

Discussion 9

Tuesday, April 20th, 2021.

Problem 1. Integral Domain. If $x \in R$, an integral domain, and $x^2 = 1$. Show that $x = 1$ or $x = -1$.

R is an integral domain

$$\Rightarrow \forall a, b \in R \text{ and } a, b \neq 0 \quad | \\ a \cdot b \neq 0$$

$$\therefore \text{ if } xy = 0 \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

Pf: $x^2 = 1 \Rightarrow x^2 - 1 = 0$

$$\Rightarrow (x-1)(x+1) = 0$$

$$\Rightarrow \begin{cases} x-1 = 0 \\ x+1 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x = 1 \\ x = -1 \end{cases} \text{ (qed)}$$

Problem 2. Ring Homomorphism and Isomorphism.

(a) Find a ring homomorphism from the ring $R[x, y]$ to $R[y]$.

Recall: $\rho: R \rightarrow R'$ is ring homomorphism:
 for $a, b \in R$, $\rho(a+b) = \rho(a) + \rho(b)$ ✓
 $\rho(ab) = \rho(a)\rho(b)$ ✓
 $\rho(1_R) = \rho(1_{R'})$ ✓

If: Define $\rho: R[x, y] \rightarrow R[y]$

where ρ substitutes $x = 0$

$\therefore \rho$ maps $f(x, y) \in R[x, y]$ to $f(0, y)$
 $= f(y) \in R[y]$

Consider some $f_1(x, y), f_2(x, y) \in R[x, y]$

$$\begin{aligned}\rho(f_1(x, y) + f_2(x, y)) &= f_1(0, y) + f_2(0, y) \\ &= \rho(f_1(0, y)) + \rho(f_2(0, y))\end{aligned}$$

$$\rho(f_1(x, y)f_2(x, y)) = \rho(f_1(x, y))\rho(f_2(x, y))$$

$$\rho(1_R) = \rho(1_{R'}) = 1$$

$\Rightarrow \rho: R[x, y] \rightarrow R[y]$ is homomorphic

$R[x, y] \not\cong R[y]$

- (b) Use the ring homomorphism that you found in the previous question to show the following isomorphism: $R[x, y]/(x) \cong R[y]$, where (x) is the principal ideal of $R[x, y]$ generated by $x \in R$.

Recap: First isomorphic Thm:

$$\rho: R \rightarrow R'$$

$$R/\ker(\rho) \cong \text{Im}(\rho)$$

where $\text{Im}(\rho) = \rho(R) = \{\rho(a) \mid a \in R\}$

$$\ker(\rho) = \{a \in R \mid \rho(a) = 0\}$$

$$\begin{array}{ccc} x^2 + y & \xrightarrow{\quad} & y+1 \\ x^3 + y+1 & \xrightarrow{\quad} & \end{array} \quad \begin{array}{l} I \subseteq R \\ a, b \in I, r \in R \\ a+b \in I, ar \in I, -a \in I \end{array} \quad \begin{array}{l} \text{def Ideal} \\ R[y] \end{array}$$

$$R[x, y]/(x) \cong R[y]$$

(x) : principal ideal of $R[x, y]$

$$(x) = \{x f(x, y) \mid f(x, y) \in R[x, y]\}$$

$$R = R[x, y], \ker(\rho) = (x)$$

$$\text{Im}(\rho) = R[y]$$

$$R[y] = R[0, y] \subset R[x, y]$$

(continue) $\Rightarrow \text{Im}(\ell) = R[y]$

To show: $\text{Ker}(\ell) = (x)$ (2)
 $\therefore \text{Ker}(\ell) \subseteq (x)$; $(x) \subseteq \text{Ker}(\ell)$

(1): Let $f(x, y) \in \text{Ker}(\ell)$

$$\ell(f(x, y)) = 0$$

Then, $f(x, y)$ can be written as

$$f(x, y) = xc(f_1(x, y) + \dots + f_{nA}(x, y))$$

$xc \bullet \underbrace{f'(x, y)}$

where $f'(x, y) \in R[x, y]$

$$\Rightarrow f(x, y) \in (x) \Rightarrow \text{Ker}(\ell) \subseteq (x)$$

(2) Let $f(x, y) \in (x)$

$$\Rightarrow \exists g(x, y) \in R[x, y] \text{ s.t.}$$

$$xg(x, y) = f(x, y) \text{ (by def (x))}$$

$$\ell(f(x, y)) = \ell(xg(x, y)) = 0 \cdot g(0, y) = 0$$

$$\Rightarrow g(x, y) \in \text{Ker}(\ell)$$

$$\Rightarrow (x) \subseteq \text{Ker}(\ell) \stackrel{4}{\Rightarrow} (x) = \text{Ker}(\ell)$$

By 1st \cong Thm: $R[x, y]/(x) \cong R[y]$