Correlation Clustering: Introduction and Innovations

Nathan Cordner

Boston University

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Clustering Problem

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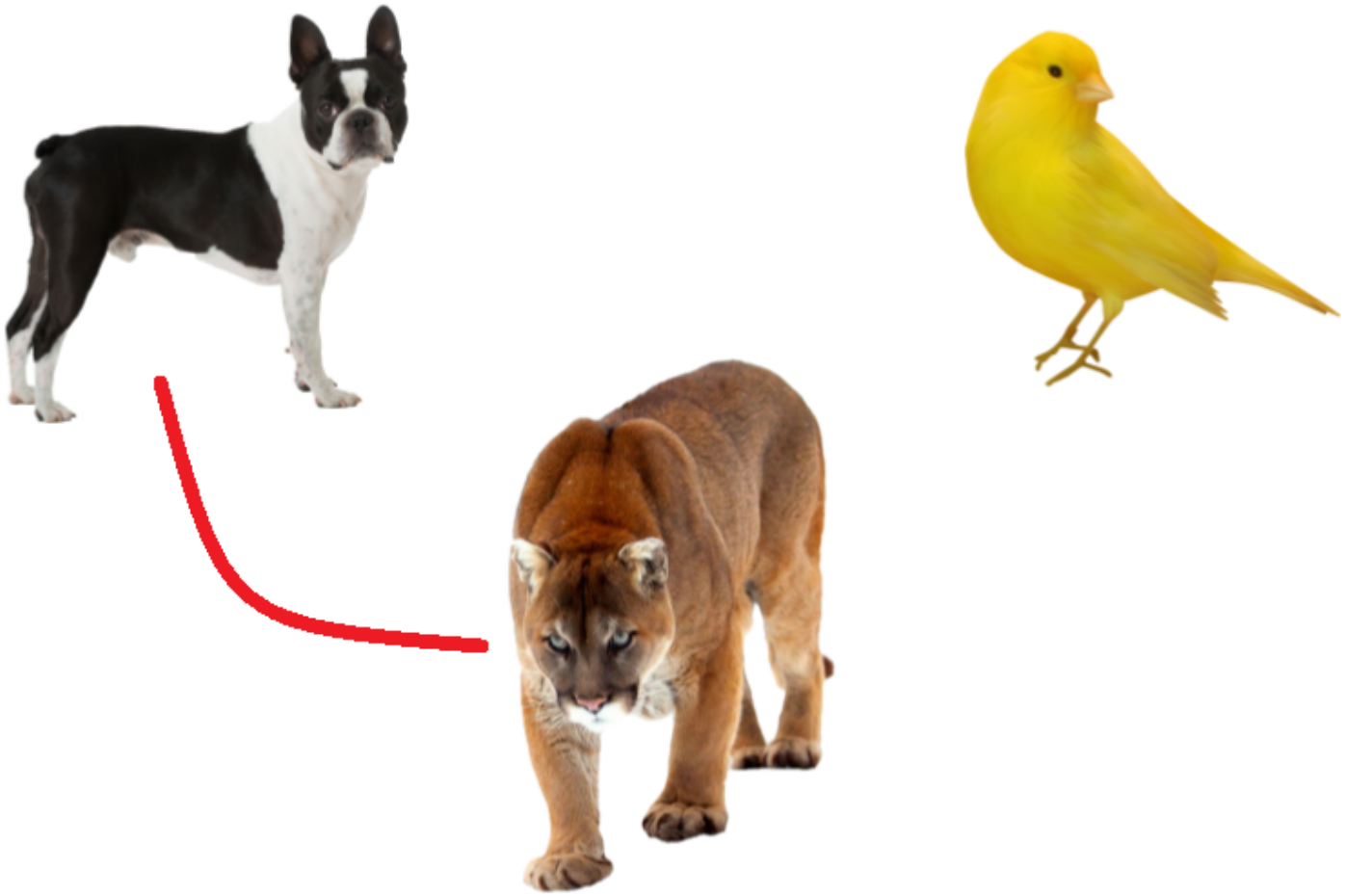

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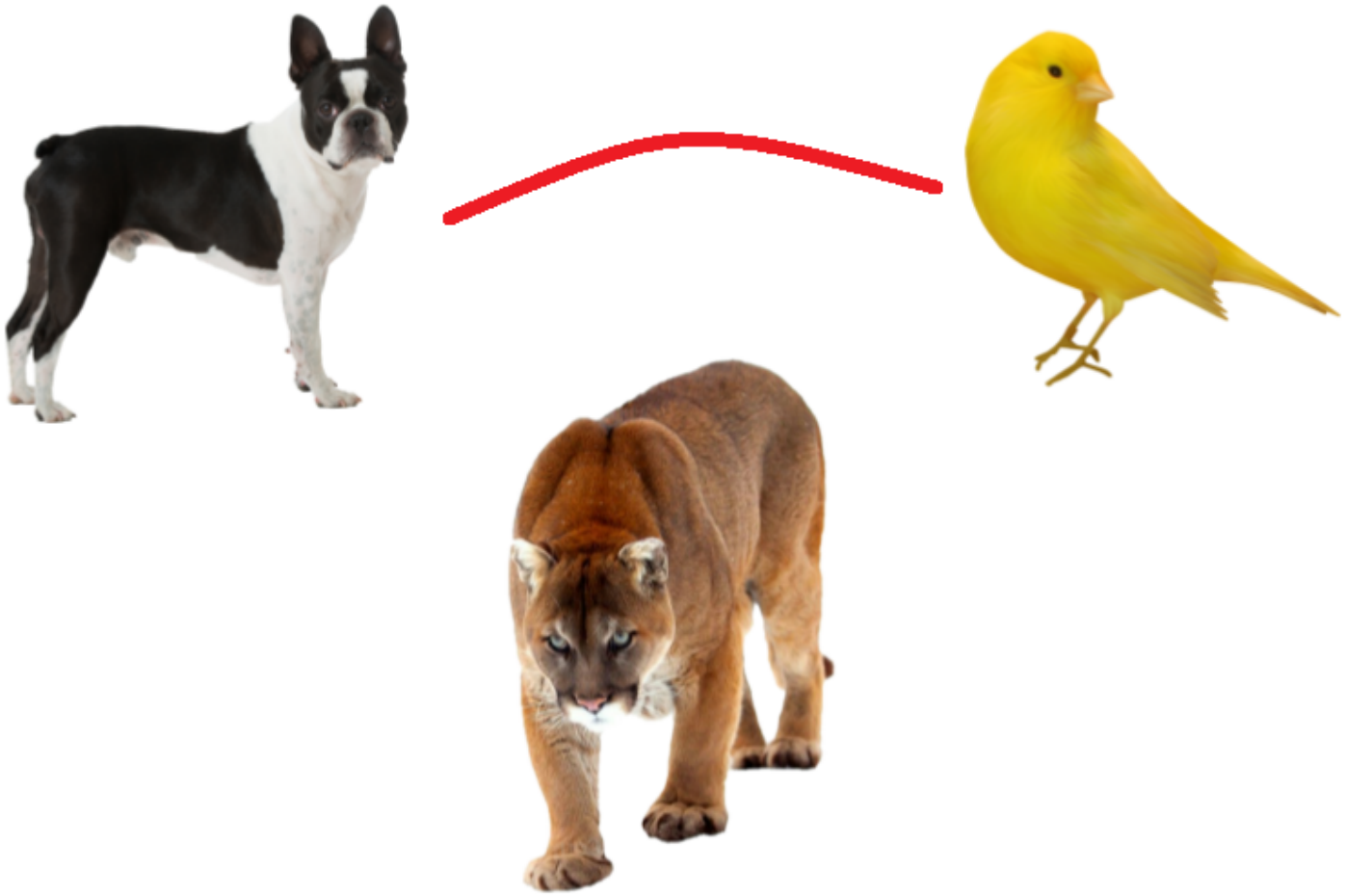

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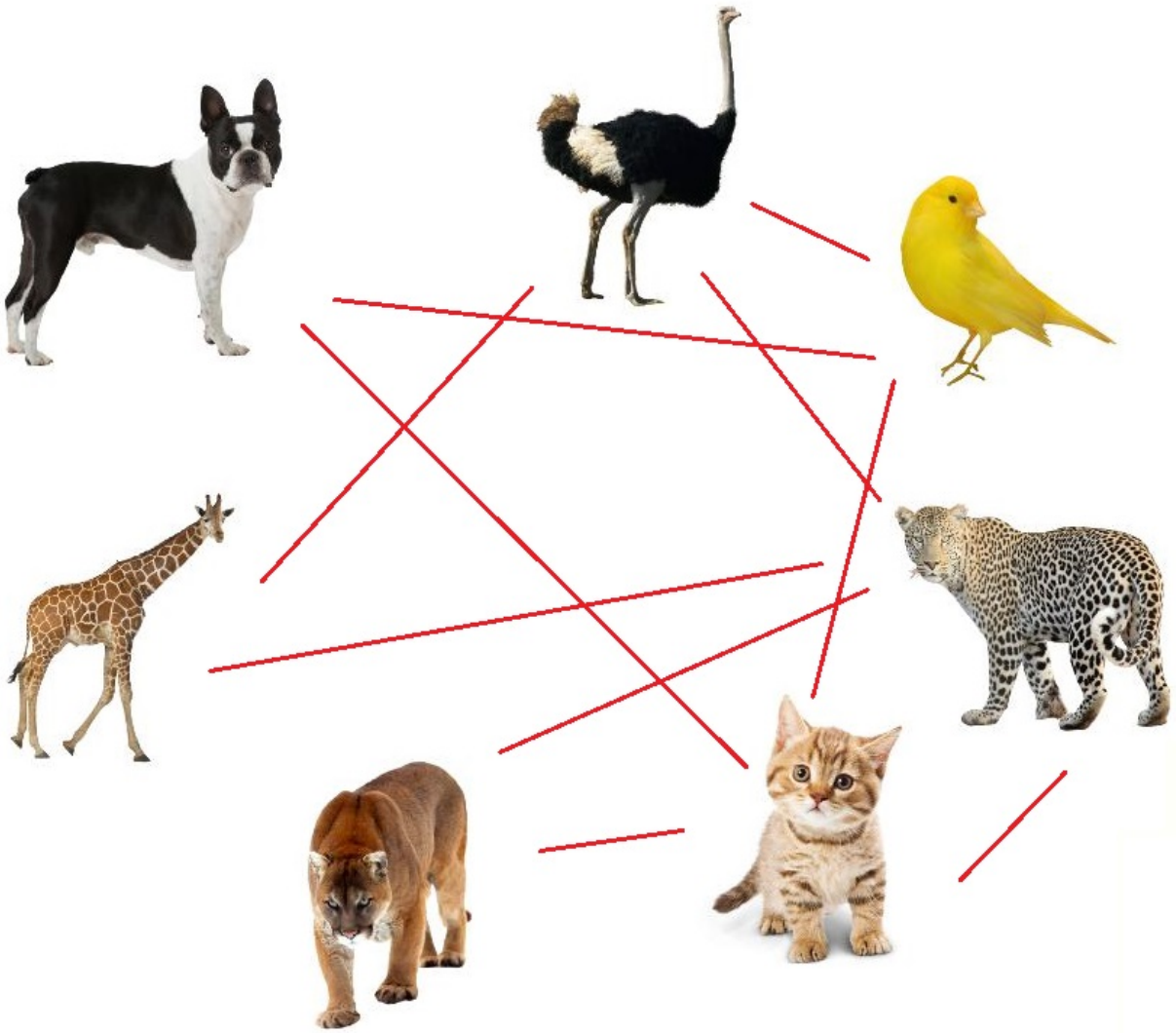

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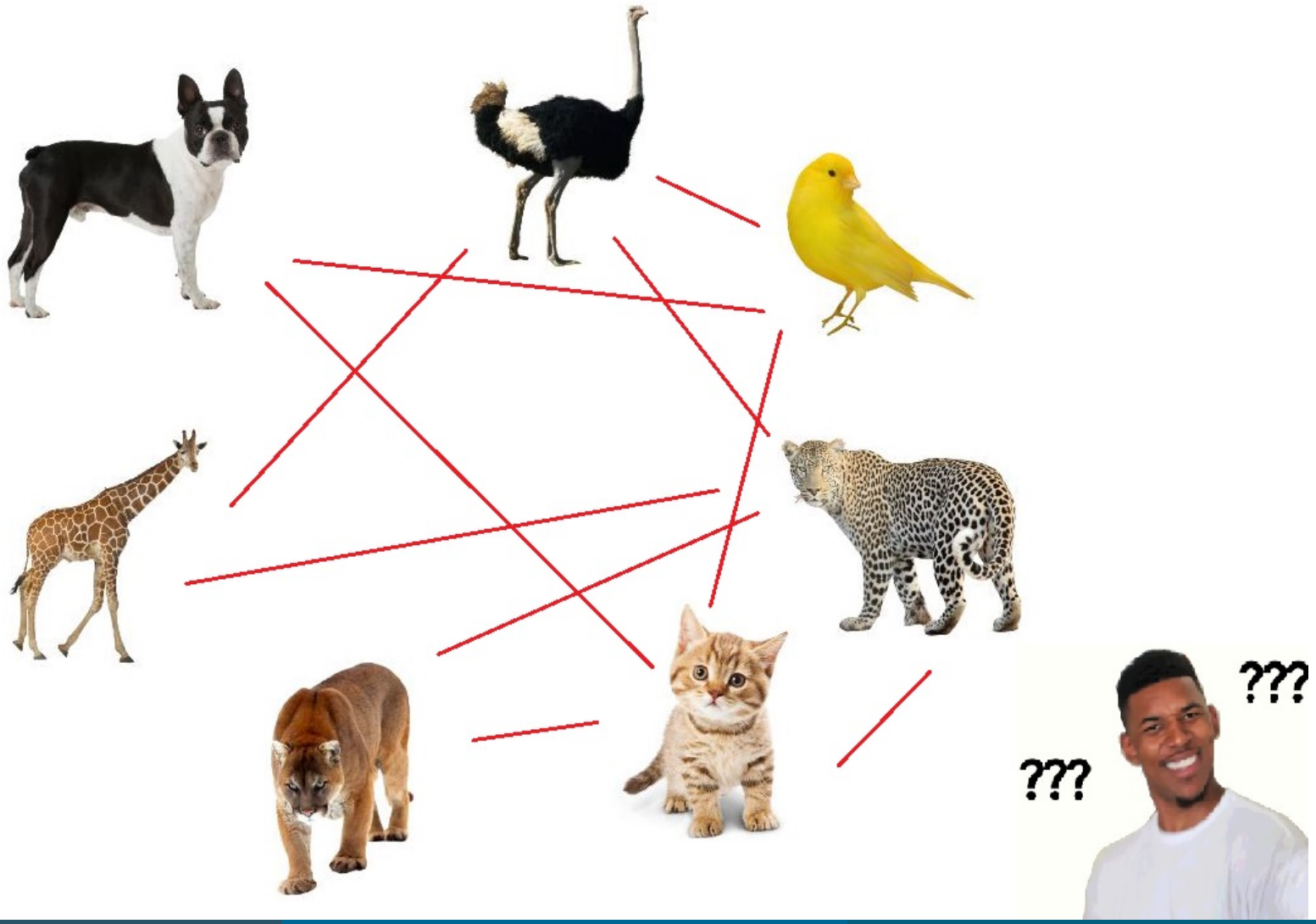


Correlation Clustering [BBC04]

Given a graph $G = (V, E)$

Want to cluster edges and separate non-edges

- Maximize Agreements
- **Minimize Disagreements**

Some Applications

- Classification
- Entity Resolution
- Communities in Social Networks

The Pivot Algorithm [ACN08]

Pivot(V, E):

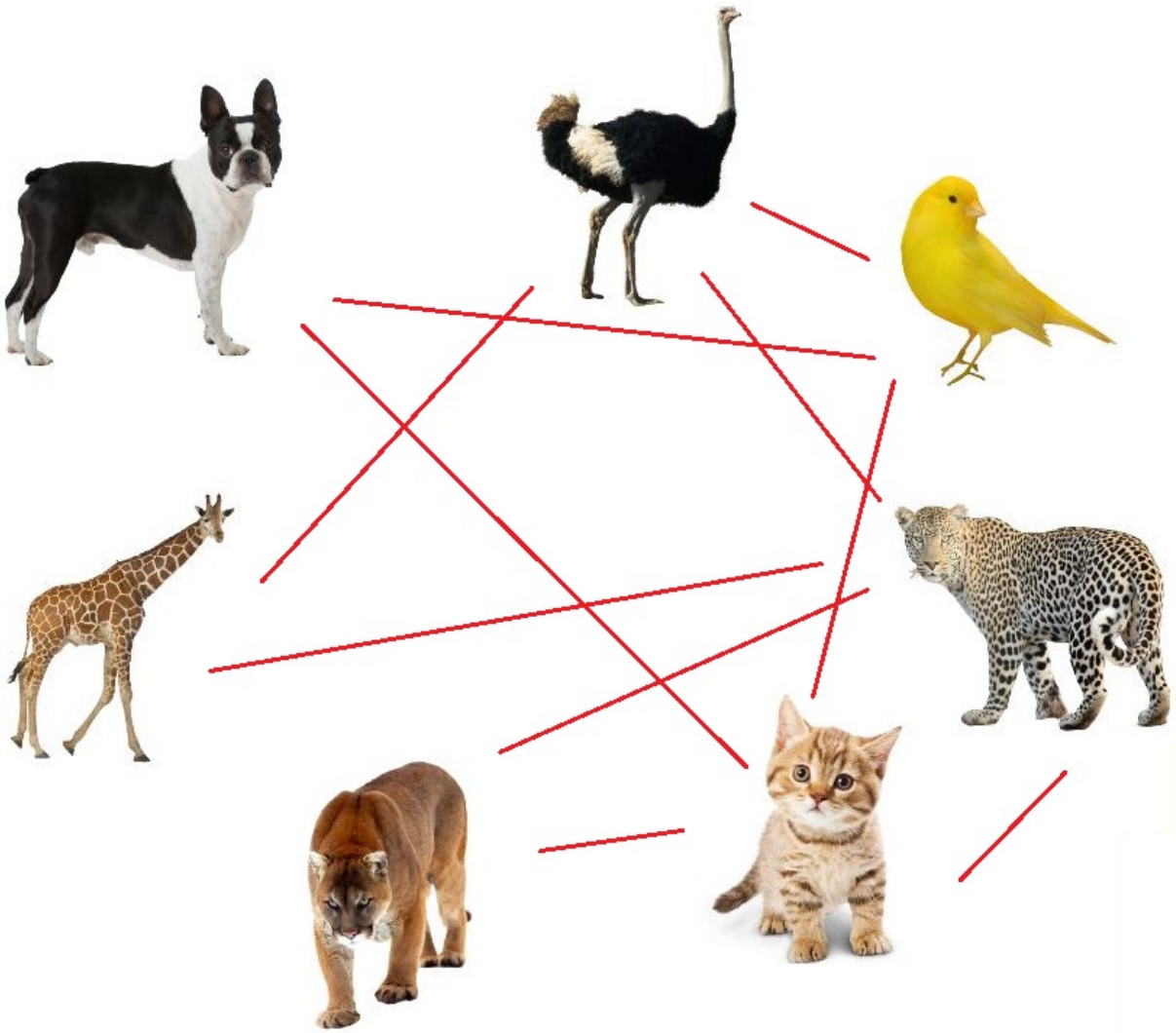
- Pick random pivot node $u \in V$
- Set $C = \{u\}$
- For all $v \in V \setminus \{u\}$:
  - If $(u, v) \in E$: Add v to C
- Repeat on $V = V \setminus C$ until empty
- Return completed clustering

Runs in $O(|V| + |E|)$ time

Randomized expected 3-approximation

The Pivot Algorithm [ACN08]

Example:



LP Methods

Linear Program for Correlation Clustering:

$$\min \sum_{(u,v)\in E} x_{uv} + \sum_{(u,v)\notin E} (1 - x_{uv})$$

$$\begin{aligned} x_{uv} + x_{vw} &\geq x_{uw} && \text{for all } u, v, w \in V \\ x_{uv} &\in [0,1] && \text{for all } u, v \in V \end{aligned}$$

- $O(|V|^2)$ variables, $O(|V|^3)$ constraints!

Rounding Methods:

- 2.5-approx [ACN08]
- Later improved to 2.06-approx [CMSY15]
  - Integrality gap is 2, so this is near optimal

Correlation Clustering Generalizations

Edge Weights [BBC04, ACN08]

- Every pair of nodes u, v has weights $w^+_{uv}, w^-_{uv} \geq 0$
- Clustering Cost:

$$\sum_{u,v \text{ in different clusters}} w^+_{uv} + \sum_{u,v \text{ in same cluster}} w^-_{uv}$$

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State of the Art: LP rounding

- 2.5-approx for probability weights [ACN08]
- $O(\log |V|)$-approx for general case [CGW05, DEFI06]

Correlation Clustering Generalizations

Limited Cluster Sizes

- Given K, each cluster allowed at most K elements

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- 6-approx [PM15]
- Later improved to 5.37-approx [JCTZ21]

Correlation Clustering Generalizations

Chromatic Correlation Clustering [BGTU15]:

- Every edge has a color from label set L
- Assign a dominant color to each cluster formed
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State of the Art: Also Pivot!

- Color-blind Pivot is a 3-approx [KSZC21]
- Other methods have better experimental results

One Algorithm to Rule Them All?

Pivot has been successfully used to cluster

- Social network graphs [KPT11]
- Protein-protein interaction graphs [KPT11; HWH15]
- Event graphs generated from news stories [CMB17]

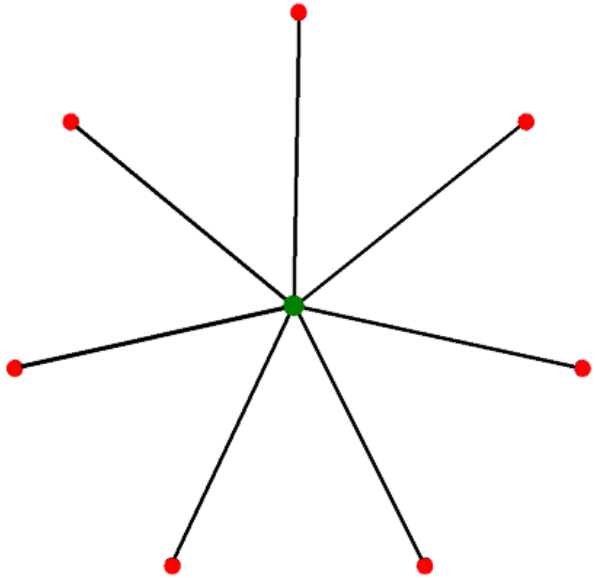
Pivot has been adapted for

- Probabilistic graphs [KPT11; MTG20]
- Fair correlation clustering [AEKM20]
- Data streaming and online settings [ACGM15; LMVW21]
- Query constraints [GKBT20]

Deterministic and parallel versions [ZW09; CDK14; PORJ15]

Drawbacks of Pivot

Pivot can form sparse, star-shaped clusters



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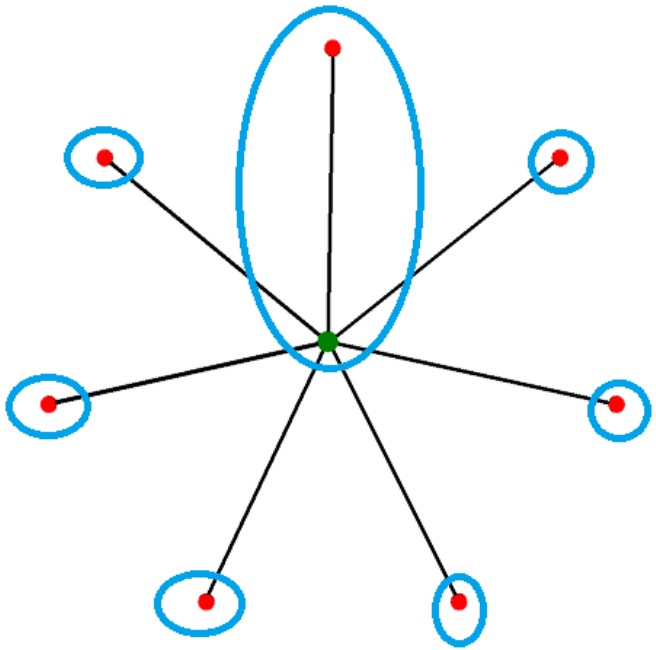

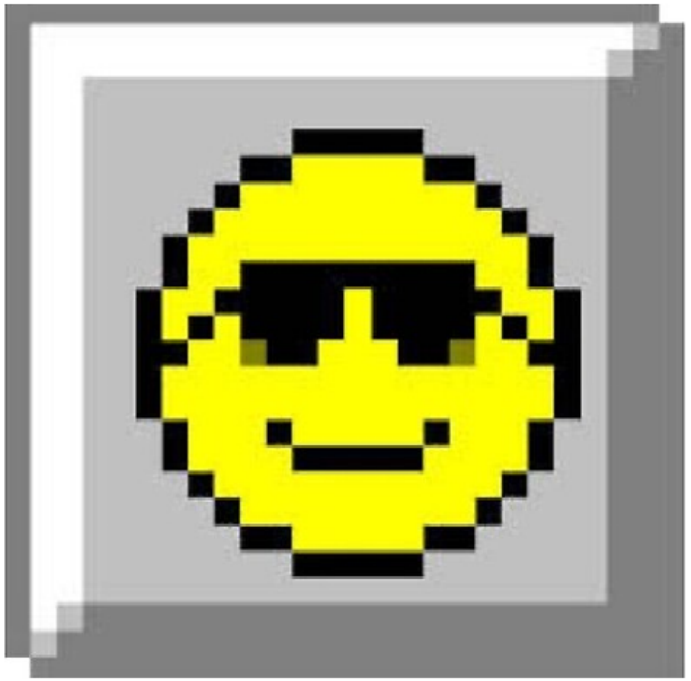

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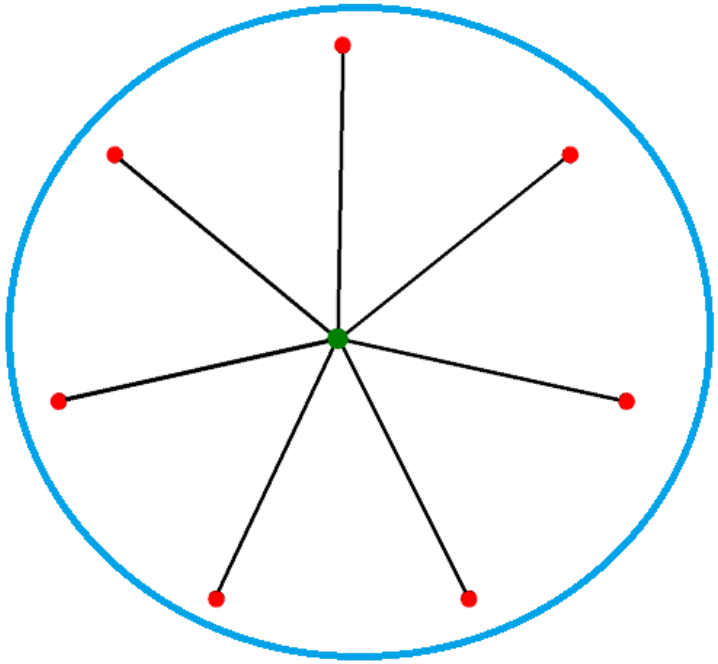

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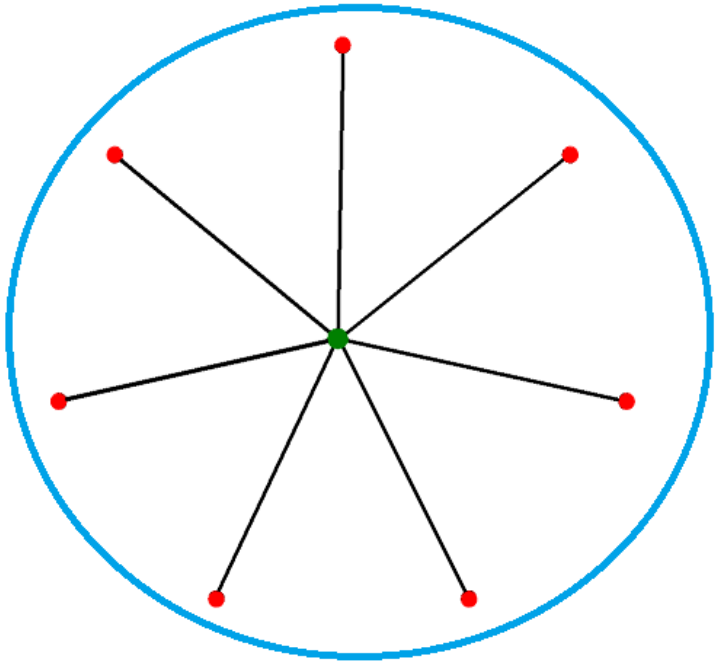

Can we do better and still maintain scalability?

Cluster Improvement

Local Search: given a clustering,

- Each node decides whether to stay or move clusters
- Iterate until improvements stop
- Slow: each iteration is $O(|V| + |E|)$
- Somewhat popular though [MTG20; AEKM20]

New Ideas:

- Idea 1: limit LS to just one iteration
- Idea 2: Run LS *inside* clusters only
  - Special ordering of nodes inside clusters maximizes expected cost improvement

Cluster Improvement: Idea 1

Clustering Precision: the ratio of edges inside clusters to the total number of node pairs within clusters

Lemma: clustering precision after one round of LocalSearch is at least 50%

Proof Idea: Consider node u

- If node u has fewer than 50% edges to other nodes in its current cluster, then separating u into its own cluster will decrease overall clustering cost
- Similarly, if u joins an existing cluster then it must have at least 50% edges present to other nodes in that cluster

Cluster Improvement: Idea 2

Focus on running LocalSearch inside a single cluster

Neighborhood: $N(u) = \{v \in V \mid (u, v) \in E\}$

- Order nodes in cluster by increasing size of $|N(u)|$
  - $O(|V| \log |V|)$ time
- Follow LocalSearch with this node order

Lemma: Following LocalSearch in this order maximizes expected cost decrease after one LocalSearch round

Proof Idea: Nodes with small neighborhood sizes decrease cost more when moved from the current cluster

Inner Local Search

Inner Local Search: on graph G

- Obtain clusters $C_1, \ldots, C_k$ from Pivot(G)
- Let G_i be the graph induced by C_i
- Return LocalSearch$(G_1), \ldots,$ LocalSearch(G_k)

Properties

- Nearly linear running time: $O(|V| \log d + |E|)$
  - d is size of largest Pivot cluster
- Easily run in parallel
- Immediately applies to CC generalizations
- Improves cluster costs from Pivot
- **Approximation Bound**: stay tuned!

Inner Local Search

Examples:

Name	$\lvert\mathbf{V}\rvert$	$\lvert\mathbf{E}\rvert$	$\mathbf{d}$	**Description**
DBLP	317080	1049866	145.1	co-authors network
Amazon	334863	925872	105.0	joint purchases
YouTube	1134890	2987624	452.6	friend network
LiveJournal	3997962	34681189	895.6	friend network
Orkut	3072441	117185083	1287.0	friend network

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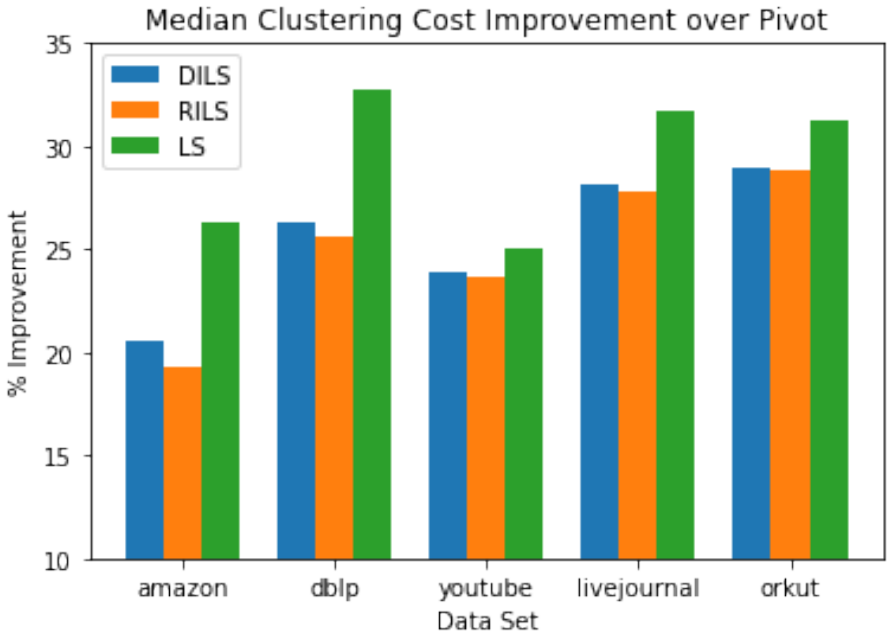
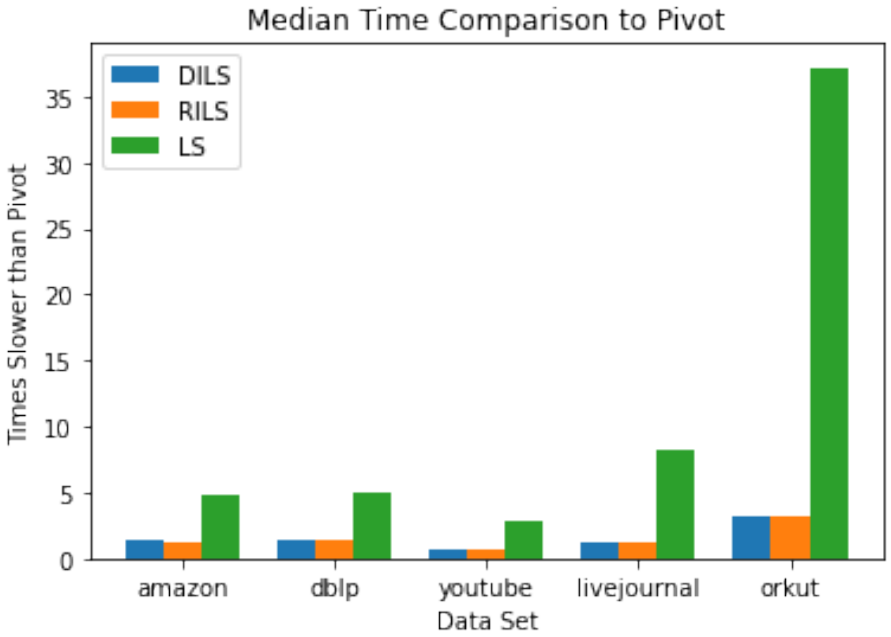
Inner Local Search

Examples: ILS gives nearly the same improvement as LS, but in a lot less time!





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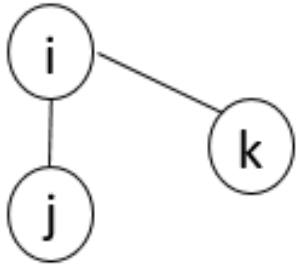
Inner Local Search

Theoretical Improvements



"Bad Triangles": i, j, k unclustered

- Two edges exist but the third is absent

Lemma [ACN08]: Approx bound of Pivot $\leq$ worst cost ratio for bad triangles

- Triangle completely inside cluster when i is chosen as pivot (1/3 chance)
- **Claim**: ILS reduces average cost of bad triangles inside Pivot clusters by half
- ILS approximation bound:

$$3((1/3)(\textbf{ILS triangle cost}) + 2/3) = 2.5$$

Applications and Limitations

ILS immediately applies to several CC generalizations:

- Weighted / probabilistic graphs
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ILS does not work as well for some other variants:

- Fair correlation clustering
- Data streaming and online settings
- Query constraints
- **Constrained number of clusters**

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Given k, find clustering with at most k clusters

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  - Pivot-like 3-approximation [BBC04]
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- General case: $(1 + \epsilon)$ PTAS [GG06]
  - *Extremely* inefficient: $|V|^{O(9^k/\epsilon^2)} \log |V|$ running time
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Goal: develop algorithm that is both time-efficient *and* has provable approximation guarantees

The Vote Algorithm [ES09]

Pick unclustered nodes one at a time

- First node creates its own cluster
- All others: add to existing cluster, or create own
- Greedily minimize increase in clustering cost

Previous Results

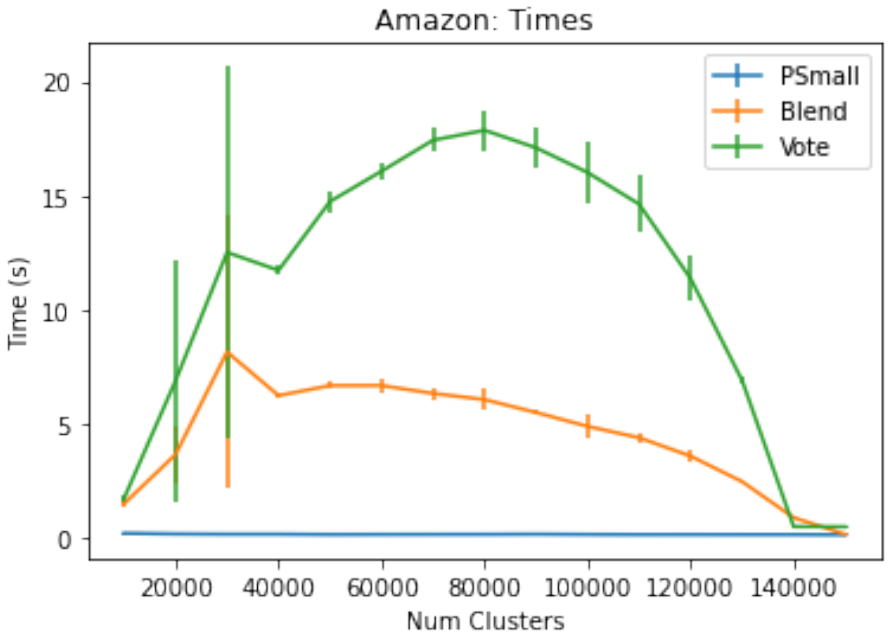
- Experimentally better than Pivot (e.g. [ES09])
- Much slower, though complexity is still $O(|V| + |E|)$

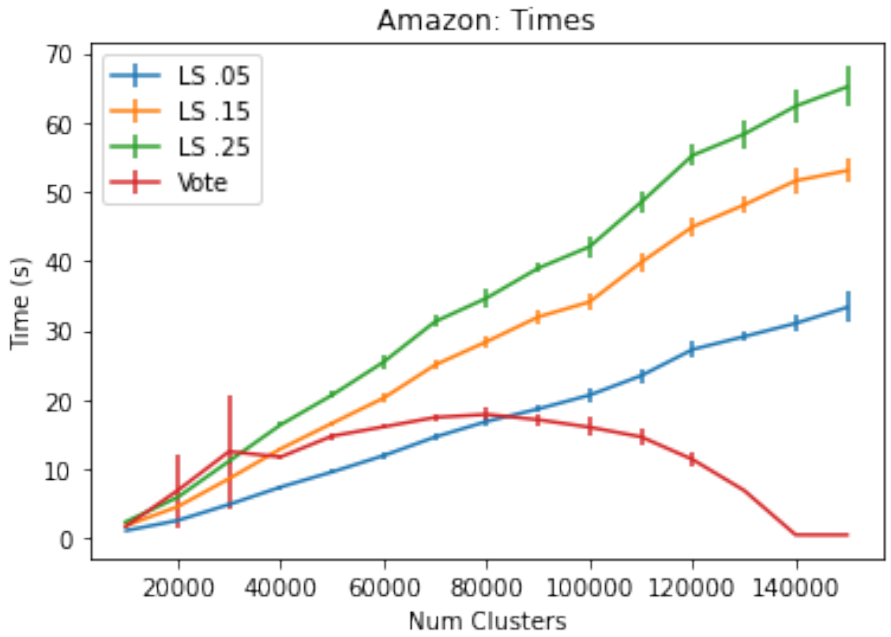
New k-CC Strategies

1. Run Pivot until k clusters are formed, then...
   - Merge new Pivot clusters into existing ones
     - Merge at random
     - Merge in order
     - **Merge to the current smallest cluster**
   - Add remaining nodes to existing clusters using the Vote algorithm
2. Run Vote until k clusters are formed, then continue without the option to form new clusters

Claim: k-Vote and k-Pivot-and-Vote ("Blend") are 7-approximation algorithms

k-CC Experiments





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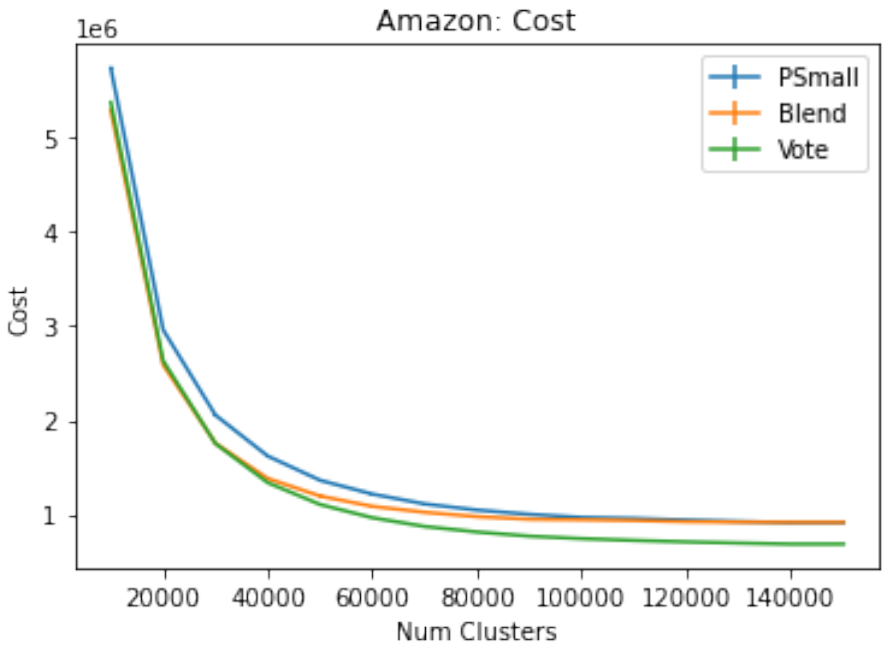

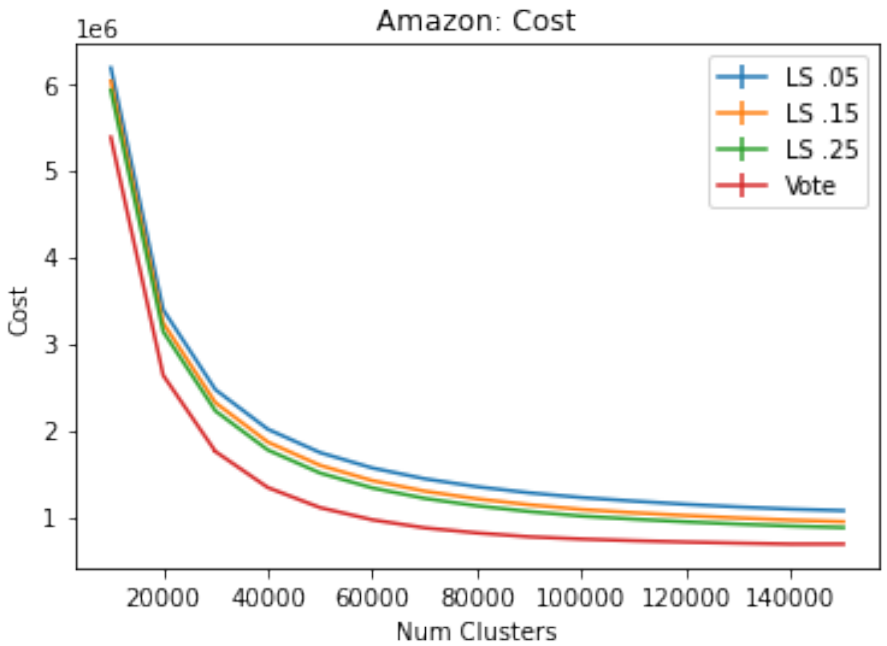

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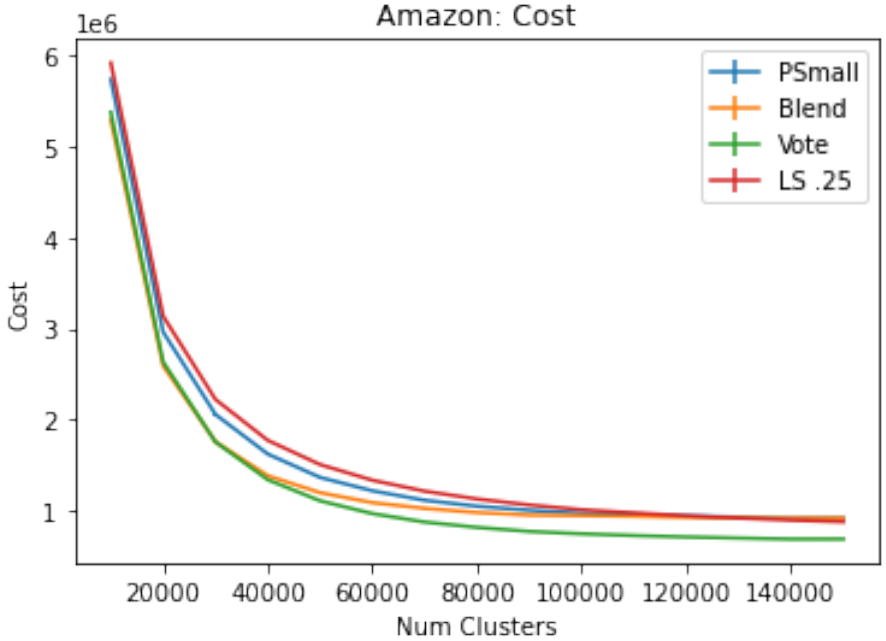

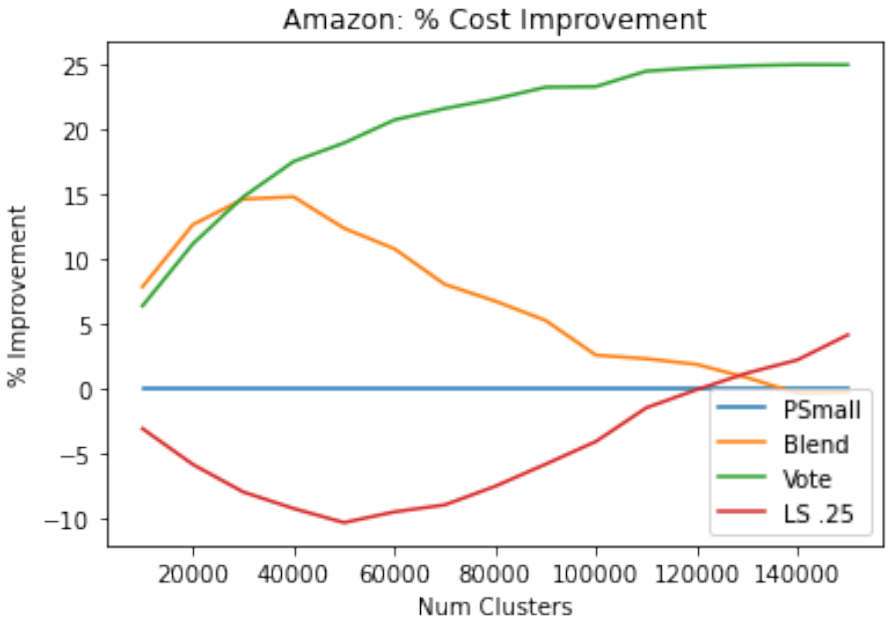

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