha_j \\
\text{subject to} \quad & \forall i \in F, j \in C : \alpha_j - \beta_{ij} \leq c_{ij}, \\
& \forall i \in F : \sum_{j \in C} \beta_{ij} \leq f_i, \\
& \forall j \in C : \alpha_j \geq 0, \\
& \forall i \in F, j \in C : \beta_{ij} \geq 0.
\end{aligned}
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$\phi(j) = i$ denotes that client j is *connected* to facility i

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Facility i is *paid for* when $\sum_{j:\phi(j)=i} \beta_{ij} = f_i$

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Total price paid by client j: $\alpha_j = \beta_{\phi(j)j} + c_{\phi(j)j}$

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Algorithm Outline: sort list of edges in increasing order

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Finish when all clients are connected

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Running Time: $O(m \log m)$, where $m = |F| \cdot |C|$

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Approximation Bound:

$$\sum_{i \in F, j \in C} c_{ij} x_{ij} + \sum_{i \in F} f_i y_i \leq 3 \cdot \mathsf{OPT}$$

Primal-Dual Clustering

Now let $C = F$, and each $f_i = \lambda$

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LP relaxation and dual programs are similar

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Compare with K-means

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Compare with K-means

- PD approach takes a little longer, but can give better results

Primal-Dual Clustering

Examples: 1200 points

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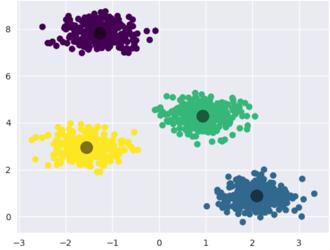
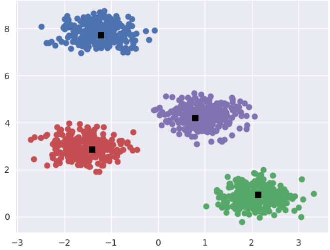
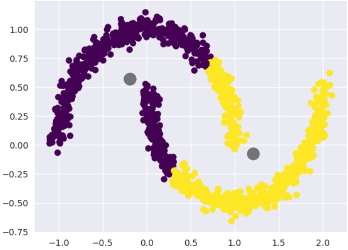


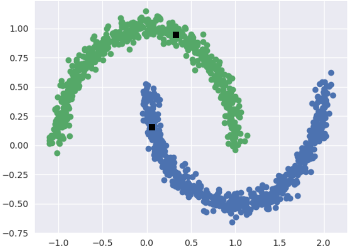


K-means: 0.06 s, $k = 4$ / PD: 0.77 s, $\lambda = 7$

Primal-Dual Clustering

Examples: 1200 points





K-means: 0.04 s, $k = 2$ / PD: 0.59 s, $\lambda = 3$

Primal-Dual Clustering

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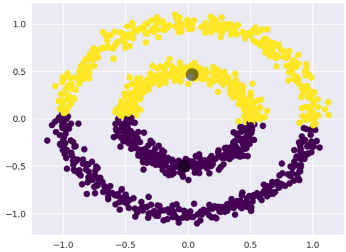

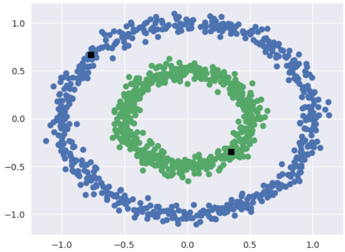

K-means: 0.04 s, $k = 2$ / PD: 0.54 s, $\lambda = 2$

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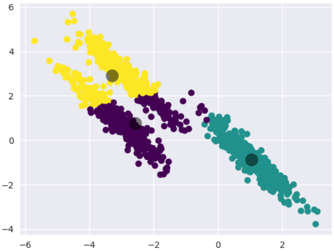

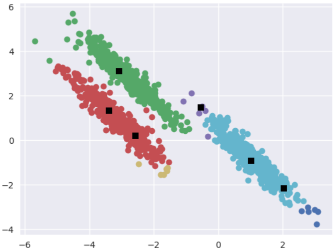

K-means: 0.04 s, $k = 3$ / PD: 0.83 s, $\lambda = 5$

Primal-Dual Clustering

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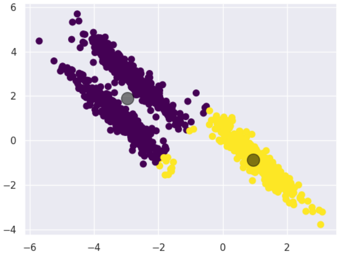

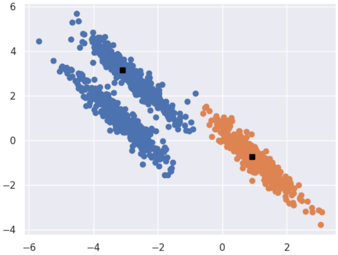

K-means: 0.12 s, $k = 2$ / PD: 1.15 s, $\lambda = 10$

Primal-Dual Feature Allocation

Similar to clustering, with relaxed constraint for x_{ij}

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- same as clustering, except we do not remove any contributions from clients when they connect to facilities

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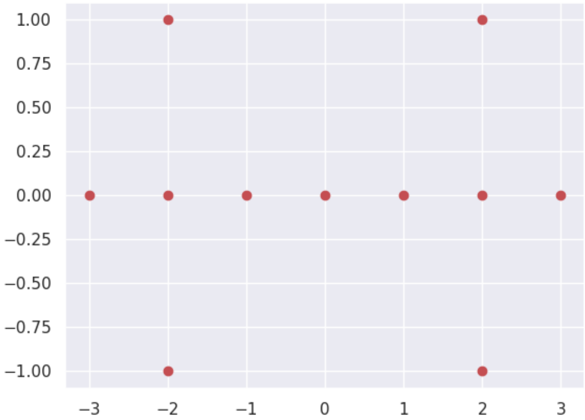
A client may contribute to more than one facility, and thus have multiple features

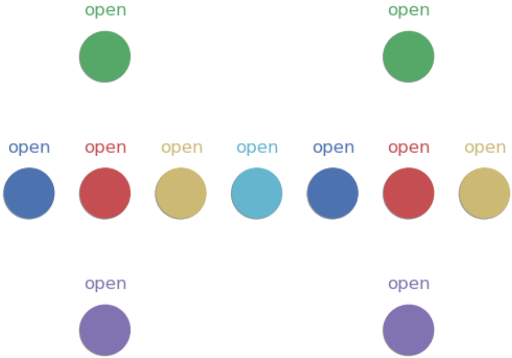
Primal-Dual Feature Allocation

Example:



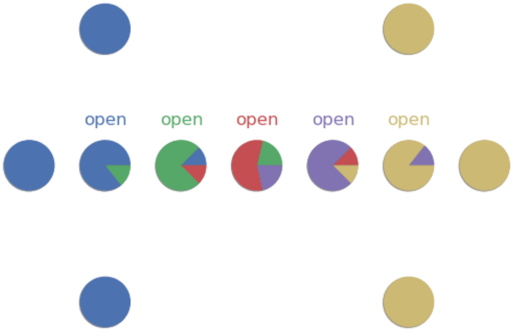
Primal-Dual Feature Allocation

Example: $\lambda = 1$



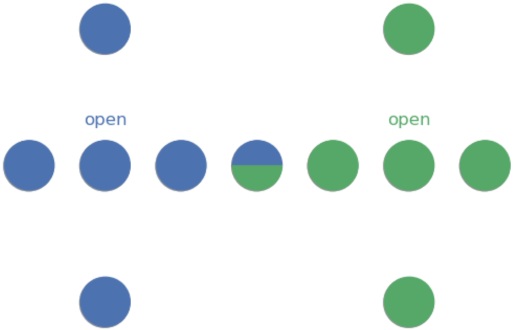
Primal-Dual Feature Allocation

Example: $\lambda = 2$



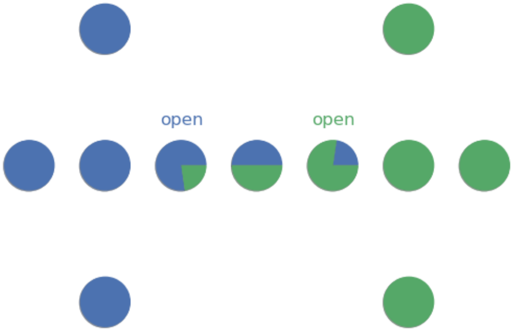
Primal-Dual Feature Allocation

Example: $\lambda = 7$



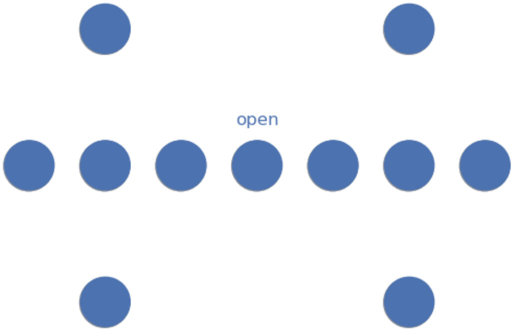
Primal-Dual Feature Allocation

Example: $\lambda = 11$



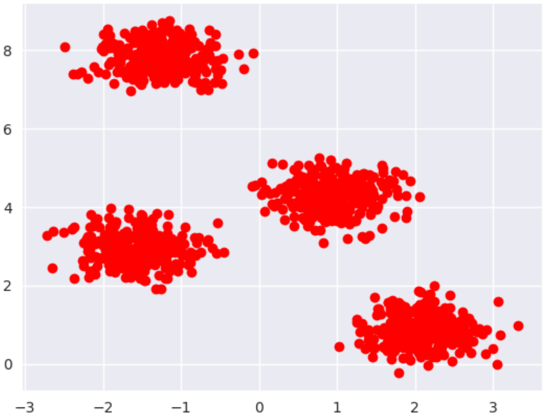
Primal-Dual Feature Allocation

Example: $\lambda = 15$



Primal-Dual Feature Allocation

Compare with BP Means [BKJ]:



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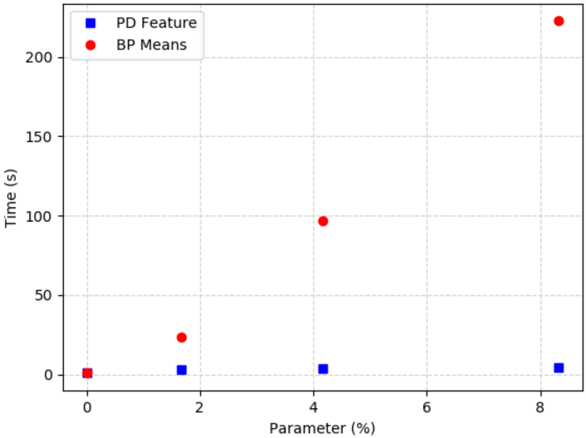

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- Worst case approximation bound: ongoing work

Leaving Out λ

We require $\lambda \geq 0$

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Theorem: The smallest value of λ that allows all edges to go tight is $\lambda^* = \max_i \left(n \cdot \max_j (c_{ij}) - \sum_j c_{ij} \right)$

Leaving Out λ

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Theorem: The smallest value of λ that allows all edges to go tight is $\lambda^* = \max_i \left(n \cdot \max_j (c_{ij}) - \sum_j c_{ij} \right)$

Theorem: For $\lambda \geq \lambda_* = \min_i \left(n \cdot \max_j (c_{ij}) - \sum_j c_{ij} \right)$, OPT $= \min_i \left(\sum_j c_{ij} \right) + \lambda$

Leaving Out λ

Running PD algorithms for multiple values of λ:

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Running PD algorithms without first choosing λ:

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Running PD algorithms for multiple values of λ:

- Before the first facility gets paid for, the algorithm's computations are the same for each value of λ
- Branch computations for a particular λ when it reaches its first paid facility event

Running PD algorithms without first choosing λ:

- Algorithm chooses values of $\lambda \in [0, \lambda^*]$ to test

Leaving Out λ

Same strategies work for running both clustering and feature allocation together

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- Keep the best looking result?

Leaving Out λ

Same strategies work for running both clustering and feature allocation together

- Keep the best looking result?
- Run clustering and feature allocation together, without λ!