Random Walks as a Stable Analogue of Eigenvectors
(with Applications to Nearly-Linear-Time Graph Partitioning)

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Based on joint works with Michael Mahoney (Stanford), Sushant Sachdeva (Yale) and Nisheeth Vishnoi (MSR India).
Why Spectral Algorithms for Graph Problems ... 

... in practice?
  • Simple to implement
  • Can exploit very efficient linear algebra routines
  • Perform well in practice for many problems

... in theory?
  • Connections between spectral and combinatorial objects
  • Connections to Markov Chains and Probability Theory
  • Intuitive geometric viewpoint

RECENT ADVANCES:
  Fast algorithms for fundamental combinatorial problems
  rely on spectral and optimization ideas
Spectral Algorithms for Graph Partitioning

Spectral algorithms are widely used in many graph-partitioning applications: clustering, image segmentation, community-detection, etc.

CLASSICAL VIEW:
- Based on Cheeger’s Inequality
- Eigenvectors sweep-cuts reveal sparse cuts in the graph
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NEW TREND:
- Random walk vectors replace eigenvectors:
  • Fast Algorithms for Graph Partitioning
  • Local Graph Partitioning
  • Real Network Analysis
- Different random walks: PageRank, Heat-Kernel, etc.
Advantages of Random Walks:

1) **Quick approximation to eigenvector** in massive graphs

   \[ A = \text{adjacency matrix} \quad D = \text{diagonal degree matrix} \]

   \[ W = AD^{-1} = \text{natural random walk matrix} \quad L = D - A = \text{Laplacian matrix} \]

   **Second Eigenvector of the Laplacian** can be computed by iterating \( W \) :

   For random \( y_0 \) s.t. \( y_0^T D^{-1} 1 = 0 \), compute

   \[ D^{-1} W^t y_0 \]
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Why Random Walks? A Practitioner’s View

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**Heuristic:** For massive graphs, pick \( t \) as large as computationally affordable.
Why Random Walks? A Practitioner’s View

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1) Quick approximation to eigenvector in massive graphs
2) Statistical robustness

Real-world graphs are noisy

GROUND TRUTH GRAPH
Why Random Walks? A Practitioner’s View

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**GOAL:** estimate eigenvector of ground-truth graph.
Why Random Walks? A Practitioner’s View

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GROUND-TRUTH GRAPH

INPUT GRAPH

GOAL: estimate eigenvector of ground-truth graph.

OBSERVATION: eigenvector of input graph can have very large variance, as it can be very sensitive to noise

RANDOM-WALK VECTORS provide better, more stable estimates.
This Talk

**QUESTION:**

Why random-walk vectors in the design of fast algorithms?
This Talk

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**ANSWER:** Stable, regularized version of the eigenvector
This Talk

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Why random-walk vectors in the design of fast algorithms?

ANSWER: Stable, regularized version of the eigenvector

GOALS OF THIS TALK:
- Show optimization perspective on why random walks arise
- Application to nearly-linear-time balanced graph partitioning
Random Walks as Regularized Eigenvectors
What is Regularization?

Regularization is a fundamental technique in optimization.

**OPTIMIZATION PROBLEM** \[\rightarrow\] **WELL-BEHAVED OPTIMIZATION PROBLEM**

- Stable optimum
- Unique optimal solution
- Smoothness conditions
...

OPTIMIZATION PROBLEM
What is Regularization?

Regularization is a fundamental technique in optimization.

\[ \min_{x \in H} L(x) \quad \Rightarrow \quad \min_{x \in H} L(x) + \lambda \cdot F(x) \]

Benefits of Regularization in Learning and Statistics:

- Increases stability
- Decreases sensitivity to random noise
- Prevents overfitting
Instability of Eigenvector
Instability of Eigenvector

Current eigenvector

1

$\epsilon$

$\epsilon$

$\epsilon$

$\epsilon$

EXPANDER
Instability of Eigenvector

Eigenvector Changes Completely!
The Laplacian Eigenvalue Problem

Quadratic Formulation

\[ \frac{1}{d} \min_{x} x^T L x \]

s.t. \( \|x\|_2 = 1 \)

\[ x^T 1 = 0 \]

For simplicity, take G to be d-regular.
The Laplacian Eigenvalue Problem

### Quadratic Formulation

$$\frac{1}{d} \min x^T L x$$

s.t. $\|x\|_2 = 1$

$$x^T 1 = 0$$

### SDP Formulation

$$\frac{1}{d} \min (L \cdot X)$$

s.t. $I \cdot X = 1$

$$11^T \cdot X = 0$$

$X \succeq 0$
The Laplacian Eigenvalue Problem

**Quadratic Formulation**

\[
\frac{1}{d} \min x^T L x \quad \text{s.t. } \|x\|_2 = 1, \quad x^T 1 = 0
\]

**SDP Formulation**

\[
\frac{1}{d} \min L \cdot X \quad \text{s.t. } I \cdot X = 1, \quad 11^T \cdot X = 0, \quad X \succeq 0
\]

Programs have **same optimum**. Take optimal solution

\[X^* = x^* (x^*)^T\]
Instability of Linear Optimization

Consider a convex set $S \subset \mathbb{R}^n$ and a linear optimization problem:

$$f(c) = \arg \min_{x \in S} c^T x$$

The optimal solution $f(c)$ may be very unstable under perturbation of $c$:

$$\| c' - c \| \leq \delta \quad \text{and} \quad \| f(c') - f(c) \| \gg \delta$$
Regularization Helps Stability

Consider a convex set $\mathcal{S} \subset \mathbb{R}^n$ and a regularized linear optimization problem

$$f(c) = \arg \min_{x \in \mathcal{S}} c^T x + F(x)$$

where $F$ is $\sigma$-strongly convex.

Then:

$$\|c' - c\| \leq \delta \quad \text{implies} \quad \|f(c) - f(c')\| \leq \frac{\delta}{\sigma}$$
Consider a convex set $\mathcal{S} \subset \mathbb{R}^n$ and a regularized linear optimization problem

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slope $\leq \delta$
Regularized Spectral Optimization

SDP Formulation

\[
\frac{1}{d} \min L \cdot X \\
\text{s.t.} \quad I \cdot X = 1 \\
11^T \cdot X = 0 \\
X \succeq 0
\]

Density Matrix

Eigenvector decomposition of \(X\):

\[
X = \sum p_i v_i v_i^T
\]

\[\forall i, p_i \geq 0, \quad \sum p_i = 1, \quad \forall i, v_i^T 1 = 0.\]

Eigenvalues of \(X\) define probability distribution
Regularized Spectral Optimization

SDP Formulation

\[
\frac{1}{d} \min L \cdot X \\
\text{s.t.} \quad I \cdot X = 1 \quad \quad J \cdot X = 0 \\
X \succeq 0
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Density Matrix

Eigenvalues of \( X \) define probability distribution

\[
X^* = x^* (x^*)^T
\]

TRIVIAL DISTRIBUTION
Regularized Spectral Optimization

\[
\frac{1}{d} \min \quad L \bullet X + \eta \cdot F(X)
\]

s.t.

\[ I \bullet X = 1 \]

\[ 11^T \bullet X = 0 \]

\[ X \succeq 0 \]

The regularizer $F$ forces the distribution of eigenvalues of $X$ to be non-trivial.

\[
X^* = x^*(x^*)^T
\]

\[
X^* = \sum p_i v_i v_i^T
\]
Regularizers

Regularizers are **SDP-versions** of common regularizers

- von Neumann Entropy
  \[
  F_H(X) = \text{Tr}(X \log X) = \sum p_i \log p_i
  \]

- $p$-Norm, $p > 1$
  \[
  F_p(X) = \frac{1}{p} \|X\|^p_p = \frac{1}{p} \text{Tr}(X^p) = \frac{1}{p} \sum p_i^p
  \]

- And more, e.g. log-determinant.
Our Main Result

**RESULT:** Explicit correspondence between **regularizers and random walks**

**REGULARIZER**

\[
F = F_H \quad \text{Entropy} \quad X^* \propto H_G^t \quad \text{where } t \text{ depends on } \eta
\]

\[
F = F_p \quad \text{p-Norm} \quad X^* \propto (qI + (1 - q)W)^{\frac{1}{p-1}} \quad \text{where } q \text{ depends on } \eta
\]

**OPTIMAL SOLUTION OF REGULARIZED PROGRAM**

\[
\frac{1}{d} \min \frac{L \cdot X + \eta \cdot F(X)}{d}
\text{ s.t. } I \cdot X = 1
\text{ and } J \cdot X = 0
\]

\[
X \succeq 0
\]

**Regularized SDP**

\[
\frac{1}{d} \min \frac{L \cdot X + \eta \cdot F(X)}{d}
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Our Main Result

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\frac{1}{d} \min_{X} \quad L \cdot X + \eta \cdot F(X)
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s.t.

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**RESULT:** Explicit correspondence between regularizers and random walks

**REGULARIZER**

\[
F = F_H \quad \text{Entropy} \quad \rightarrow \quad X^* \propto H_G^t
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where \( t \) depends on \( \eta \)

**HEAT KERNEL**

\[
F = F_p \quad \text{p-Norm} \quad \rightarrow \quad X^* \propto (qI + (1 - q)W)^{\frac{1}{p-1}}
\]

**LAZY RANDOM WALK** where \( q \) depends on \( \eta \)
Background: Heat-Kernel Random Walk

For simplicity, take $G$ to be \textbf{d-regular}.

- The Heat-Kernel Random Walk is a \textit{Continuous-Time Markov Chain} over $V$, modeling the \textit{diffusion of heat} along the edges of $G$.

- Transitions take place in \textit{continuous time} $t$, with an \textit{exponential} distribution.

\[
\frac{\partial p(t)}{\partial t} = -L \frac{p(t)}{d}
\]

\[
p(t) = e^{-\frac{t}{d}L} p(0)
\]

- The Heat Kernel can be interpreted as \textit{Poisson distribution} over number of steps of the natural random walk $W=AD^{-1}$:

\[
e^{-\frac{t}{d}L} = e^{-t} \sum_{k=1}^{\infty} \frac{t^k}{k!} W^k
\]
For simplicity, take $G$ to be d-regular.

- The Heat-Kernel Random Walk is a Continuous-Time Markov Chain over $V$, modeling the diffusion of heat along the edges of $G$.

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\[
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p(t) = e^{-\frac{t}{d}L} p(0) =: \begin{bmatrix} H^t_G \end{bmatrix} p(0)
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\[
e^{-\frac{t}{d}L} = e^{-t} \sum_{k=1}^{\infty} \frac{t^k}{k!} W^k
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Heat Kernel Walk: Stability Analysis

Consider a convex $sS \subset R^n$ and a **regularized** linear optimization problem

$$f(c) = \arg\min_{x \in S} c^T x + F(x)$$

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Analogous statement for Heat Kernel:

$$\|G' - G\|_\infty \leq \delta \quad \text{implies} \quad \left\| \frac{H_{G'}}{I \bullet H_{G'}} - \frac{H_{G}}{I \bullet H_{G}} \right\|_1 \leq \tau \cdot \delta$$
Applications to Graph Partitioning: Nearly-Linear-Time Balanced Cut
Partitioning Graphs - Conductance

Undirected unweighted \( G = (V, E), |V| = n, |E| = m \)

Conductance of \( S \subseteq V \)

\[
\phi(S) = \frac{|E(S, \bar{S})|}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}}
\]
Partitioning Graphs – Balanced Cut

NP-HARD DECISION PROBLEM
Does $G$ have a $b$-balanced cut of conductance $<$ $\gamma$?

![Diagram](image)

$\phi(S) < \gamma$

$\frac{1}{2} > \frac{\text{vol}(S)}{\text{vol}(V)} > b$
NP-HARD DECISION PROBLEM
Does G have a $b$-balanced cut of conductance $< \gamma$?

$\phi(S) < \gamma$

$\frac{1}{2} > \frac{\text{vol}(S)}{\text{vol}(V)} > b$

- Important primitive for many recursive algorithms.
- Applications to clustering and graph decomposition.
### Spectral Approximation Algorithms

Does $G$ have a $b$-balanced cut of conductance $< \gamma$ ?

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Recursive Eigenvector Algorithm

**INPUT:** $(G, b, \gamma)$  
**DECISION:** does there exists $b$-balanced $S$ with $\phi(S) < \gamma$?
Recursive Eigenvector Algorithm

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- If \(S_1\) is \((b/2)\)-balanced, Output \(S_1\). Otherwise, consider the graph \(G_1\) induced by \(G\) on \(V-S_1\) with self-loops replacing the edges going to \(S_1\).
Recursive Eigenvector Algorithm

INPUT: \((G, b, \gamma)\)  
DECISION: does there exist a \(b\)-balanced set \(S\) with \(\phi(S) < \gamma\) ?

- Compute eigenvector of \(G\) and corresponding Laplacian eigenvalue \(\lambda_2\).
- If \(\lambda_2 \geq \gamma\), output NO. Otherwise, sweep eigenvector to find \(S_1\) such that \(\phi(S_1) \leq O(\sqrt{\gamma})\).
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- Recurse on $G_1$. 

\[ (G, b, \gamma) \rightarrow S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow S_4 \]
Recursive Eigenvector Algorithm

**INPUT:** \((G, b, \gamma)\)  \hspace{1cm} **DECISION:** does there exists \(b\)-balanced \(S\) with \(\phi(S) < \gamma\)?

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- Recurse on \(G_1\).

**Graph:**

- \(S_1\)
- \(S_2\)
- \(S_3\)
- \(S_4\)

**Equation:**

\[\lambda_2(G_5) \geq \gamma\]

**Conclusion:**

LARGE INDUCED EXPANDER = NO-CERTIFICATE
Recursive Eigenvector Algorithm

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- Recurse on \(G_1\).

**RUNNING TIME:** \(\tilde{O}(m)\) per iteration, \(O(n)\) iterations. Total: \(\tilde{O}(mn)\)
Recursive Eigenvector: The Worst Case

\[ \Omega(n) \] nearly-disconnected components

Varying conductance
Recursive Eigenvector: The Worst Case

NB: Recursive Eigenvector eliminates one component per iteration. \( \Omega(n) \) iterations are necessary. Each iteration requires \( \Omega(m) \) time.
Recursive Eigenvector: The Worst Case

NB: Recursive Eigenvector eliminates one component per iteration. \( \Omega(n) \) iterations are necessary. Each iteration requires \( \Omega(mn) \) time.

GOAL: Eliminate unbalanced low-conductance cuts faster.
Recursive Eigenvector: The Worst Case

STABILITY VIEW:

• Ideally, we would like to enforce progress: \( \lambda_2(G_{t+1}) \gg \lambda_2(G_t) \)

• Eigenvector may change completely at every iteration. Impossible to enforce any non-trivial relation between \( \lambda_2(G_{t+1}) \) and \( \lambda_2(G_t) \)
# Our Algorithm: Contributions

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**MAIN FEATURES:**
- Compute $O(\log n)$ global heat-kernel random-walk vectors at each iteration
- Unbalanced cuts are removed in $O(\log n)$ iterations
- Method to compute heat-kernel vectors in nearly-linear time

**TECHNICAL COMPONENTS:**
1) New iterative algorithm with a simple random walk interpretation
2) Novel analysis of Lanczos methods for computing heat-kernel vectors
Eliminating Unbalanced Cuts

• The graph eigenvector may be correlated with only one sparse unbalanced cut.
Eliminating Unbalanced Cuts

• The graph eigenvector may be correlated with only one sparse unbalanced cut.

• Consider the Heat-Kernel random walk-matrix $H_G^\tau$ for $\tau = \log n/\gamma$.

$H_G^\tau e_i$ Probability vector for random walk started at vertex $i$

$H_G^\tau e_j$ Long vectors are slow-mixing random walks
Eliminating Unbalanced Cuts

- The graph eigenvector may be correlated with only one sparse unbalanced cut.

- Consider the Heat-Kernel random walk-matrix $H^\tau_G$ for $\tau = \log n/\gamma$. 

Unbalanced cuts of conductance $\leq \sqrt{\gamma}$
The graph eigenvector may be correlated with only one sparse unbalanced cut.

Consider the Heat-Kernel random walk-matrix $H_G^\tau$ for $\tau = \log n/\gamma$.  

Unbalanced cuts of conductance $< \sqrt{\gamma}$
Eliminating Unbalanced Cuts

• The graph eigenvector may be correlated with only one sparse unbalanced cut.

SINGLE VECTOR
SINGLE CUT

AFTER CUT REMOVAL ...

... eigenvector can change completely

• Consider the Heat-Kernel random walk-matrix $H_G^\tau$ for $\tau = \log n/\gamma$.

VECTOR
EMBEDDING
MULTIPLE CUTS

... vectors do not change a lot
Our Algorithm for Balanced Cut

**IDEA BEHIND OUR ALGORITHM:**
Replace eigenvector in recursive eigenvector algorithm with Heat-Kernel random walk $H_G^\tau$ for $\tau = \log n/\gamma$

Consider the embedding $\{v_i\}$ given by $H_G^\tau$:

$$v_i = H_G^\tau e_i$$
Our Algorithm for Balanced Cut

IDEA BEHIND OUR ALGORITHM:
Replace eigenvector in recursive eigenvector algorithm with
Heat-Kernel random walk $H_G^\tau$ for $\tau = \log n / \gamma$

Consider the embedding $\{v_i\}$ given by $H_G^\tau$:

$\begin{align*}
v_i &= H_G^\tau e_i
\end{align*}$

Chosen to emphasize cuts of conductance $\approx \gamma$

Stationary distribution is uniform as $G$ is regular
Our Algorithm for Balanced Cut

IDEA BEHIND OUR ALGORITHM:
Replace eigenvector in recursive eigenvector algorithm with Heat-Kernel random walk $H_G^T$ for $\tau = \log n / \gamma$

Consider the embedding $\{v_i\}$ given by $H_G^T$:

$$v_i = H_G^T e_i$$

MIXING:
Define the total deviation from stationary for a set $S \subseteq V$ for walk

$$\Psi(H_G^T, S) = \sum_{i \in S} \|v_i - \bar{1}/n\|^2$$

FUNDAMENTAL QUANTITY TO UNDERSTAND CUTS IN $G$
Our Algorithm: Case Analysis

Recall:
\[
\tau = \log n / \gamma \quad \quad \Psi(H_G^\tau, S) = \sum_{i \in S} ||H_G^\tau e_i - \vec{1}/n||^2
\]

**CASE 1: Random walks have mixed**

\[
v_i = H_G^\tau e_i
\]

**ALL VECTORS ARE SHORT**

\[
\Psi(H_G^\tau, V) \leq \frac{1}{\text{poly}(n)}
\]
Our Algorithm: Case Analysis

Recall:
\[ \tau = \log n / \gamma \]
\[ \Psi(H_G^\tau, S) = \sum_{i \in S} ||H_G^\tau e_i - \vec{1}/n||^2 \]

**CASE 1:** Random walks have mixed

\[ v_i = H_G^\tau e_i \]

**ALL VECTORS ARE SHORT**

\[ \Psi(H_G^\tau, V) \leq \frac{1}{\text{poly}(n)} \]

By definition of \( \tau \)

\[ \lambda_2 \geq \Omega(\gamma) \]

\[ \phi_G \geq \Omega(\gamma) \]
Our Algorithm

\[ \tau = \log n / \gamma \]

\[ \Psi(H_G^\tau, S) = \sum_{i \in S} ||H_G^\tau e_i - \bar{1}/n||^2 \]

\[ v_i = H_G^\tau e_i \]

**CASE 2:** Random walks have **not mixed**

\[ \Psi(H_G^\tau, V) > \frac{1}{\text{poly}(n)} \]

We can either find an \( \Omega(b) \)-balanced cut with conductance \( O(\sqrt{\gamma}) \)
Our Algorithm

\[ \tau = \log n / \gamma \]

\[ \Psi(H_G^\tau, S) = \sum_{i \in S} \|H_G^\tau e_i - \bar{1}/n\|^2 \]

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\[ \tau = \log n / \gamma \]

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\( S_1 \)

CASE 2: Random walks have not mixed

\[ \Psi(H_G^\tau, V) > \frac{1}{\text{poly}(n)} \]

We can either find an \( \Omega(b) \)-balanced cut with conductance \( O(\sqrt{\gamma}) \)

OR a ball cut yields \( S_1 \) such that \( \phi(S_1) \leq O(\sqrt{\gamma}) \) and

\[ \Psi(H_G^\tau, S_1) \geq \frac{1}{2} \Psi(H_G^\tau, V). \]
Our Algorithm: Iteration

\[ \tau = \log n / \gamma \]

\[ \Psi(H_G^\tau, S) = \sum_{i \in S} ||H_G^\tau e_i - \bar{1}/n||^2 \]

CASE 2: We found an unbalanced cut \( S_1 \) with \( \phi(S_1) \leq O(\sqrt{\gamma}) \) and

\[ \Psi(H_G^\tau, S_1) \geq \frac{1}{2} \Psi(H_G^\tau, V). \]

Modify \( G = G^{(1)} \) by adding edges across \((S_1, \bar{S}_1)\) to construct \( G^{(2)} \).

Analogous to removing unbalanced cut \( S_1 \) in Recursive Eigenvector algorithm.
Our Algorithm: Modifying G

**CASE 2:** We found an unbalanced cut $S_1$ with $\phi(S_1) \leq O(\sqrt{\gamma})$ and

$$\Psi(H_G^*, S_1) \geq \frac{1}{2} \Psi(H_G^*, V).$$

Modify $G = G^{(1)}$ by **adding edges** across $(S_1, \bar{S}_1)$ to construct $G^{(2)}$. 
Our Algorithm: Modifying $G$

**CASE 2:** We found an unbalanced cut $S_1$ with $\phi(S_1) \leq O(\sqrt{\gamma})$ and

$$\Psi(H^\tau_G, S_1) \geq \frac{1}{2} \Psi(H^\tau_G, V).$$

Modify $G = G^{(1)}$ by **adding edges** across $(S_1, \bar{S}_1)$ to construct $G^{(2)}$.

$$G^{(t+1)} = G^{(t)} + \gamma \sum_{i \in S_t} \text{Star}_i$$

where $\text{Star}_i$ is the **star graph** rooted at vertex $i$. 

[Diagram of a star graph and a cut set $S_1$.]
Our Algorithm: Modifying $G$

**CASE 2:** We found an unbalanced cut $S_1$ with $\phi(S_1) \leq O(\sqrt{\gamma})$ and

$$\Psi(H_G, S_1) \geq \frac{1}{2} \Psi(H_G, V).$$

Modify $G = G^{(1)}$ by adding edges across $(S_1, \bar{S}_1)$ to construct $G^{(2)}$.

The random walk can now escape $S_1$ more easily.
Our Algorithm: Iteration

\[ \tau = \log n / \gamma \]

\[ \Psi(H_G^\tau, S) = \sum_{i \in S} \|H_G^\tau e_i - \bar{1}/n\|^2 \]

\[ S_1 \]

CASE 2: We found an unbalanced cut \( S_1 \) with \( \phi(S_1) \leq O(\sqrt{\gamma}) \) and

\[ \Psi(H_G^\tau, S_1) \geq \frac{1}{2} \Psi(H_G^\tau, V). \]

Modify \( G = G^{(1)} \) by adding edges across \( (S_1, \bar{S}_1) \) to construct \( G^{(2)} \).

POTENTIAL REDUCTION:

\[ \Psi(H_{G(t+1)}^\tau, V) \leq \Psi(H_{G(t)}^\tau, V) - \frac{1}{2} \Psi(H_{G(t)}^\tau, S_t) \leq \frac{3}{4} \Psi(H_{G(t)}^\tau, V) \]
Our Algorithm: Iteration

\[ \tau = \log n / \gamma \]

\[ \Psi(H_G^\tau, S) = \sum_{i \in S} ||H_G^\tau e_i - \frac{1}{n}||^2 \]

**CASE 2:** We found an unbalanced cut \( S_1 \) with \( \phi(S_1) \leq O(\sqrt{\gamma}) \) and

\[ \Psi(H_G^\tau, S_1) \geq \frac{1}{2} \Psi(H_G^\tau, V). \]

Modify \( G = G^{(1)} \) by **adding edges** across \( (S_1, \bar{S}_1) \) to construct \( G^{(2)} \).

**POTENTIAL REDUCTION:**

\[ \Psi(H_{G(t+1)}^\tau, V) \leq \Psi(H_{G(t)}^\tau, V) - \frac{1}{2} \Psi(H_{G(t)}^\tau, S_t) \leq \frac{3}{4} \Psi(H_{G(t)}^\tau, V) \]

**CRUCIAL APPLICATION OF STABILITY OF RANDOM WALK**
Summary and Potential Analysis

IN SUMMARY:
At every step $t$ of the recursion, we either
1. Produce a $\Omega(b)$-balanced cut of the required conductance, OR
Potential Reduction

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At every step $t$ of the recursion, we either

1. Produce a $\Omega(b)$-balanced cut of the required conductance, OR
2. Find that

$$\Psi(H_{G(t)}^\tau, V) \leq \frac{1}{\text{poly}(n)}, \text{ OR}$$
Potential Reduction

IN SUMMARY:
At every step $t$ of the recursion, we either

1. Produce a $\Omega(b)$-balanced cut of the required conductance, OR

2. Find that

$$\Psi(H_{G(t)}^\tau, V) \leq \frac{1}{\text{poly}(n)}, \text{ OR}$$

3. Find an unbalanced cut $S_t$ of the required conductance, such that for the graph $G^{(t+1)}$, modified to have increased edges from $S_t$,

$$\Psi(H_{G^{(t+1)}}^\tau, V) \leq \frac{3}{4} \Psi(H_{G^{(t)}}^\tau, V)$$
Potential Reduction

IN SUMMARY:
At every step \( t-1 \) of the recursion, we either
1. Produce a \( \Omega(b) \)-balanced cut of the required conductance, OR
2. Find that
   \[
   \Psi(H_{G(t)}^\tau, V) \leq \frac{1}{\text{poly}(n)} \text{, OR}
   \]
3. Find an unbalanced cut \( S_t \) of the required conductance, such that for the process \( P^{(t+1)} \), modified to have increased transitions from \( S_t \),
   \[
   \Psi(H_{G(t+1)}^\tau, V) \leq \frac{3}{4} \Psi(H_{G(t)}^\tau, V)
   \]
After \( T=O(\log n) \) iterations, if no balanced cut is found:
   \[
   \Psi(H_{G(T)}^\tau, V) \leq \frac{1}{\text{poly}(n)}
   \]
From this guarantee, using the definition of \( G^{(T)} \), we derive an SDP-certificate
that no \( b \)-balanced cut of conductance \( O(\gamma) \) exists in \( G \).

NB: Only \( O(\log n) \) iterations to remove unbalanced cuts.
Heat-Kernel and Certificates

- If no balanced cut of conductance is found, our potential analysis yields:

\[ \Psi(H^\tau_{G(T)}, V) \leq \frac{1}{\text{poly}(n)} \quad \Rightarrow \quad L + \gamma \sum_{j=1}^{T-1} \sum_{i \in S_j} L(\text{Star}_i) \geq \gamma L(K_V) \]

Modified graph has \( \lambda_2 \geq \gamma \)

**CLAIM:** This is a certificate that no balanced cut of conductance \(< \gamma \) existed in \( G \).
Heat-Kernel and Certificates

- If no balanced cut of conductance is found, our potential analysis yields:

$$\Psi(H^T_{G(T)}, V) \leq \frac{1}{\text{poly}(n)} \rightarrow L + \gamma \sum_{j=1}^{T-1} \sum_{i \in S_j} L(\text{Star}_i) \geq \gamma L(K_V)$$

Modified graph has $\lambda_2 \geq \gamma$

CLAIM: This is a certificate that no balanced cut of conductance $< \gamma$ existed in $G$. 

$$\phi(T) \geq \gamma - \gamma \frac{|\bigcup S_j|}{|T|}$$
Heat-Kernel and Certificates

• If no balanced cut of conductance is found, our potential analysis yields:

$$\Psi(H^T_{G(T)}, V) \leq \frac{1}{\text{poly}(n)} \quad \rightarrow \quad L + \gamma \sum_{j=1}^{T-1} \sum_{i \in S_j} L(\text{Star}_i) \geq \gamma L(K_V)$$

Modified graph has $\lambda_2 \geq \gamma$

CLAIM: This is a certificate that no balanced cut of conductance $< \gamma$ existed in G.

$$\phi(T) \geq \gamma - \gamma \frac{|\cup S_j|}{|T|} \geq \gamma - \gamma \frac{b/2}{b} \geq \gamma / 2$$
Comparison with Recursive Eigenvector

**RECURSIVE EIGENVECTOR:**
We can only bound number of iterations by volume of graph removed.
\[ \Omega(n) \text{ iterations possible.} \]

**OUR ALGORITHM:**
Use variance of random walk as potential.
Only \( O(\log n) \) iterations necessary.

\[ \Psi(P, V) = \sum_{i \in V} ||Pe_i - \bar{1}/n||^2 \]

**STABLE SPECTRAL NOTION OF POTENTIAL**
Running Time

- Our Algorithm runs in $O(\log n)$ iterations.

- In one iteration, we perform some simple computation (projection, sweep cut) on the vector embedding $H^\tau_{G(t)}$. This takes time $\tilde{O}(md)$, where $d$ is the dimension of the embedding.

- Can use Johnson-Lindenstrauss to obtain $d = O(\log n)$.

- Hence, we only need to compute $O(\log^2 n)$ matrix-vector products $H^\tau_{G(t)}u$.

- We show how to perform one such product in time $\tilde{O}(m)$ for all $\tau$.

- **OBSTACLE:**
  $\tau$, the mean number of steps in the Heat-Kernel random walk, is $\Omega (n^2)$ for path.
Conclusion

NOVEL ALGORITHMIC CONTRIBUTIONS

• Balanced-Cut Algorithm using Random Walks in time $\tilde{O}(m)$

MAIN IDEA
Random walks provide a very useful stable analogue of the graph eigenvector via regularization

OPEN QUESTION
More applications of this idea? Applications beyond design of fast algorithms?
A Different Interpretation

THEOREM:
Suppose eigenvector $x$ yields an unbalanced cut $S$ of low conductance and no balanced cut of the required conductance.

Then,

$$\sum_{i \in S} d_i x_i^2 \geq \frac{1}{2} \sum_{i \in V} d_i x_i^2.$$

In words, $S$ contains most of the variance of the eigenvector.
A Different Interpretation

THEOREM:
Suppose eigenvector \( x \) yields an unbalanced cut \( S \) of low conductance and no balanced cut of the required conductance.

\[
\sum d_i x_i = 0
\]

Then,

\[
\sum_{i \in S} d_i x_i^2 \geq \frac{1}{2} \sum_{i \in V} d_i x_i^2.
\]

In words, \( S \) contains most of the variance of the eigenvector.

QUESTION: Does this mean the graph induced by \( G \) on \( V - S \) is much closer to have conductance at least \( \gamma \)?
A Different Interpretation

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Then,

$$\sum_{i \in S} d_i x_i^2 \geq \frac{1}{2} \sum_{i \in V} d_i x_i^2.$$ 

QUESTION: Does this mean the graph induced by $G$ on $V - S$ is much closer to have conductance at least $\gamma$?

ANSWER: NO. $x$ may contain little or no information about $G$ on $V - S$. Next eigenvector may be only infinitesimally larger.

CONCLUSION: To make significant progress, we need an analogue of the eigenvector that captures sparse
Theorems for Our Algorithm

**THEOREM 1:** (WALKS HAVE NOT MIXED)

\[ \Psi(P^{(t)}, V) > \frac{1}{\text{poly}(n)} \]

Can find cut of conductance \( O(\sqrt{\gamma}) \)
Theorems for Our Algorithm

**THEOREM 1:** (WALKS HAVE NOT MIXED)

\[ \Psi(P^{(t)}, V) > \frac{1}{\text{poly}(n)} \]

Can find cut of conductance \( O(\sqrt{\gamma}) \)

**Proof:** Recall that

\[ P^{(t)} = e^{-\tau Q^{(t)}} \quad \tau = \log n / \gamma \]

Use the definition of \( \tau \). The spectrum of \( P^{(t)} \) implies that

\[ \sum_{i,j \in E} \left\| P^{(t)} e_i - P^{(t)} e_j \right\|^2 \cdot O(\gamma) \cdot \Psi(P^{(t)}, V) \]

\[ \Psi(P, V) = \sum_{i \in V} \| P e_i - \vec{1}/n \|^2 \]
Theorems for Our Algorithm

**THEOREM 1:** \textit{(WALKS HAVE NOT MIXED)}

\[ \Psi(P^{(t)}, V) > \frac{1}{\text{poly}(n)} \]

Can find cut of conductance \(O(\sqrt{\gamma})\)

**Proof:** Recall that

\[ P^{(t)} = e^{-\tau Q^{(t)}} \quad \tau = \log n / \gamma \quad \Psi(P, V) = \sum_{i \in V} \| P e_i - \frac{1}{n} \|^2 \]

Use the definition of \(\tau\). The spectrum of \(P^{(t)}\) implies that

\[ \sum_{i,j \in E} \| P^{(t)} e_i - P^{(t)} e_j \|^2 \cdot O(\gamma) \cdot \Psi(P^{(t)}, V) \]

Hence, by a random projection of the embedding \(\{P e_i\}\), followed by a sweep cut, we can recover the required cut.

SDP Rounding Technique
THEOREM 2: (WALKS HAVE MIXED)

\[ \Psi(P^{(t)}, V) \cdot \frac{1}{\text{poly}(n)} \rightarrow \text{No } \Omega(b)\text{-balanced cut of conductance } O(\gamma) \]
Theorems for Our Algorithm

**THEOREM 2:** (WALKS HAVE MIXED)

\[ \Psi(P^{(t)}, V) \cdot \frac{1}{\text{poly}(n)} \rightarrow \text{No } \Omega(b)\text{-balanced cut of conductance } O(\gamma) \]

**Proof:** Consider \( S = \bigcup S_i \). Notice that \( S \) is unbalanced.
Assumption is equivalent to

\[ L(K_V) \cdot e^{-\tau L - O(\log n)} \sum_{i \in S} L(S_i) \cdot \frac{1}{\text{poly}(n)}. \]
Theorems for Our Algorithm

**THEOREM 2:** (WALKS HAVE MIXED)

\[ \Psi(P^{(t)}, V) \cdot \frac{1}{\text{poly}(n)} \Rightarrow \text{No } \Omega(b)\text{-balanced cut of conductance } O(\gamma) \]

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\[
L(K_V) \cdot e^{-\tau L - O(\log n) \sum_{i \in S} L(S_i)} \cdot \frac{1}{\text{poly}(n)}.
\]

By taking logs,

\[
L + O(\gamma) \sum_{i \in S} L(S_i) \geq \Omega(\gamma)L(K_V).
\]

SDP DUAL CERTIFICATE
Theorems for Our Algorithm

THEOREM 2: (WALKS HAVE MIXED)

\[ \Psi(P^{(t)}, V) \cdot \frac{1}{\text{poly}(n)} \rightarrow \text{No } \Omega(b)\text{-balanced cut of conductance } O(\gamma) \]

Proof: Consider \( S = \bigcup S_i \). Notice that \( S \) is unbalanced.

Assumption is equivalent to

\[ L(K_V) \cdot e^{-\tau L - O(\log n) \sum_{i \in S} L(S_i)} \cdot \frac{1}{\text{poly}(n)}. \]

By taking logs,

\[ L + O(\gamma) \sum_{i \in S} L(S_i) \geq \Omega(\gamma)L(K_V). \]

This is a certificate that no \( \Omega(1) \)-balanced cut of conductance \( O(\gamma) \) exists, as evaluating the quadratic form for a vector representing a balanced cut \( U \) yields

\[ \phi(U) \geq \Omega(\gamma) - \frac{\text{vol}(S)}{\text{vol}(U)}O(\gamma) \geq \Omega(\gamma) \]

as long as \( S \) is sufficiently unbalanced.
SDP Interpretation

\[
\begin{align*}
&\mathbb{E}_{\{i,j\} \in E_G} \|v_i - v_j\|^2 \cdot \gamma, \\
&\mathbb{E}_{\{i,j\} \in V \times V} \|v_i - v_j\|^2 = \frac{1}{2m}, \\
&\forall i \in V \quad \mathbb{E}_{j \in V} \|v_i - v_j\|^2 \cdot \frac{1}{b} \cdot \frac{1}{2m}.
\end{align*}
\]

**SHORT EDGES**

**FIXED VARIANCE**

**LENGTH OF STAR EDGES**

\[
\begin{align*}
0 & \quad \text{STOP!}
\end{align*}
\]
SDP Interpretation

\[ \mathbb{E}_{\{i,j\} \in E_G} \| v_i - v_j \|^2 \cdot \gamma, \]

\[ \mathbb{E}_{\{i,j\} \in V \times V} \| v_i - v_j \|^2 = \frac{1}{2m}, \]

\[ \forall i \in V \quad \mathbb{E}_{j \in V} \| v_i - v_j \|^2 \cdot \frac{1}{b} \cdot \frac{1}{2m}. \]
For simplicity, take $G$ to be \textbf{d-regular}.

- The Heat-Kernel Random Walk is a \textbf{Continuous-Time Markov Chain} over $V$, modeling the \textbf{diffusion of heat} along the edges of $G$.

- Transitions take place in \textbf{continuous time} $t$, with an \textbf{exponential distribution}.

\[ \frac{\partial p(t)}{\partial t} = -L p(t) \]

\[ p(t) = e^{-\frac{t}{d}L} p(0) =: H_G^t p(0) \]

- The Heat Kernel can be interpreted as \textbf{Poisson distribution} over number of steps of the natural random walk $W \sim \mathcal{D}^d$.

\[ e^{-\frac{t}{d}L} = e^{-t} \sum_{k=1}^{\infty} \frac{t^k}{k!} W^k \]

- In practice, can replace Heat-Kernel with natural random walk $ W^t$