Lecture 7: Effective Resistance

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1 Review: Graphs as electric circuits

- The weights of the graph edges $w_{i,j}$ are the electrical conductance of the respective connections, i.e., the reciprocal of the resistances of the edges. We denote by W the diagonal matrix of edge conductances.
- The vector $\chi_{st} = e_s - e_t$ is the demand vector when one unit of current flows from vertex s to vertex t.
- B is the $m \times n$ incidence matrix of the graph. The row corresponding to edge $\{i, j\}$ is χ_{ij}. An arbitrary orientation of the edges is chosen (in the previous example, from i to j).
- Ohm's Law: $f = WBv$, where f is the flow
- Conservation Law: $B^T f = \chi_{st}$, flow is created only at the sources and sinks.
- From this we have $B^T(WBv) = \chi_{st} = Lv$, where L is the graph Laplacian.
- Pseudo-Inverse: When the spectral decomposition is $L = \sum_{i=1}^{n} \lambda_i v_i v_i^T$, then the pseudo inverse is $L^+ = \sum_{i=1}^{n} \frac{1}{\lambda_i} v_i v_i^T$
- This implies that $LL^+ = (I - \pi_{\text{null}(L)})$, where $\pi_{\text{null}(L)}$ is the orthogonal projection on the null space of L, i.e, $\pi_{\text{null}(L)} = \sum_{i:\lambda_i \neq 0} v_i v_i^T$.
- This last theorem, together with the fact that χ_{st} is orthogonal to the null space of L (assuming G is connected) implies that the voltage vector of the electrical flow sending one unit of current from s to t can be solved as:
$$v = L^+ \chi_{st}.$$

2 Optimization Characterization of Electrical Flows

We start this lecture by providing an optimization view of electrical flows in terms of an energy minimzation problem over voltages. While in the last lecture, we fixed the current going from s to t, we are now going to fix the voltage gap between s and t to one.

In this scenario, always assuming G is connected, some current will flow from s to t, inducing some voltages over G. It turns out that the energy consumed by edge $\{i, j\}$ with a voltage gap of

$(x_i - x_j)$ can be expressed as $w_{ij}(x_i - x_j)^2$. Optimizing over the total energy of the graph, while keeping the gap from s to t fixed, yields the following optimization problem.

$$\min_{x \neq 0} \sum_{\{i,j\} \in E} w_{i,j}(x_i - x_j)^2 = \min_{x \neq 0} x^T L x$$
$$\text{s.t. } (x_s - x_t) = 1.$$

Theorem 1. *The optimal solution is* $x^* = \frac{1}{\chi_{st}^T L^+ \chi_{st}} L^+ \chi_{st}$. *The optimum is* $\frac{1}{\chi_{st}^T L^+ \chi_{st}}$.

Proof. Because the objective function is convex and the feasible set is the linear subspace $\chi_{st}^T x = 1$, a voltage x will be optimal if the gradient of the objective (which is $2Lx$) is parallel to χ_{st}, as at that point it will not be possible to move within the feasible set while decreasing the objective. This means that we want, for some $\lambda \in \mathbb{R}$:

$$Lx^* = \lambda \chi_{st}.$$

We must choose λ such that x^* is feasible, i.e. $\chi_{st}^T x^* = 1$. Hence, we have $\lambda = \frac{1}{\chi_{st} L^+ \chi_{st}}$ and

$$x^* = \frac{1}{\chi_{st} L^+ \chi_{st}} \cdot L^+ \chi_{st},$$

and the optimum is

$$x^{*T} L x^* = \frac{1}{\chi_{st}^T L^+ \chi_{st}}$$

□

3 Effective Resistance

The quantity $\frac{1}{\chi_{st}^T L^+ \chi_{st}}$ is known as the *effective conductance* between s and t. As shown in the previous section, it is the energy consumed by the electrical flow when a voltage gap of 1 is set between s and t. Notice that, under the same assumption, the flow going from s to t is $\frac{1}{\chi_{st} L^+ \chi_{st}}$:

$$B^T(R^{-1} B x^*) = Lx^* = \frac{1}{\chi_{st} L^+ \chi_{st}} \chi_{st}.$$

This shows that, with respect to electrical flows going from s to t, the circuit graph G behaves like a single resistor with conductance equal to the effective conductance. This also applies to the energy of the circuit, which is the voltage times the effective conductance. Usually, these relations are expressed in terms of the reciprocal of the effective conductance, i.e., the effective resistance.

Definition 1. For a connected graph G and pair of vertices s and t, the effective resistance R_{st} between s and t is $\chi_{st} L^+ \chi_{st}$.

In the next section, we will see how the electrical concepts discussed so far and in the previous lecture can cast light on the behavior of random walks over the graph G.

3.1 Eigenvalues and Random Walks

Recall that the second eigenvalue of the normalized Laplacian and its associated eigenvector are the optimum and optimal solution of the following optimization problem:

$$\lambda_2 = \min_{x^T D\mathbf{1}=0} \frac{x^T L x}{x^T D x}$$

A few lectures ago, we saw that λ_2 is related to the worst-case mixing of a random walk over G. In particular, we showed that for any initial probablity distribution p_0 over V, we have:

$$\left\|W^T \rho - \pi\right\|_E^2 \leq (1-\lambda_2)^{2t} \|p_0\|_E^2$$

A similar relation exists between the effective conductance and the behavior of a random walk from s to t. An intuition behind this is that electrons follow a natural random walk over the graph, where edges are picked based on their conductance. Consider for instance the following probability vector $h \in \mathbb{R}^V$:

$$h_u = \Pr[\text{a random walk starting at u gets to s before getting to t}]$$

We can take a recursive approach to find h. By definition, we have:

$$h_s = 1 \text{ and } h_t = 0$$

For $v \neq s$ and $v \neq t$, we can also see the effect of taking one step of the random walk:

$$h_v = \sum_{u \sim v} \frac{1}{d_v} h_u = \frac{1}{d_v} \sum_{u \sim v} h_u$$

Hence, we have:

$$h_v = (W^T h)_v$$

and, for $v \notin \{s, t\}$,

$$[(I - W^T)h]_v = (D^{-1} L h)_v = 0$$

This essentially means that h behaves like a voltage vector for an unknown vector of input/output flows, which is only supported on s and t. In other words, for some values of c and λ, we must have:

$$Lh + c\vec{1} = \lambda \chi_{st}.$$

We require c because voltages are determined up to a translation and we actually require $h_s = 1$. To figure out what λ is, we solve $h = \lambda L^+ \chi_{st}$ and notice that we require $h_s - h_t = 1$. Hence, we have

$$\lambda = \frac{1}{\chi_{st} L^+ \chi_{st}}.$$

and λ is the effective resistance between s and t. This makes sense, as we proved above that this is the quantity of flow running into s and out of t when there is a voltage gap of one unit between s and t. We can then also solve for c to get:

$$h = R_{st} L^+ \chi_{st} + (1 - R_{st} e_s^T L^+ \chi_{st})\vec{1}.$$

4 Other Properties of Effective Resistance

4.1 Monotonicity

Theorem 2. *Let v be a vector of voltages, i.e., given $v = L^+\chi_{st}$. Then, for all $c \in V$*

$$v_s \geq v_c \geq v_t$$

Proof. Notice that for all $c \neq s, t$

$$h_c = \frac{1}{d_c}\sum_{u\sim c} h_u$$

In words, the voltage at c is the harmonic mean(average) of the voltages of its neighbors.[1] By virtue of being an average, the voltage at c can neither be the maximum nor the minimum value among its neighbors. This implies that the only vertex whose voltage is maximum among its neighbors is s and the only one for which it is minimum is t. Because G is connected, this implies the statement of the theorem. □

4.2 Metric Property

Theorem 3. *The effective resistance R^{eff} is a metric: for any $a, b, c \in V$, we have*

$$R_{a,b} \leq R_{a,c} + R_{c,b}$$

Proof. The theorem statement is equivalent to:

$$(e_a - e_b)^T L^+ (e_a - e_b) \leq (e_a - e_c)^T L^+ (e_a - e_c) + (e_c - e_b)^T L^+ (e_c - e_b)$$

The latter is equivalent to:

$$-2e_a^T L^+ e_b \leq 2e_c^T L^+ e_c - 2e_a^T L^+ e_c - 2e_b^T L^+ e_c$$

Hence, it suffices to prove that:

$$e_c^T L^+ e_c - e_a^T L^+ e_c \geq e_b^T L^+ e_c - e_b^T L^+ e_a$$

But this is the same as:

$$e_c^T L^+ (e_c - e_a) \geq e_b^T L^+ (e_c - e_a)$$

This is easily seen to be true by the monotonicity law for an electrical flow of one unit going from c to a. □

Notice that this is a very special property because effective resistance is an ℓ_2^2-seminorm, i.e., a distance of the form:

$$d_{i,j} = \|x_i - x_j\|^2$$

where $x_i = L^{-\frac{1}{2}} e_i$. We saw when discussing Cheeger's Inequality that, in general, this kind of distance do not satisfy the triangle inequality and hence, are not metrics.

[1] Functions defined over graphs that have this property are said to be *harmonic*.