Problem 1. Let $R$ be a non-trivial ring. Show that for some $a, b \in R$ such that $ab = 1$, if either $a$ or $b$ is a Zero Divisor then $ba = 1$.
Problem 2. Let $S$ and $T$ be subrings of ring $R$. Show that $S \cap T$ is also a subring of $R$.

$S$ is (an additive) subring of $R$:

(i) $S$ is an additive subgroup of $R$

i.1) $a + b \in S$, for $a, b \in S$.

i.2) $-a \in S$, for $a \in S$.

(ii) $ab \in S$ for $a, b \in S$.

(iii) $1_R \in S$.

Let $a, b \in S \cap T \Rightarrow a, b \in S; a, b \in T$

Since $S, T$ are subrings of $R$, we have:

\[
\begin{align*}
  a + b & \in S \cap T & a, b & \in S \\
  ab & \in S \cap T & a, b & \in T
\end{align*}
\]

$\Rightarrow a + b \in S \cap T \& ab \in S \cap T$

Let $x \in S \cap T \Rightarrow x \in S \& x \in T$

Since $S, T$ are subrings, we have:

$-x \in T \& -x \in S \Rightarrow -x \in S \cap T$

$\Rightarrow$ every element in $S \cap T$ has an additive inverse.

Similarly, $1_R \in S \cap T$.

Hence $S \cap T$ is a subring.

What about $S \cup T$? Not quite. $S \cup T$ is a subring ($\Rightarrow \bigcap_{t \in T} S \cap T$).
Problem 3. Show that if $F$ is a field, the units in $F[X]$ are exactly nonzero elements of $F$.

$F$ is a field $\implies F$ is a ring and every element in $F$ has a multiplicative inverse:
\[ \text{i.e. } \alpha \in F, \exists \beta = \alpha^{-1}, \alpha \beta = 1 \]
\[ \implies \beta \text{ is the unit.} \]

Ring / field $F$ has elements which are numbers, $F[X]$ has \textit{polynomials} (e.g. $X^2 - 2X + 1 = f(X)$).

\[ \text{Let } f(X) \in F[X] \text{ of degree } n \]
Then $f(X)$ is a unit if $\exists g(X) \in F[X]$ of degree $m$ s.t. $f(X) \cdot g(X) = 1$
\[ \deg(f \cdot g) = \deg(f) + \deg(g) = n + m \]
\[ \deg(1) = 0 \implies n + m = 0 \]
Since $n, m > 0 \implies n = m = 0$
\[ \implies f(X) \text{ and } g(X) \text{ are constant functions} \]
\[ \implies f \cdot g = 1 \implies f \text{ & } g \text{ are units of } F \]

$F \subset F[X]$.