On Partitioning Graphs via Single Commodity Flows

Lorenzo Orecchia
UC Berkeley

Leonard J. Schulman
Caltech

Umesh V. Vazirani
UC Berkeley

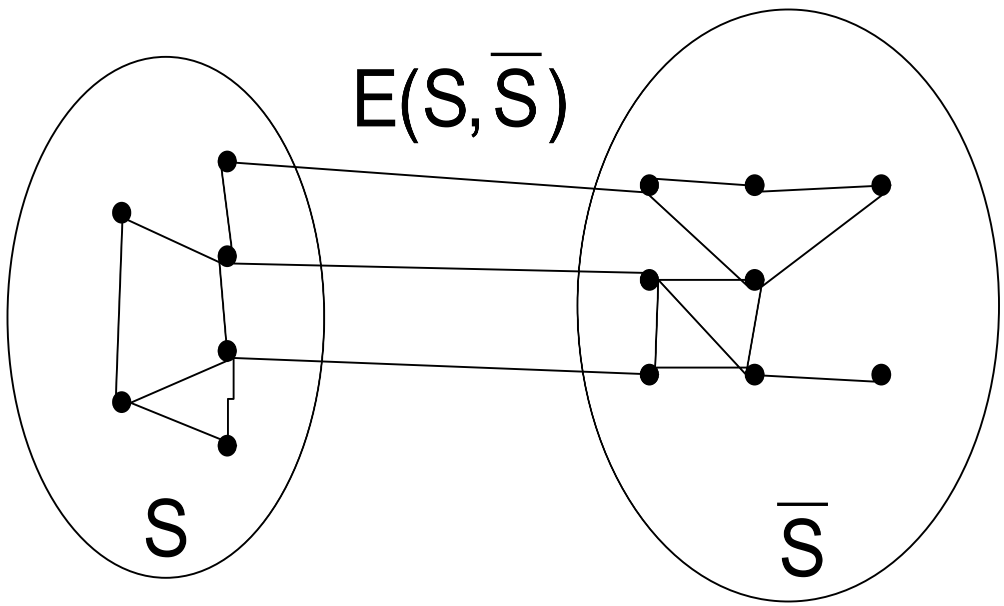
Nisheeth K. Vishnoi
IBM Delhi – work done while visiting UC Berkeley

STOC 2008 , Victoria

The SPARSEST CUT problem

Given a graph G=(V,E) and partition $(S,\bar{S})$

$$\text{Expansion of } (S,\bar{S}) = \frac{|E(S,\bar{S})|}{\min\{|S|,|\bar{S}|\}}$$



The SPARSEST CUT problem

Given a graph G=(V,E) and partition $(S,\overline{S})$

$$\text{Expansion of } (S,\overline{S}) = \frac{|E(S,\overline{S})|}{\min\{|S|,|\overline{S}|\}}$$

SPARSEST CUT:

find $(S,\overline{S})$ with <u>minimum</u> expansion $\phi(G)$.

Applications: Divide-and-Conquer, Image Segmentation, VLSI design, Clustering.

Theoretical Importance: Metric Embeddings, Spectral Methods.

The SPARSEST CUT problem

Given a graph G=(V,E) and partition $(S,\overline{S})$

$$\text{Expansion of } (S,\overline{S}) = \frac{|E(S,\overline{S})|}{\min\{|S|,|\overline{S}|\}}$$

SPARSEST CUT:

find $(S,\overline{S})$ with <u>minimum</u> expansion $\phi(G)$.

Applications: Divide-and-Conquer, Image Segmentation, VLSI design, Clustering.

Theoretical Importance: Metric Embeddings, Spectral Methods.

The **SPARSEST CUT** problem is **NP-hard**.

Approximation Algorithms for SPARSEST CUT

Algorithm	Output Expansion	Running Time *
Spectral	$2\sqrt{d\phi}$ **	$O\left(\frac{d^2 n}{\phi^2}\right)$ **
Leigthon-Rao	$\phi \log n$	$\tilde{O}(n^2)$
ARV	$\phi\sqrt{\log n}$	[ARV] poly(n) [AHK] $\tilde{O}(n^2)$

*All graphs have been sparsified to $\tilde{O}(n)$ edges. ** For a d-regular graph G.

Approximation Algorithms for SPARSEST CUT

Algorithm	Output Expansion	Running Time *
Spectral	$2\sqrt{d\phi}$ **	$O\left(\frac{d^2 n}{\phi^2}\right)$ **
Leigthon-Rao	$\phi \log n$	$\tilde{O}(n^2)$
ARV	$\phi\sqrt{\log n}$	[ARV] poly(n) [AHK] $\tilde{O}(n^2)$

IN PRACTICE: Too slow for massive data sets.

Spectral and heuristics like METIS used instead.

*All graphs have been sparsified to $\tilde{O}(n)$ edges. ** For a d-regular graph G.

Fast Approximation Algorithms

Algorithm	Output Expansion	Running Time *
Spectral	$2\sqrt{d\phi}$ **	$O\left(\frac{d^2 n}{\phi^2}\right)$ **
Leigthon-Rao	$\phi \log n$	$\tilde{O}(n^2)$
ARV	$\phi\sqrt{\log n}$	[ARV] poly(n) [AHK] $\tilde{O}(n^2)$
KRV	$\phi(\log n)^2$	$\tilde{O}(n^{3/2})$

CUT-MATCHING GAME: FRAMEWORK FOR COMPUTING APPROX USING s-t MAXFLOW COMPUTATIONS

*All graphs have been sparsified to $\tilde{O}(n)$ edges. ** For a d-regular graph G.

Fast Approximation Algorithms

Algorithm	Output Expansion	Running Time
Leigthon-Rao	$\phi \log n$	$\tilde{O}(n^2)$
ARV	$\phi\sqrt{\log n}$	[ARV] poly(n) [AHK] $\tilde{O}(n^2)$
KRV	$\phi(\log n)^2$	$\tilde{O}(n^{3/2})$
AK	$\phi \log n$	$\tilde{O}(n^{3/2})$

*All graphs have been sparsified to $\tilde{O}(n)$ edges. ** For a d-regular graph G.

Our Contribution

Algorithm	Output Expansion	Running Time
KRV	$\phi(\log n)^2$	$\tilde{O}(n^{3/2})$
AK	$\phi \log n$	$\tilde{O}(n^{3/2})$
THIS PAPER	$\phi \log n$	$\tilde{O}(n^{3/2})$

IN KRV CUT-MATCHING GAME FRAMEWORK

Our Contribution

Algorithm	Output Expansion	Running Time
KRV	$\phi(\log n)^2$	$\tilde{O}(n^{3/2})$
AK	$\phi \log n$	$\tilde{O}(n^{3/2})$
THIS PAPER	$\phi \log n$	$\tilde{O}(n^{3/2})$

IN KRV CUT-MATCHING GAME FRAMEWORK

LOWER BOUND

No better approx than $\Omega((\log n)^{1/2})$ in KRV framework.

Our Contribution

Algorithm	Output Expansion	Running Time
KRV	$\phi(\log n)^2$	$\tilde{O}(n^{3/2})$
AK	$\phi \log n$	$\tilde{O}(n^{3/2})$
THIS PAPER	$\phi \log n$	$\tilde{O}(n^{3/2})$

LOWER BOUND

No better approx than $\Omega((\log n)^{1/2})$ in KRV framework.

Best integrality gap for SDP is $\Omega(\log\log(n))$.

Our Contribution

Algorithm	Output Expansion	Running Time
KRV	$\phi(\log n)^2$	$\tilde{O}(n^{3/2})$
AK	$\phi \log n$	$\tilde{O}(n^{3/2})$
THIS PAPER	$\phi \log n$	$\tilde{O}(n^{3/2})$

LOWER BOUND

No better approx than $\Omega((\log n)^{1/2})$ in KRV framework.

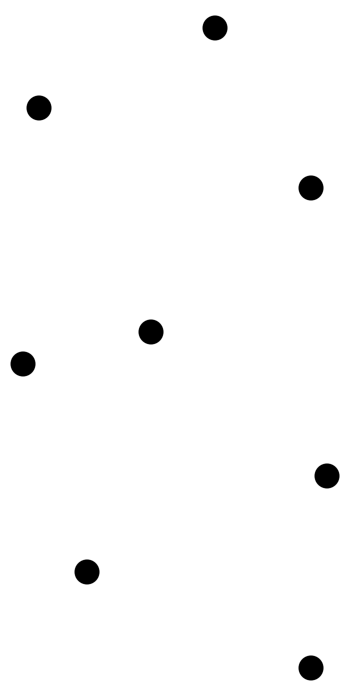
Best integrality gap for SDP is $\Omega(\log\log(n))$.

CUT-MATCHING RIGHT ABSTRACTION?

The KRV Cut-Matching Game

CUT PLAYER **MATCHING PLAYER**

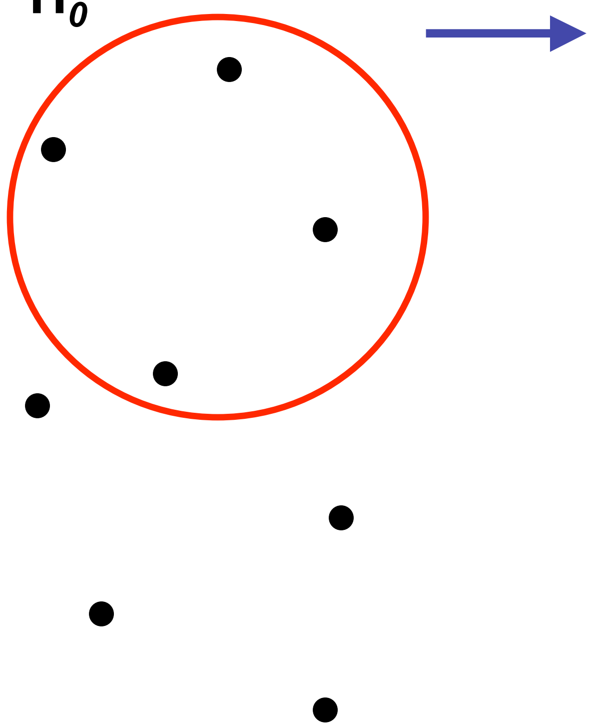
H_0



The KRV Cut-Matching Game

CUT PLAYER **MATCHING PLAYER**

H_0

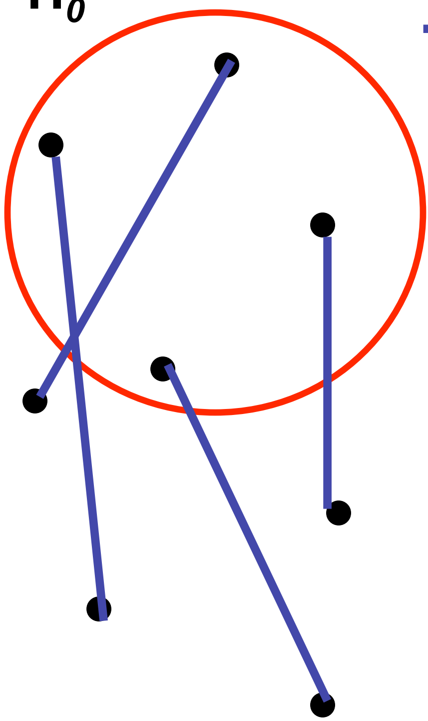


$(S_1, \overline{S}_1)$

50-50 Cut

The KRV Cut-Matching Game

H_0



CUT PLAYER

$(S_1, \overline{S}_1)$

50-50 Cut

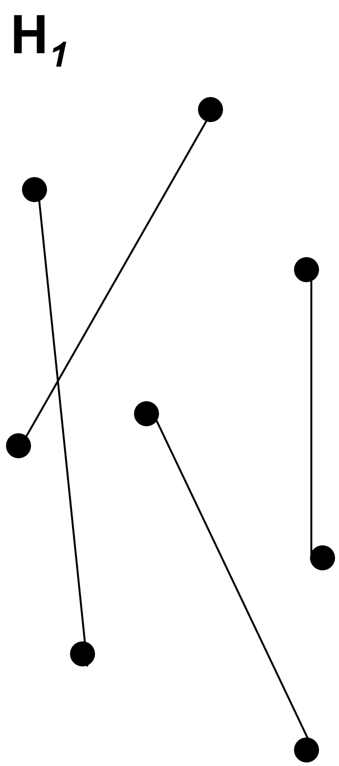
MATCHING PLAYER

M_1

Perfect Matching

The KRV Cut-Matching Game

CUT PLAYER **MATCHING PLAYER**



$(S_1, \overline{S}_1) \longrightarrow M_1$

50-50 Cut

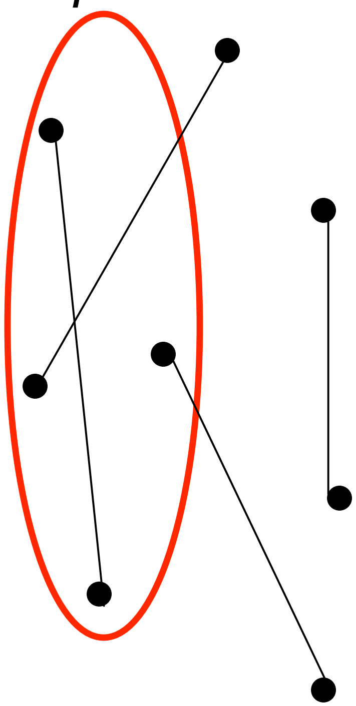
Perfect Matching

The KRV Cut-Matching Game

CUT PLAYER

MATCHING PLAYER

H_1



$(S_1, \overline{S}_1)$

50-50 Cut

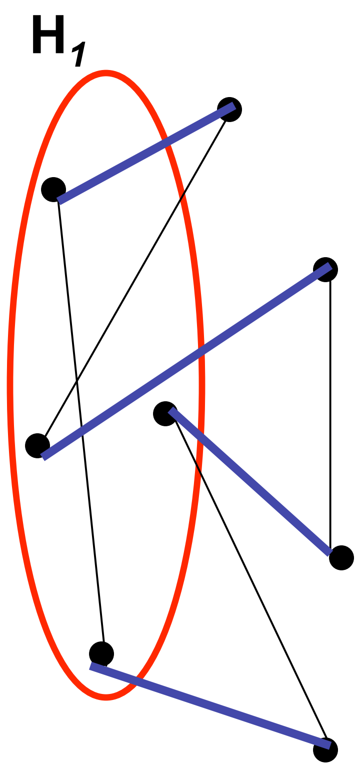
M_1

Perfect Matching

$(S_2, \overline{S}_2)$

50-50 Cut

The KRV Cut-Matching Game



CUT PLAYER | **MATCHING PLAYER**

$(S_1, \overline{S}_1)$ M_1

50-50 Cut **Perfect Matching**

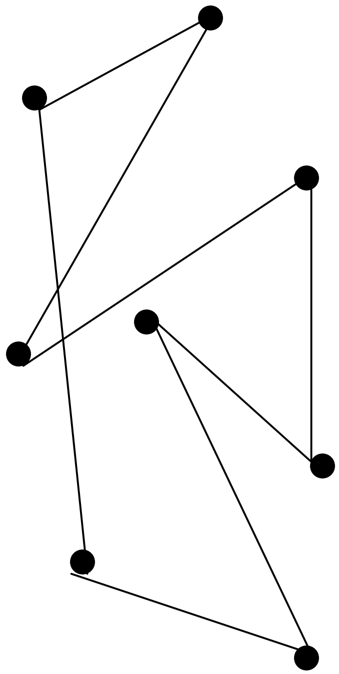
$(S_2, \overline{S}_2)$ M_1

50-50 Cut **Perfect Matching**

The KRV Cut-Matching Game

	CUT PLAYER		MATCHING PLAYER
	$(S_1, \overline{S}_1)$ **50-50 Cut**	$\rightarrow$	M_1 **Perfect Matching**
	$(S_2, \overline{S}_2)$ **50-50 Cut**	$\rightarrow$	M_1 **Perfect Matching**
		…	

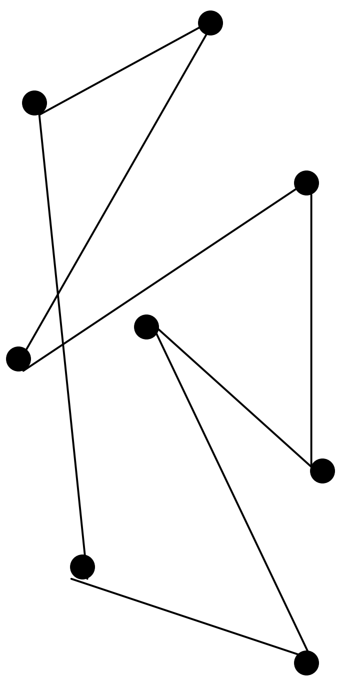
H_2



The KRV Cut-Matching Game

CUT PLAYER | **MATCHING PLAYER**

H_2



$(S_1, \overline{S}_1)$ → M_1

50-50 Cut | **Perfect Matching**

$(S_2, \overline{S}_2)$ → M_1

50-50 Cut | **Perfect Matching**

...

Go until time T when $\phi(H_T) \geq \frac{1}{4}$

GOAL:

Minimize T

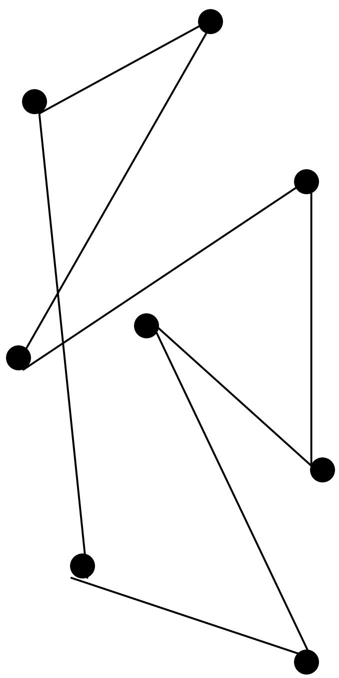
GOAL:

Maximize T

The KRV Cut-Matching Game

CUT PLAYER **MATCHING PLAYER**

H_2



$\longrightarrow (S_1, \overline{S}_1) \longrightarrow M_1$

50-50 Cut **Perfect Matching**

$\longrightarrow (S_2, \overline{S}_2) \longrightarrow M_1$

50-50 Cut **Perfect Matching**

…

Go until time T when $\phi(H_T) \geq \frac{1}{4}$

GOAL:	GOAL:
Minimize T	**Maximize T**

KRV: there exists a cut strategy achieving **$T = O((\log n)^2)$**.

The KRV Cut-Matching Game

Runs in time **c(n)** per iteration

CUT PLAYER STRATEGY

T = t(n)

The KRV Cut-Matching Game

Runs in time **c(n)** per iteration

CUT PLAYER STRATEGY

$T = t(n)$

$\Downarrow$

Running time:
t(n)· ($T_{maxflow}$ + c(n))

APPROXIMATION ALGORITHM

Approx Ratio**:**
t(n)

The KRV Cut-Matching Game

Runs in time **c(n)** per iteration

CUT PLAYER STRATEGY

$T = t(n)$

Running time:
t(n)· (T$_{\textbf{maxflow}}$ + c(n))

APPROXIMATION ALGORITHM

Approx Ratio: **t(n)**

Time to compute s-t maxflow in **G**

$\tilde{O}(n^{3/2})$

The KRV Cut-Matching Game

Runs in time **c(n)** per iteration

CUT PLAYER STRATEGY

$T = t(n)$

Running time:
$\mathbf{t(n)\cdot (T_{maxflow} + c(n))}$

APPROXIMATION ALGORITHM

Approx Ratio: **t(n)**

$\tilde{O}(n^{3/2})$

KRV strategy has **c(n) =** $\tilde{O}(n)$ and **t(n)= O((log n)²).**

TOTAL RUNNING TIME: $\tilde{O}(n^{3/2})$

Our Version of the Cut-Matching Game

- MODIFIED GAME

1. No Stopping Condition
2. Value of Game is $\dfrac{\phi(H_T)}{T}$

Our Version of the Cut-Matching Game

- MODIFIED GAME

1.No Stopping Condition

2.Value of Game is $\frac{\phi(H_T)}{T}$

- STILL YIELDS APPROX ALGORITHM

Approximation Ratio = $\frac{\phi(H_T)}{T}$

Our Version of the Cut-Matching Game

- MODIFIED GAME

1. No Stopping Condition
2. Value of Game is $\dfrac{\phi(H_T)}{T}$

- STILL YIELDS APPROX ALGORITHM

Approximation Ratio = $\dfrac{\phi(H_T)}{T}$

- RESULTS

CUT STRATEGY: $$\frac{\phi(H_T)}{T} = \frac{\Omega(\log n)}{O(\log^2 n)} = \Omega\left(\frac{1}{\log n}\right)$$

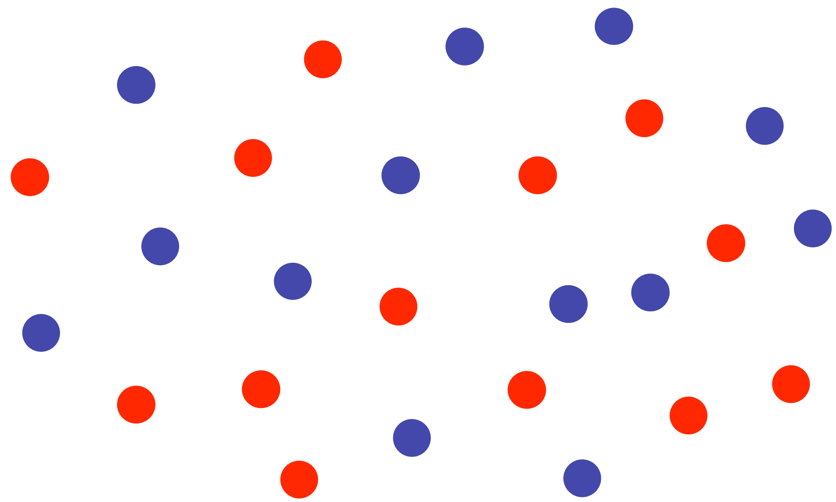
MATCHING STRATEGY: $$\frac{\phi(H_T)}{T} = O\left(\frac{1}{\sqrt{\log n}}\right)$$

Cut Strategies: Finding Cuts Quickly

After t iterations, $\mathbf{H_t} = \{ M_1, M_2, \ldots, M_t \}$.

= **+1** charge **Random assignment of charge**

= **–1** charge

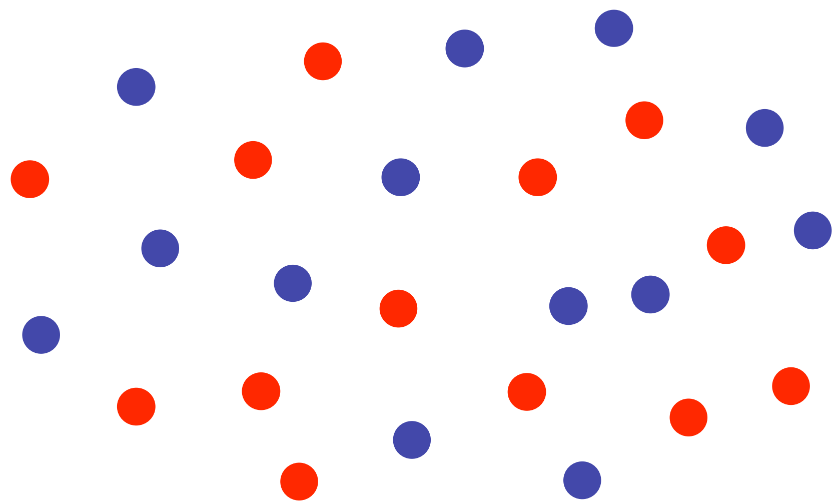


Cut Strategies: Finding Cuts Quickly

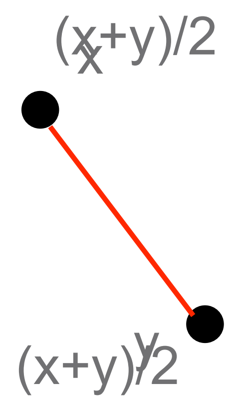
After t iterations, **H_t = { M_1, M_2, …, M_t }.**

= **+1** charge **Random assignment of charge**

= **–1** charge



Mix the charges along the matchings { M_1, M_2, …, M_t}



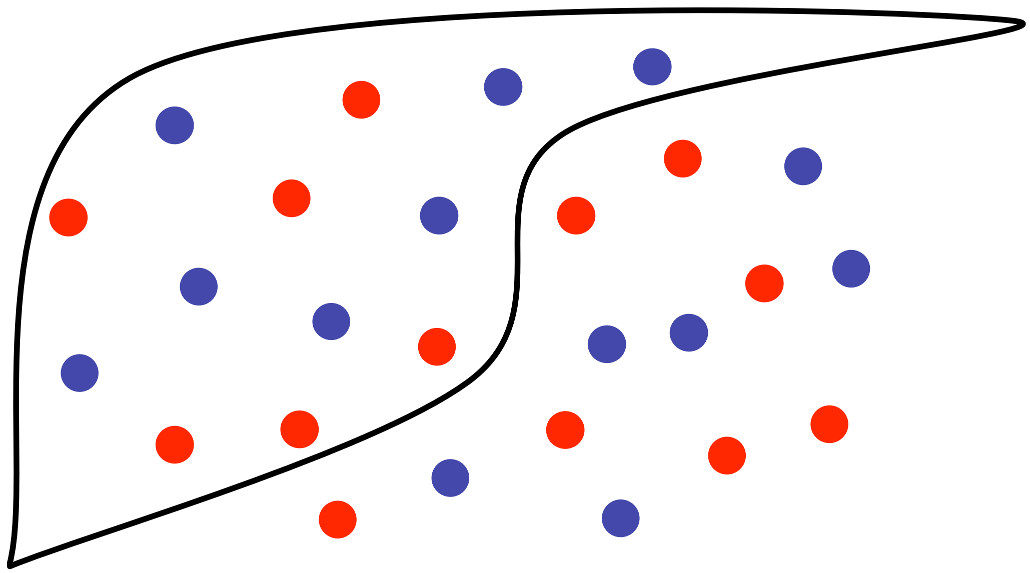
Cut Strategies: Finding Cuts Quickly

After t iterations, $\mathbf{H_t} = \{ M_1, M_2, \ldots, M_t \}$.

= **+1** charge

= **–1** charge

Random assignment of charge



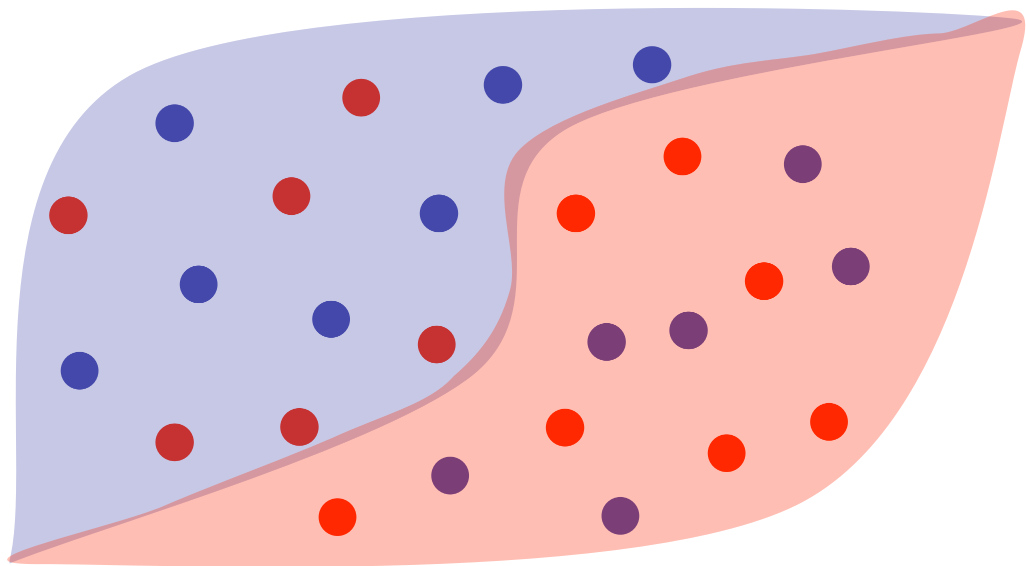
Cut Strategies: Finding Cuts Quickly

After t iterations, $\mathbf{H_t} = \{ M_1, M_2, \ldots, M_t \}$.

 = **+1** charge

 = **–1** charge

Random assignment of charge



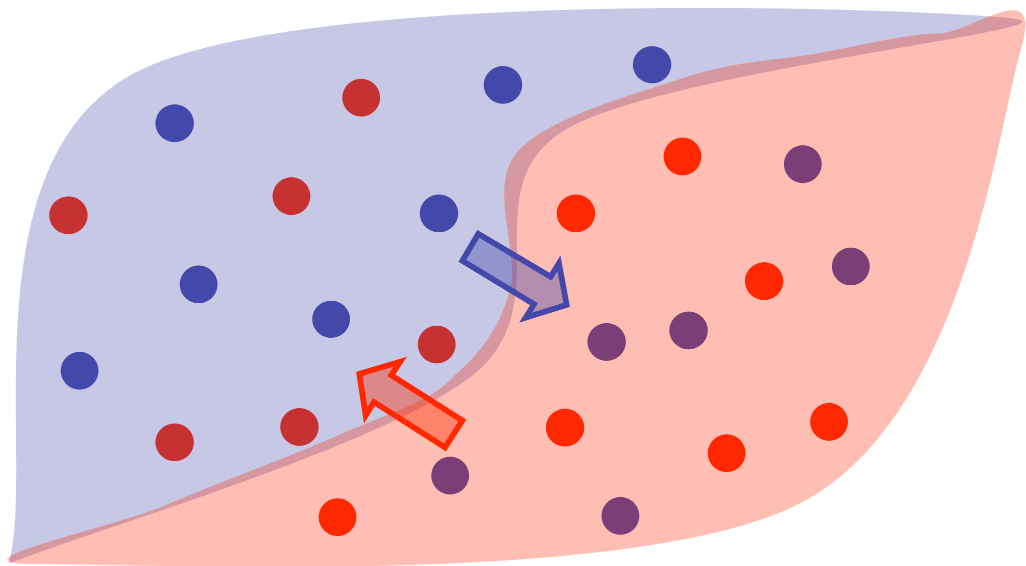
Cut Strategies: Finding Cuts Quickly

After t iterations, $\mathbf{H_t}$ **=** **{ M_1, M_2, …, M_t }.**

● = **+1** charge

● = **–1** charge

Random assignment of charge



Mix the charges along
the matchings { M_1, M_2, …, M_t }

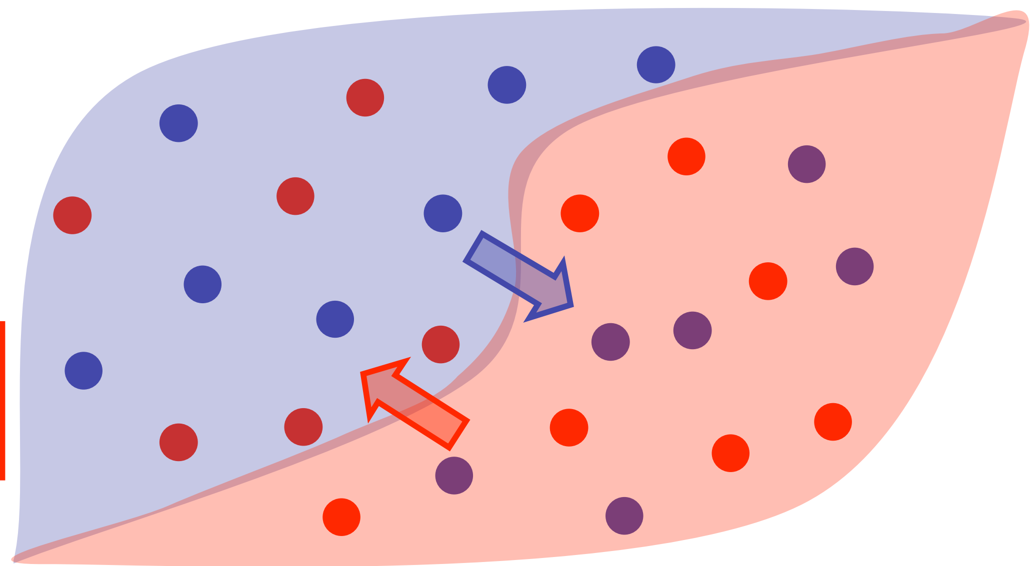
Cut Strategies: Finding Cuts Quickly

After t iterations, $\mathbf{H_t} = \{ M_1, M_2, \ldots, M_t \}$.

= **+1** charge

= **–1** charge

Random assignment of charge

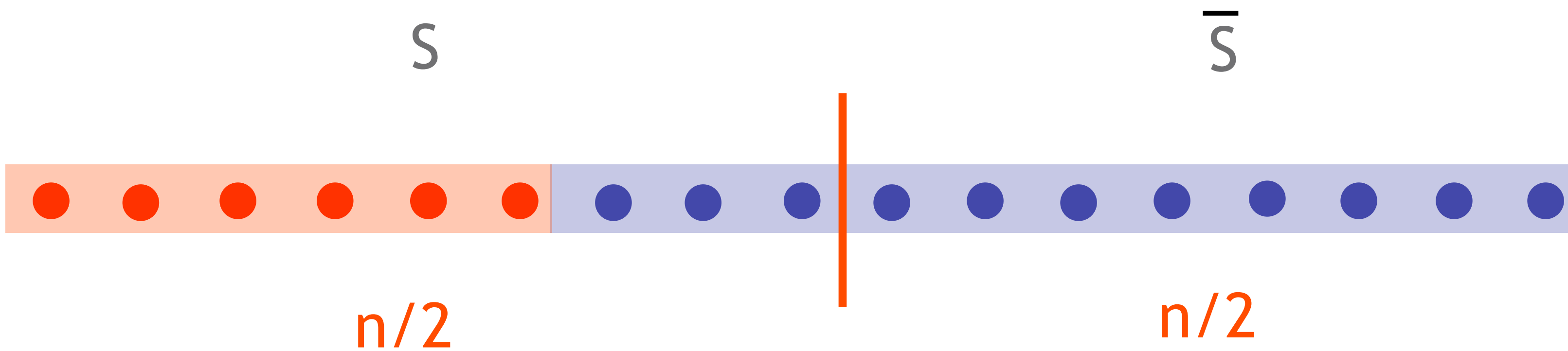


If cut is small, unbalance remains.

Mix the charges along the matchings $\{ M_1, M_2, \ldots, M_t \}$

Cut Strategies: Finding Cuts Quickly

Order the vertices according to the final charge present and cut in half.



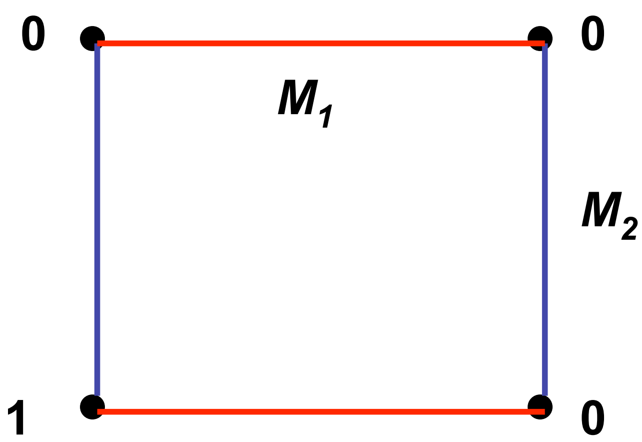
The KRV mixing walk

KRV-walk

- At round **t:**

$$P(t) = \left(\frac{I + M_{t-1}}{2}\right)\left(\frac{I + M_{t-2}}{2}\right)\cdots\left(\frac{I + M_1}{2}\right)$$

Lazy random walk traversing matchings in order.



The KRV mixing walk

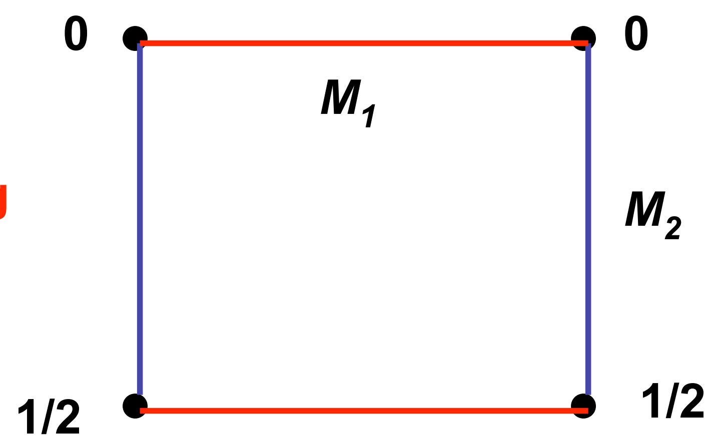
KRV-walk

- At round **t:**

$$P(t) = \left(\frac{I + M_{t-1}}{2}\right)\left(\frac{I + M_{t-2}}{2}\right)\cdots\left(\frac{I + M_1}{2}\right)$$

Lazy random walk traversing matchings in order.

Averaging along M_1



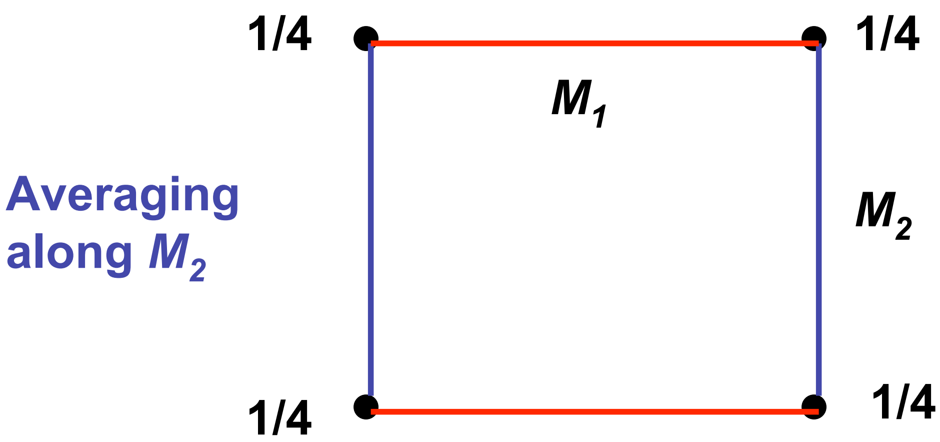
The KRV mixing walk

KRV-walk

- At round **t:**

$$P(t) = \left(\frac{I + M_{t-1}}{2}\right)\left(\frac{I + M_{t-2}}{2}\right)\cdots\left(\frac{I + M_1}{2}\right)$$

Lazy random walk traversing matchings in order.



Sketch of KRV Analysis

1. Mixing of P(t) measured by potential function

$$\Psi_t = \| P(t) - J/n \|_F^2$$

2. If P(t) mixes well, H_t has good expansion.

   Possible to embed K_n in H_t.

3. Potential Reduction at every iteration

$$\Psi_t = \Psi_{t-1} - \boxed{L(M_t)\cdot P(t)}$$

   Mixing due to matching M_t

   Decomposition possible as KRV walks matchings in order.

4. Cut-finding procedure reduces potential by a fixed factor

$$\Psi_t = \Psi_{t-1}\left(1 - \frac{1}{\log n}\right)$$

   Yields expander in $O((\log n)^2)$ rounds

Why KRV cannot do better

Recall:

Approximation is

$$\frac{\phi(H_T)}{T}$$

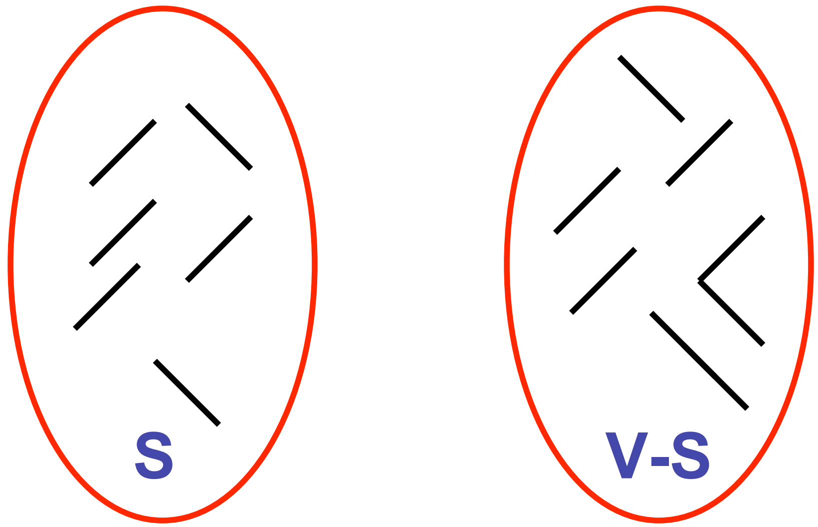
Why KRV cannot do better

Recall:

Approximation is

Can KRV get better than O(1) expansion?

Why KRV cannot do better



Recall:

Approximation is

Can KRV get better than O(1) expansion?

SUPPOSE:

Walk on M_1, M_2, …, M_k mixes perfectly on S and V-S

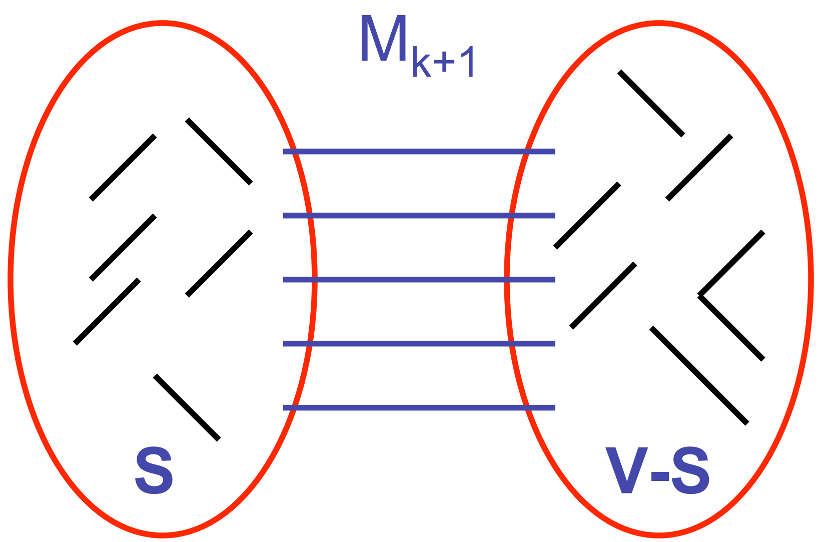
and

no edge cross (S,V-S)

Why KRV cannot do better

Now walk mixes perfectly.

<u>Expansion is **O(1).**</u>



Recall:

Approximation is

Can KRV get better than O(1) expansion?

SUPPOSE:

Walk on M_1, M_2, …, M_k mixes perfectly on S and V-S

and

no edge cross (S,V-S)

Our Cut Strategy: a Different Walk

IDEA: use lazy natural random walk

$$P(t) = \frac{I}{2} + \frac{M_1 + M_2 + \ldots + M_{t-1}}{2(t-1)}$$

ADVANTAGES:

- Eliminates bad case: possible to get better expansion.
- Better handle on expansion through mixing by Cheeger's Inequality.

CHALLENGE:

- **Impossible to decompose potential as in KRV.**

$$\Psi_t = \Psi_{t-1} - L(M_t) \cdot P(t)$$

Additional matching modifies all steps of walk.

Our Cut Strategy: a Different Walk

IDEA: use lazy natural random walk

$$P(t) = \left(\frac{I}{2} + \frac{M_1 + M_2 + \ldots + M_{t-1}}{2(t-1)} \right)^d$$

ADVANTAGES:

- Eliminates bad case: possible to get better expansion.
- Better handle on expansion through mixing by Cheeger's Inequality.

CHALLENGE:

- **Impossible to decompose potential as in KRV.**

$$\cancel{\Psi_t = \Psi_{t-1} - L(M_t) \cdot P(t)}$$

Additional matching modifies all steps of walk.

Modified Walk and Matrix Inequalities

CHALLENGE:

Impossible to decompose potential as in KRV.

Additional matching modifies all steps of walk.

SOLUTION:

Use round-robin walk close to natural walk:

$$\boxed{N_i = \frac{d}{d+1} I + \frac{1}{d+1} M_i}$$

$$P(t) = (N_1 N_2 \dots N_{t-1} N_{t-1} N_{t-2} \dots N_1)^d$$

Apply matrix inequality:

$$\left\| (ABA)^t \right\| \leq \left\| A^t B^t A^t \right\|$$

Modified Walk and Matrix Inequalities

CHALLENGE:

Impossible to decompose potential as in KRV.
Additional matching modifies all steps of walk.

SOLUTION:

Use round-robin walk close to natural walk:

$$\boxed{N_i = \frac{d}{d+1} I + \frac{1}{d+1} M_i}$$

$$P(t) = (N_1 N_2 \ldots N_{t-1} N_{t-1} N_{t-2} \ldots N_1)^d$$

Apply matrix inequality:

$$\left\| (ABA)^t \right\| \leq \left\| A^t B^t A^t \right\|$$

$$\cancel{\Psi_t = \Psi_{t-1} - L(M_t) \cdot P(t)}$$

Modified Walk and Matrix Inequalities

CHALLENGE:

Impossible to decompose potential as in KRV.

Additional matching modifies all steps of walk.

SOLUTION:

Use round-robin walk close to natural walk:

$$\boxed{N_i = \frac{d}{d+1} I + \frac{1}{d+1} M_i}$$

$$P(t) = (N_1 N_2 \ldots N_{t-1} N_{t-1} N_{t-2} \ldots N_1)^d$$

Apply matrix inequality:

$$\left\| (ABA)^t \right\| \leq \left\| A^t B^t A^t \right\|$$

$$\Psi_t = \Psi_{t-1} - L(M_t) \cdot P(t)$$

Modified Walk and Matrix Inequalities

SOLUTION:

Use round-robin walk close to natural.
Apply matrix inequality.

Yields same potential reduction as KRV.

But our walk is better related to expansion:

In $O((\log n)^2)$ rounds,
conductance $(1 \backslash \log n)$ by Cheeger.

Modified Walk and Matrix Inequalities

SOLUTION:

Use round-robin walk close to natural.
Apply matrix inequality.

Yields same potential reduction as KRV.

But our walk is better related to expansion:

In **$O((\log n)^2)$** rounds,
conductance **(1\log n)** by Cheeger.

$\Omega(\log n)$ expansion in $O((\log n)^2)$ rounds.

Modified Walk and Matrix Inequalities

SOLUTION:

Use round-robin walk close to natural.
Apply matrix inequality.

Yields same potential reduction as KRV.

But our walk is better related to expansion:

In $O((\log n)^2)$ rounds,
conductance $(1 \backslash \log n)$ by Cheeger.

$\Omega(\log n)$ expansion in $O((\log n)^2)$ rounds.

TIME: only polylog factors worse than KRV

Lower Bound

Matching player yielding

$$\frac{\phi(H_T)}{T} = O\left(\frac{1}{\sqrt{\log n}}\right)$$

against any Cut player.

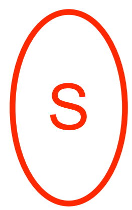
No better approximation than $O((\log n)^{1/2})$

in KRV Cut-Matching game

Lower Bound Idea

A NAÏVE MATCHING PLAYER:

Fix a cut (S,V-S). Keep it as sparse as possible.

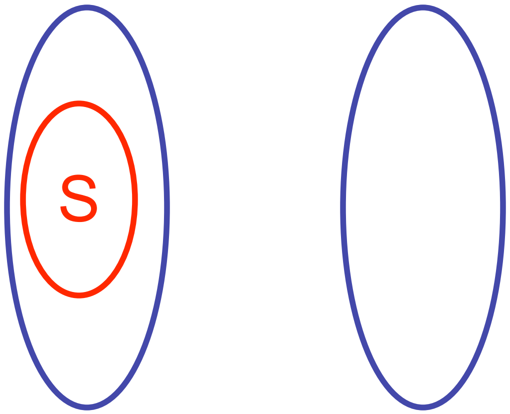


Lower Bound Idea

A NAÏVE MATCHING PLAYER:

Fix a cut (S,V-S). Keep it sparse.

Cut player plays…

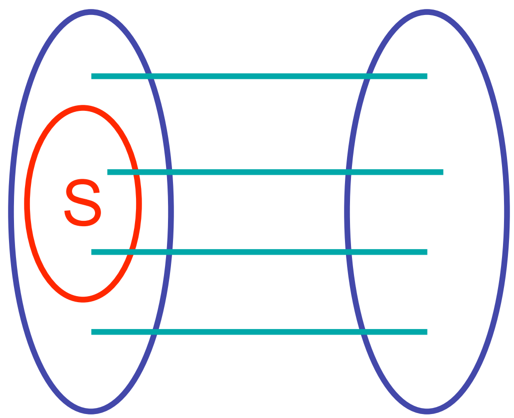


Lower Bound Idea

A NAÏVE MATCHING PLAYER:

Fix a cut (S,V-S). Keep it sparse.

Cut player plays…



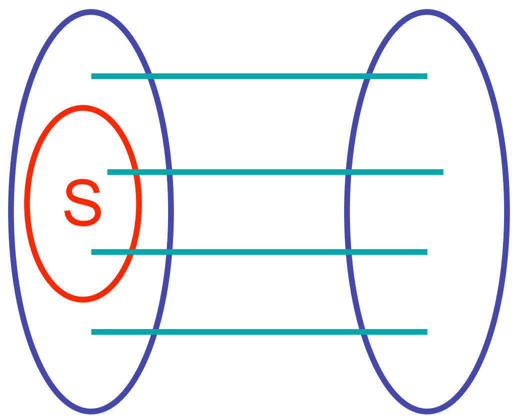
GAME OVER

Lower Bound Idea

A NAÏVE MATCHING PLAYER:

Fix a cut (S,V-S). Keep it sparse.

Cut player plays…



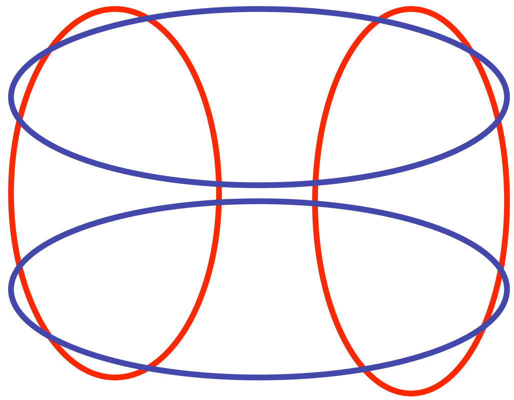
GAME OVER

<u>IDEA:</u> hedge over many cuts

Lower Bound Idea

THE REAL PLAYER - AT START:

Matching player selects log(n) ‘orthogonal’ 50-50 cuts in V.



Orthogonal 50-50 cuts

Minimum correlation

Cut player cannot ‘kill’ many cuts at once

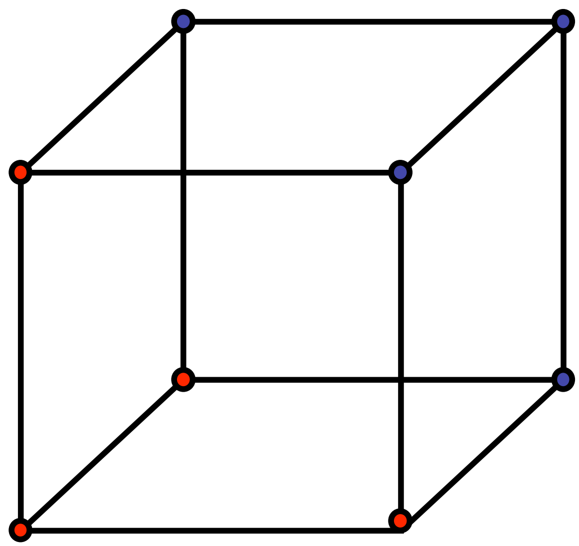
THE REAL PLAYER - THROUGHOUT THE GAME:

Matching player adds matchings to <u>minimize average expansion</u>.

Main Lemma

$\forall$ 50-50 cut (**S**,**V-S**),

$$H_d = \{-1,+1\}^d$$

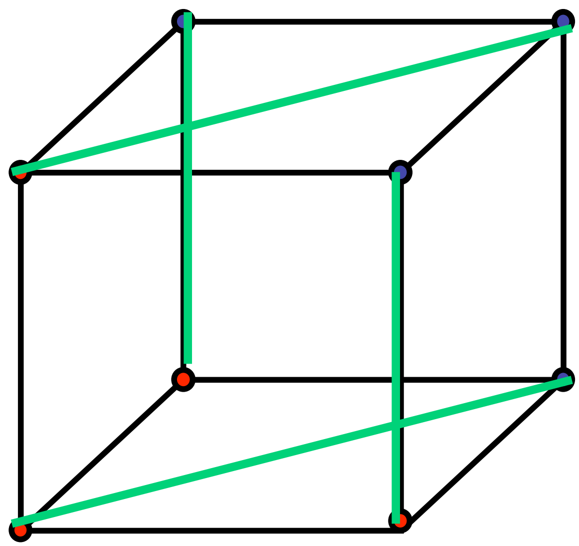


Main Lemma

$\forall$ 50-50 cut (**S**,**V-S**),

$\exists$ a perfect matching **M**, s.t.

$$H_d = \{-1,+1\}^d$$



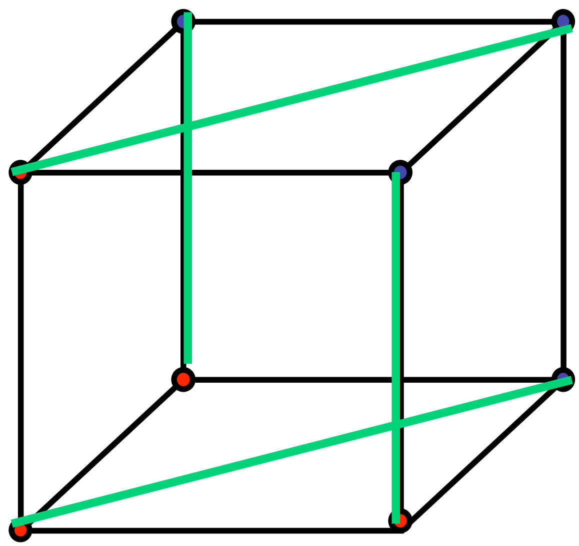
Main Lemma

$\forall$ 50-50 cut (**S**,**V-S**),

$\exists$ a perfect matching **M**, s.t.

$$\sum_{(u,v)\in M} |u-v|_1 = O\left(\sqrt{d}\right)$$

$$H_d = \{-1,+1\}^d$$



Conclusion and Open Problems

POWER OF CUT-MATCHING GAME:

Simple yet powerful framework for SPARSEST CUT.

OPEN QUESTION:

Can we use Cut-Matching to get fast $(\log n)^{1/2}$ approximation?