A Nearly-linear-time Spectral Algorithm for Balanced Graph Partitioning

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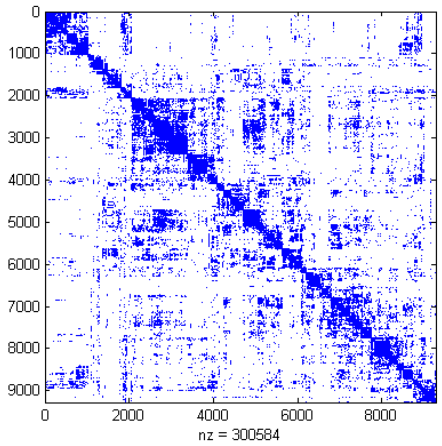
Joint work with
Nisheeth Vishnoi (MSR India) and Sushant Sachdeva (Princeton)

Spectral Algorithms for Graph Problems

What are they?

Spectral algorithms see the input graph as a linear operator



Adjacency matrix of a graph representing similarities between music artists.
Image by David Gleich.

Algorithmic primitives:

- matrix multiplication,
- eigenvector decomposition,
- Gaussian elimination

Spectral Algorithms for Graph Problems

Why are they useful …

… in practice?

- Simple to implement
- Can exploit very efficient linear algebra routines
- Perform provably well for many problems

… in theory?

- Many new interesting results based on spectral techniques:

  Laplacian Solvers, Sparsification, s-t Maxflow, etc.
- Geometric viewpoint
- Connections to Markov Chains and Probability Theory

Spectral Algorithms for Graph Partitioning

Spectral algorithms are widely used in many graph-partitioning applications:
clustering, image segmentation, community-detection, etc.

CLASSICAL VIEW:

- Based on Cheeger's Inequality

- Eigenvectors sweep-cuts reveal sparse cuts in the graph

NEW TREND:

- Random walk vectors replace eigenvectors:

  • Fast Algorithms for Graph Partitioning

  • Local Graph Partitioning

  • Real Network Analysis

- Different random walks: PageRank, Heat-Kernel, etc.

Why Random Walks? A Practitioner's View

Advantages of Random Walks:

1) Quick approximation to eigenvector in massive graphs

A= adjacency matrix

D = diagonal degree matrix

$W = AD^{-1}$= natural random walk matrix

$L = D - A$ = Laplacian matrix

Second Eigenvector of the Laplacian can be computed by iterating W:

For random y_0 s.t. $y_0^T D^{-1} 1 = 0$, compute

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Heuristic: For massive graphs, pick *t* as large as computationally affordable.

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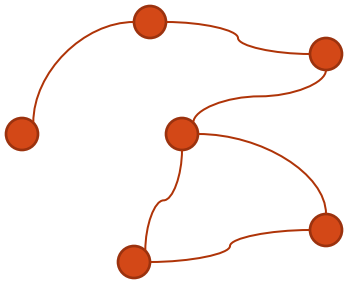
Real-world graphs are noisy



GROUND TRUTH GRAPH

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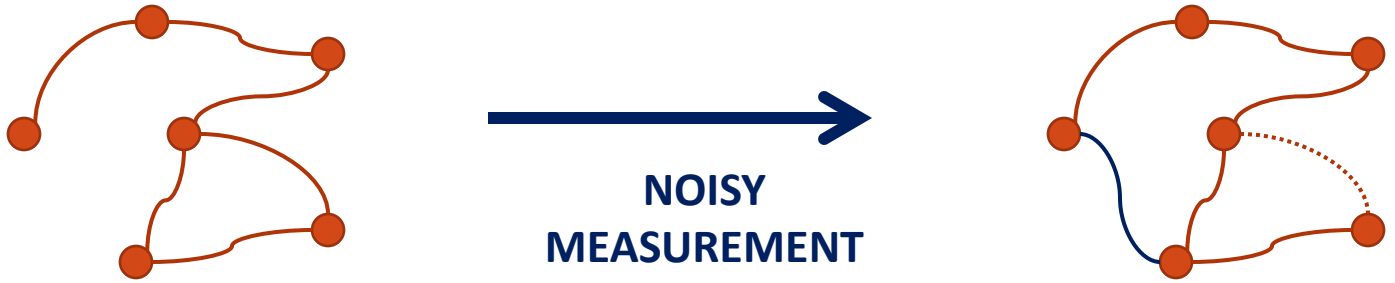

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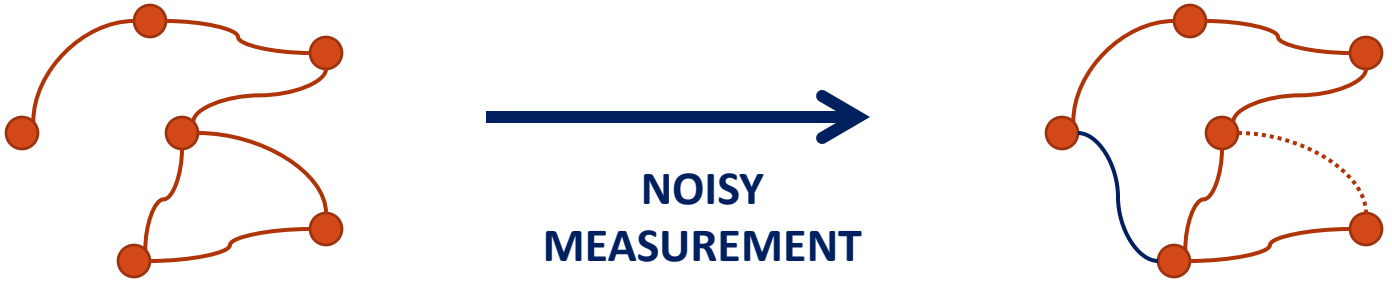

GROUND-TRUTH GRAPH **INPUT GRAPH**

GOAL: estimate eigenvector of ground-truth graph.

OBSERVATION: eigenvector of input graph can have very large variance, as it can be very sensitive to noise

RANDOM-WALK VECTORS provide more stable estimates.

This Talk

QUESTION:

Why random-walk vectors in the design of fast algorithms?

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GOALS OF THIS TALK:

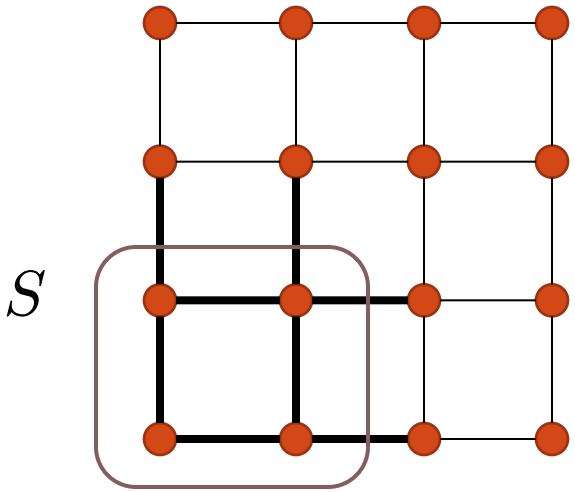
1) Showcase how these ideas apply to balanced graph partitioning

2) Show optimization perspective on why random walks arise

Problem Definition
and
Overview of Results

Partitioning Graphs - Conductance

Undirected unweighted $G = (V, E), |V| = n, |E| = m$



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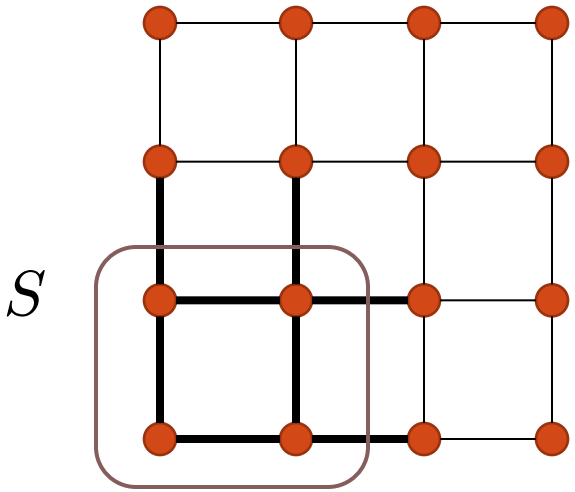

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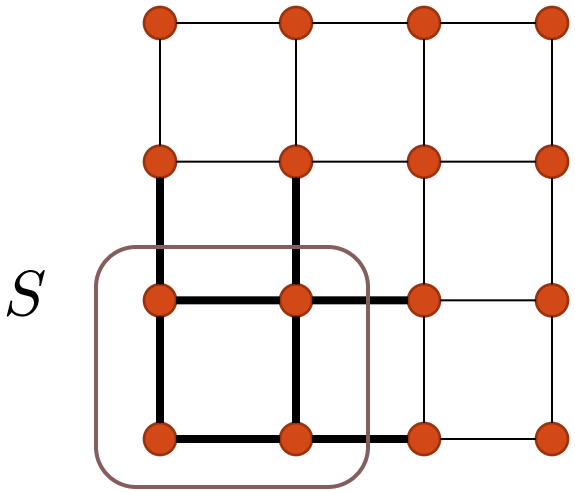

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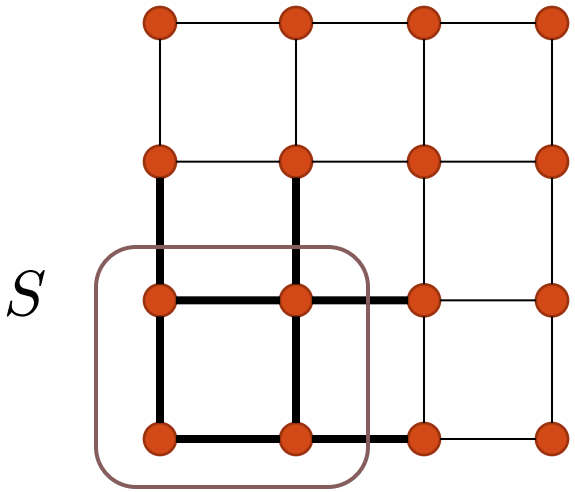

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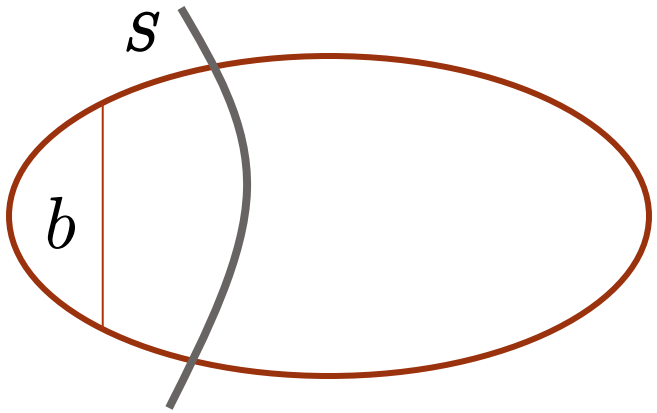
Partitioning Graphs – Balanced Cut

NP-HARD DECISION PROBLEM

Does G have a b-balanced cut of conductance < γ ?



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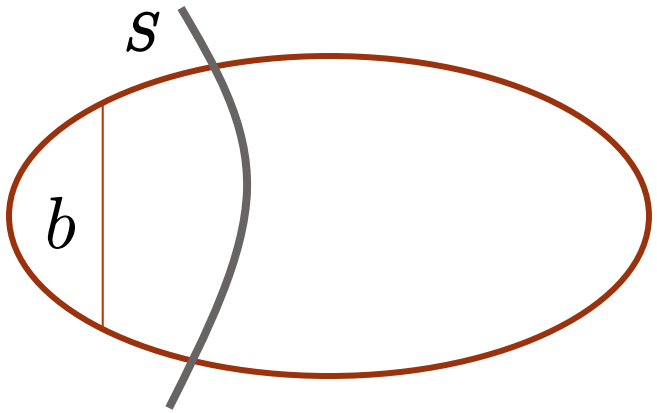

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- Important primitive for many **recursive** algorithms.
- Applications to **clustering** and **graph decomposition**.

Approximation Algorithms

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Approximation Algorithms

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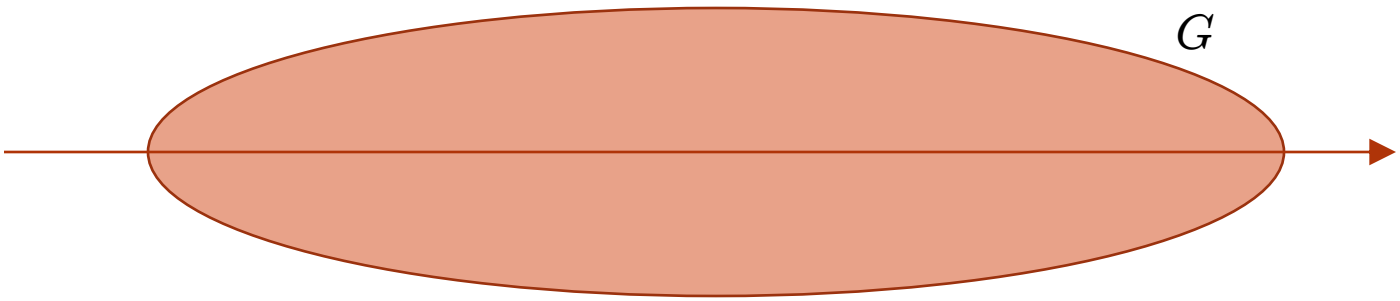

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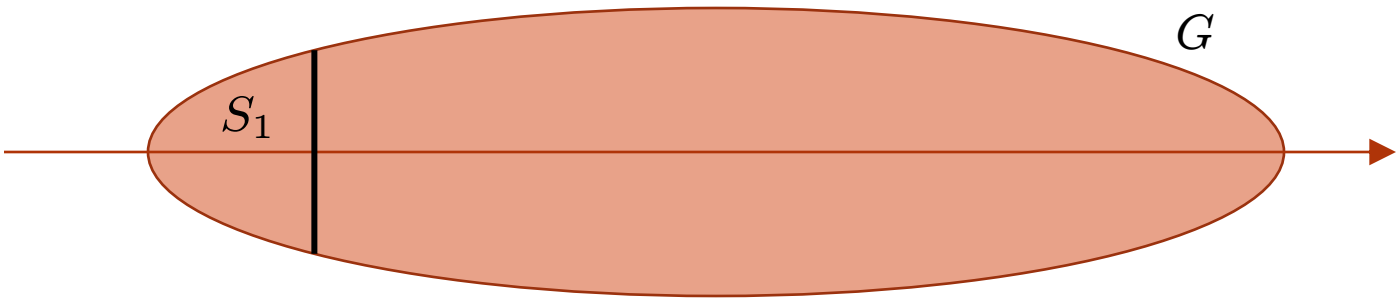

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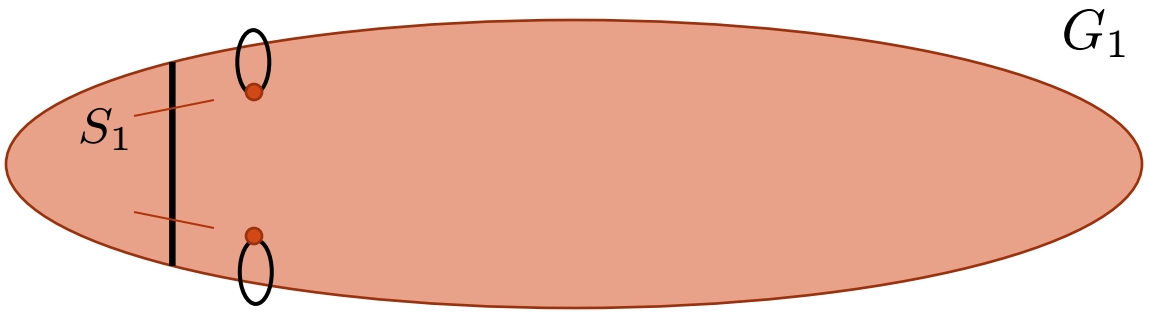

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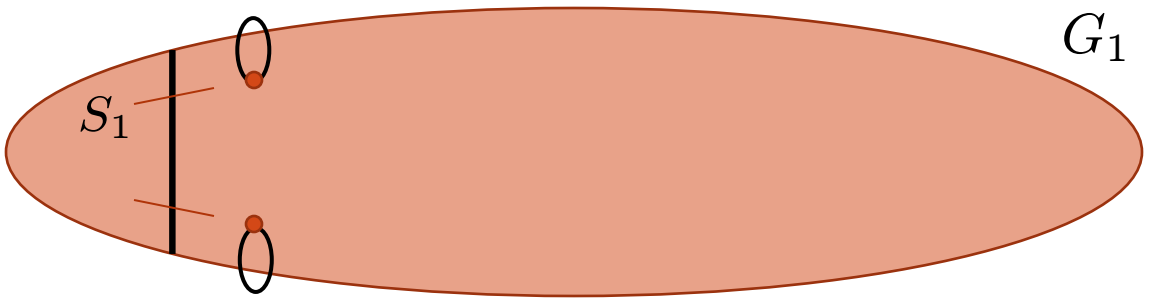

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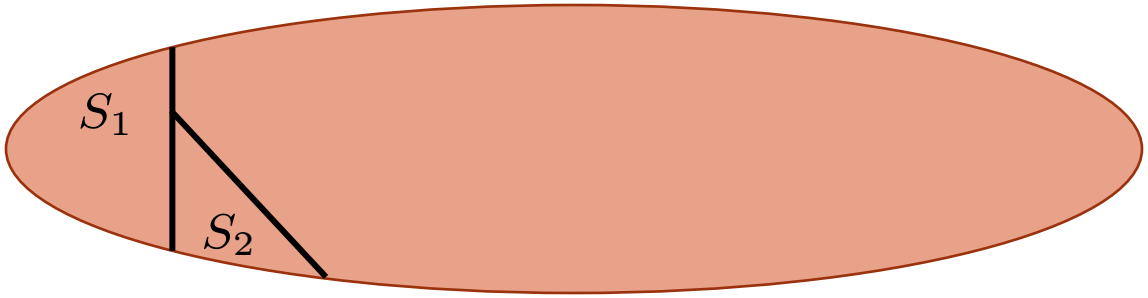

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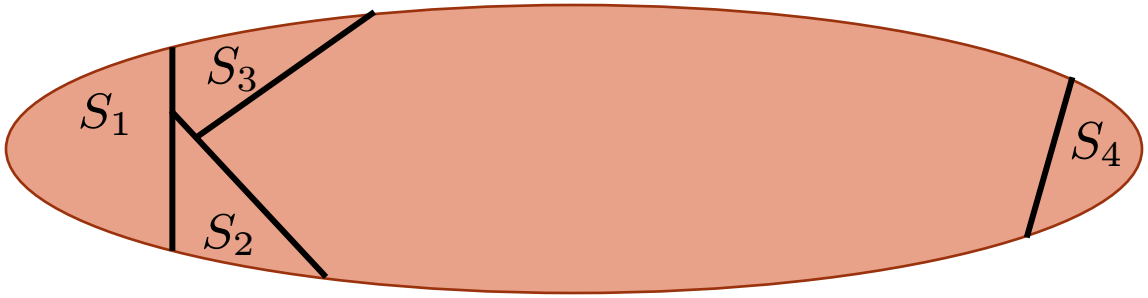

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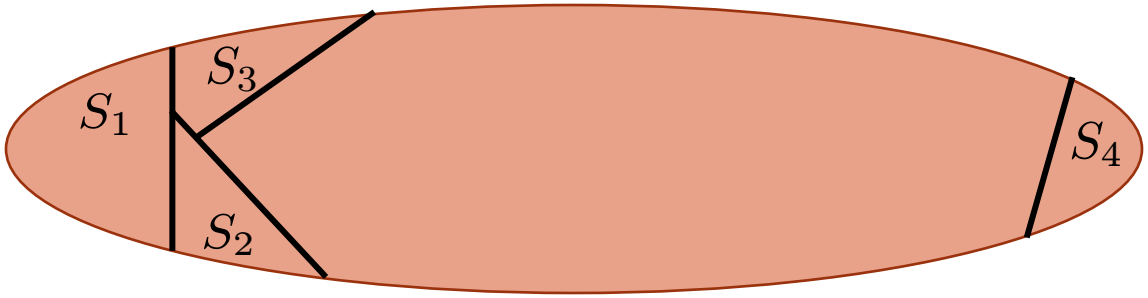

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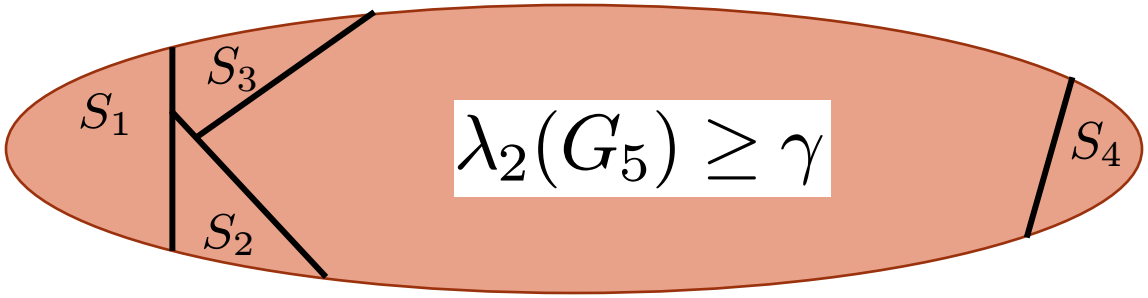

LARGE INDUCED EXPANDER = **NO-CERTIFICATE**

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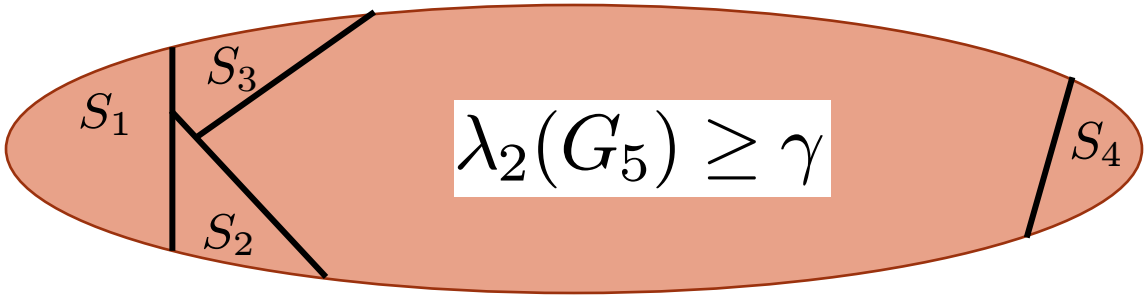
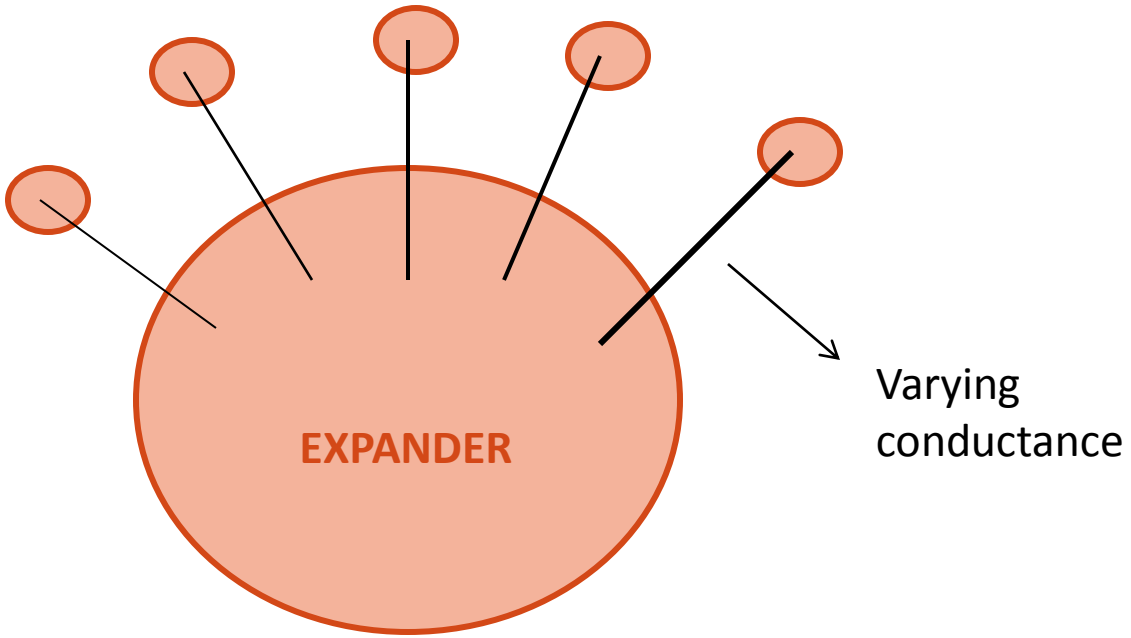


RUNNING TIME:$\tilde{O}(mn)$. $\tilde{O}(m)$ per iteration, $O(n)$ iterations.

Recursive Eigenvector: The Worst Case



- (n) nearly-disconnected components

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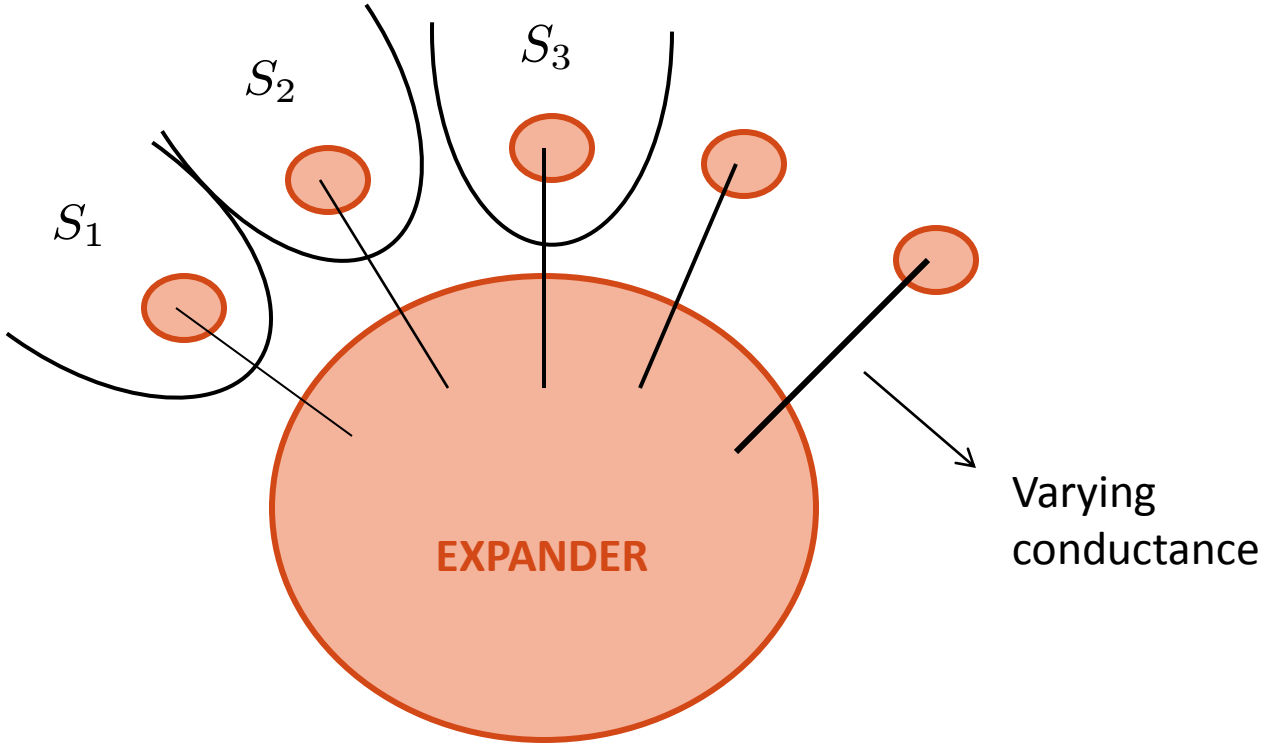

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NB: Recursive Eigenvector eliminates one component per iteration.

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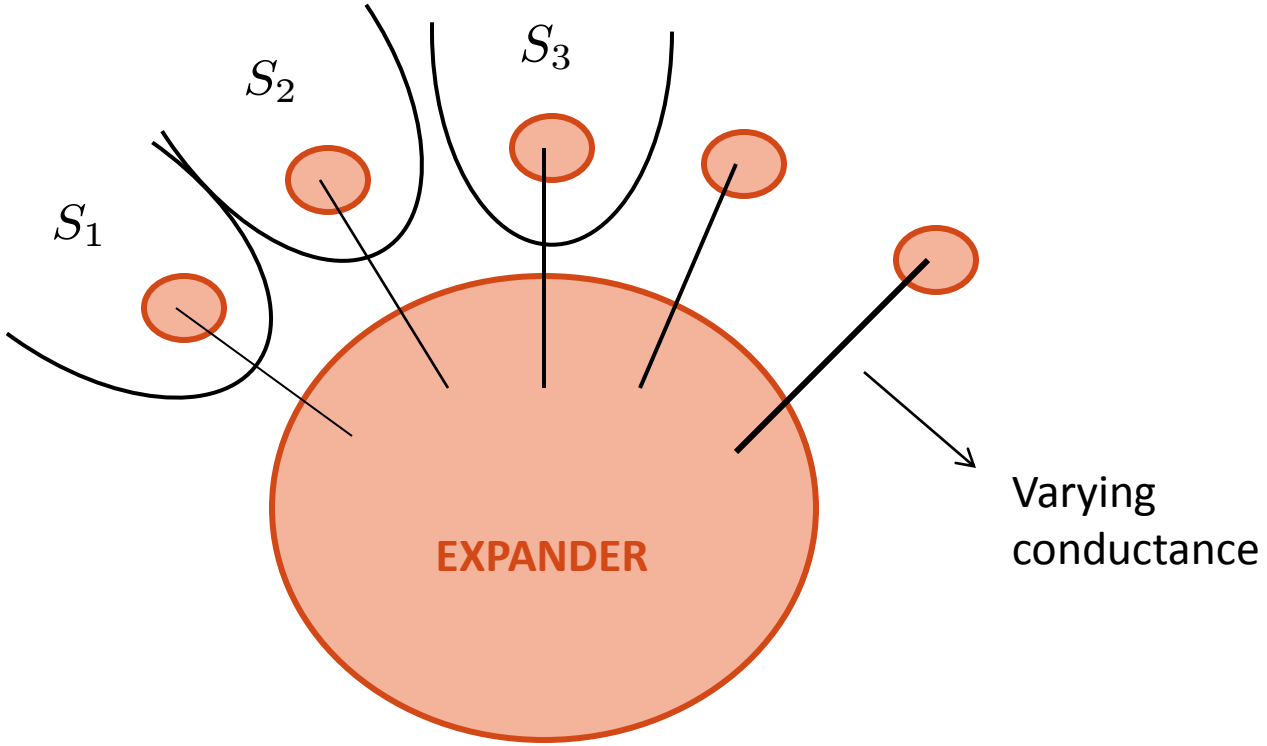

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GOAL: Eliminate unbalanced low-conductance cuts faster.

Our Algorithm: Contributions

Algorithm	Method	Distinguishes $\geq \gamma$ and	Time
Recursive Eigenvector	Eigenvector	$O(\sqrt{\gamma})$	$\tilde{O}(mn)$
OUR ALGORITHM	Random Walks	$O(\sqrt{\gamma})$	$\tilde{O}(m)$

MAIN FEATURES:

- Compute $O(\log n)$ global heat-kernel random-walk vectors at each iteration
- Unbalanced cuts are removed in $O(\log n)$ iterations
- Method to compute heat-kernel vectors, i.e. $e^{-tL}v$, in time $\tilde{O}(m)$

TECHNICAL COMPONENTS:

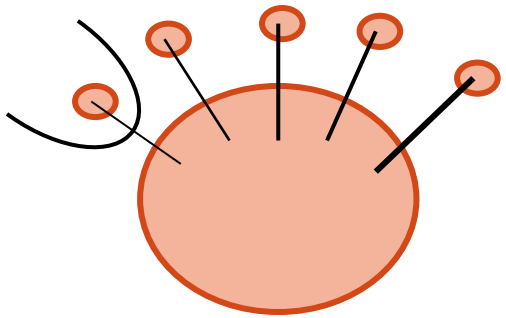
1) New iterative algorithm with a simple random walk interpretation

2) Novel analysis of Lanczos methods for computing heat-kernel vectors

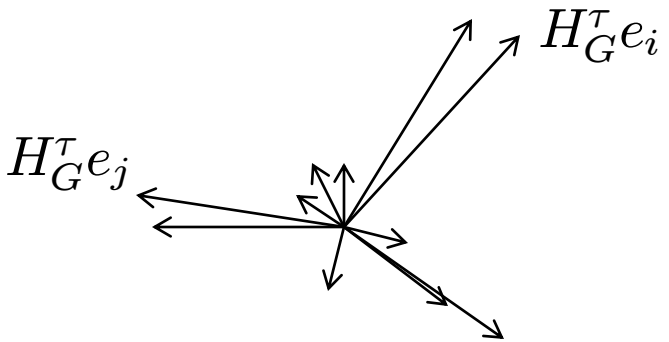
Why Random Walks

GOAL: Eliminate unbalanced cuts of low-conductance efficiently

- The graph eigenvector may be correlated with only one sparse unbalanced cut.



- Consider the Heat-Kernel random walk H_G^τ for $\tau = \log n/\gamma$.

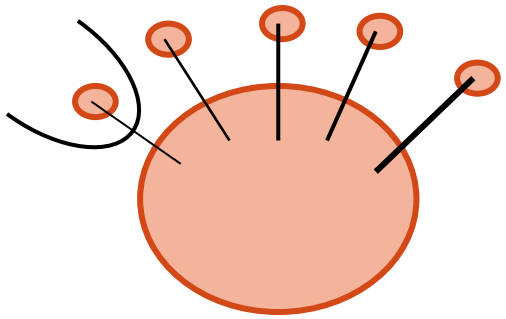


Probability vector for random walk started at vertex i

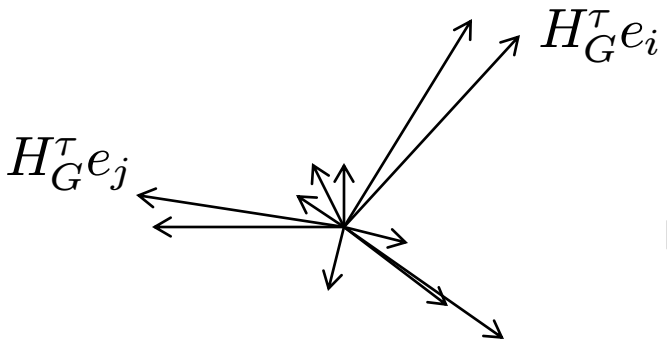
Why Random Walks

GOAL: Eliminate unbalanced cuts of low-conductance efficiently

- The graph eigenvector may be correlated with only one sparse unbalanced cut.



- Consider the Heat-Kernel random walk-matrix H_G^τ for $\tau = \log n/\gamma$.



Probability vector for random walk started at vertex i

Long vectors are slow-mixing random walks

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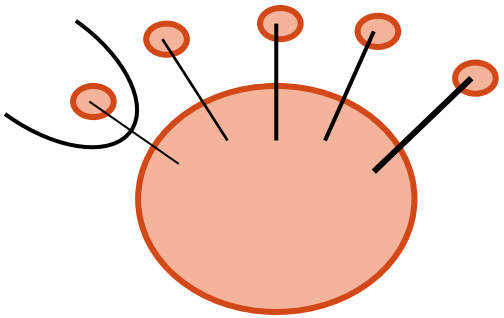

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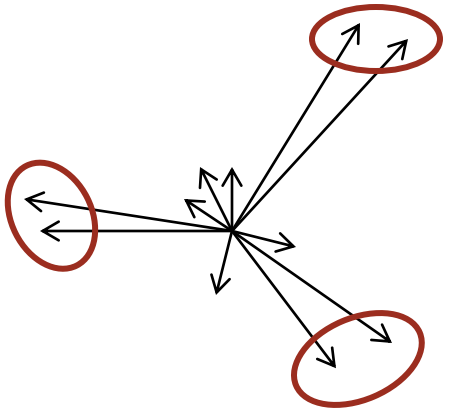

Unbalanced cuts of conductance $< \sqrt{\gamma}$

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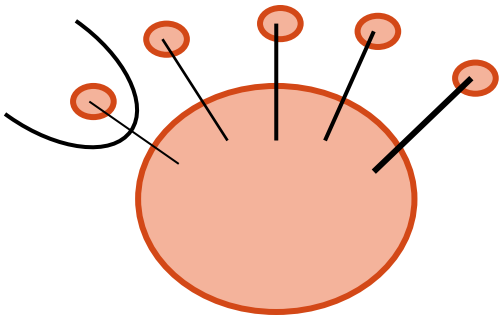
SINGLE VECTOR
SINGLE CUT



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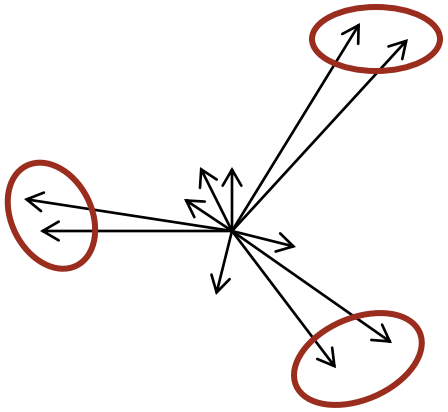
VECTOR EMBEDDING
MULTIPLE CUTS



Unbalanced cuts of conductance $< \sqrt{\gamma}$

Heat-Kernel Random Walk: A Regularization View

Background: Heat-Kernel Random Walk

For simplicity, take G to be **d-regular**.

• The Heat-Kernel Random Walk is a Continuous-Time Markov Chain over V, modeling the diffusion of heat along the edges of G.

• Transitions take place in continuous time t, with an exponential distribution.

$$\frac{\partial p(t)}{\partial t} = -L\frac{p(t)}{d}$$

$$p(t) = e^{-\frac{t}{d}L}p(0)$$

• The Heat Kernel can be interpreted as Poisson distribution over number of steps of the natural random walk $W{=}AD^{-1}$:

$$e^{-\frac{t}{d}L} = e^{-t}\sum_{k=1}^{\infty}\frac{t^k}{k!}W^k$$

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• In practice, can replace Heat-Kernel with natural random walk W^t

Regularized Spectral Optimization

Eigenvector Program

$$\frac{1}{d} \min x^T L x$$
$$\text{s.t. } ||x||_2 = 1$$
$$x^T 1 = 0$$

$\longleftrightarrow$

SDP Formulation

$$\frac{1}{d} \min \quad L \bullet X$$
$$\text{s.t.} \quad I \bullet X = 1$$
$$11^T \bullet X = 0$$
$$X \succeq 0$$

Programs have same optimum. Take optimal solution

$$X^* = x^* (x^*)^T$$

Regularized Spectral Optimization

SDP Formulation

$$\frac{1}{d}\min \quad L \bullet X$$

$$\text{s.t.} \quad I \bullet X = 1$$

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Density Matrix

Eigenvector decomposition of X:

$$X = \sum p_i v_i v_i^T$$

$$\forall i, p_i \geq 0,$$

$$\sum p_i = 1,$$

$$\forall i, v_i^T 1 = 0.$$

Eigenvalues of X define probability distribution

Regularized Spectral Optimization

SDP Formulation

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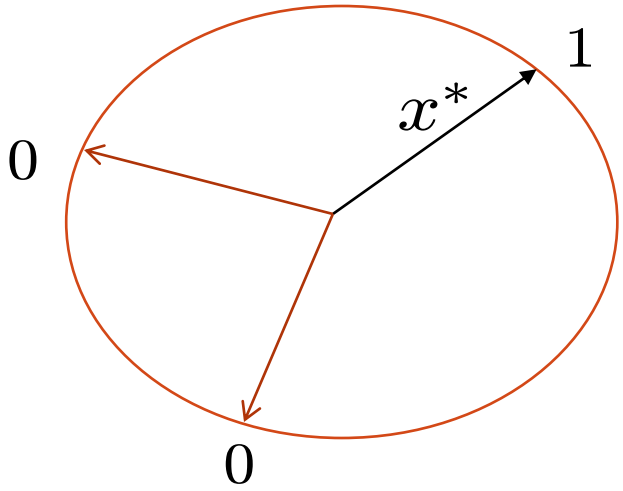
TRIVIAL DISTRIBUTION



Regularized Spectral Optimization

Regularized SDP

$$\frac{1}{d}\min \; L \bullet X + \boxed{\eta \cdot F(X)}$$

Regularizer F
Parameter η

$$\text{s.t.} \quad I \bullet X = 1$$

$$J \bullet X = 0$$

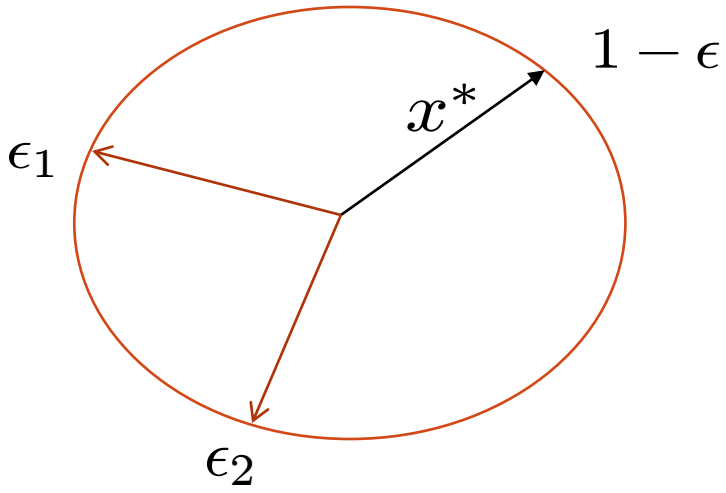
$$X \succeq 0$$

The regularizer F forces the distribution of eigenvalues of X to be non-trivial

$$X^* = x^*(x^*)^T$$

REGULARIZATION

$$X^* = \sum p_i v_i v_i^T$$



Regularizers

Regularizers are SDP-versions of common regularizers

- von Neumann Entropy

$$F_H(X) = \mathrm{Tr}(X \log X) = \sum p_i \log p_i$$

- p-Norm, $p>1$

$$F_p(X) = \frac{1}{p}||X||_p^p = \frac{1}{p}\mathrm{Tr}(X^p) = \frac{1}{p}\sum p_i^p$$

- And more, e.g. log-determinant.

Our Main Result

Regularized SDP

$$\frac{1}{d}\min \; L \bullet X + \eta \cdot F(X)$$
$$\text{s.t.} \quad I \bullet X = 1$$
$$J \bullet X = 0$$
$$X \succeq 0$$

RESULT:

Explicit correspondence between **regularizers and random walks**

REGULARIZER		OPTIMAL SOLUTION OF REGULARIZED PROGRAM	
$F = F_H$	Entropy →	$X^\star \propto H_t$	where t depends on η
			where α depends on η
$F = F_p$	p-Norm →	$X^\star \propto T_{q,\frac{1}{p-1}}$	where q depends on η

Discussion: Vector vs Density Matrix

Variable is density matrix, **not** vector

Q: Can we produce a single vector?

A: Density Matrix X describes distributions over vectors.

Assuming distribution is Gaussian, sample a vector

$$x = X^{\frac{1}{2}} u$$

where $u \sim N(0, I_{n-1})$

For example, the vector

$$x = H_{t/2} u$$

Random walk on random seed

is a sample from the solution of an entropy-regularized problem

PART 1
The New Iterative Procedure

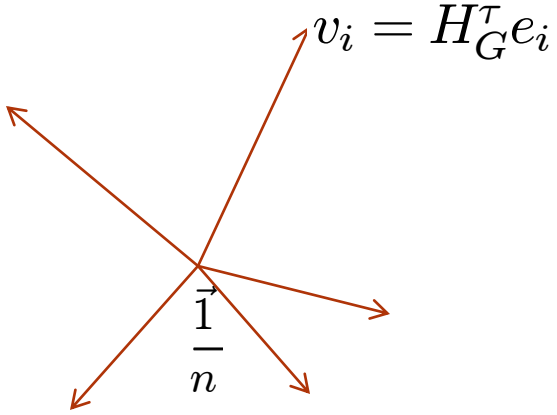
Our Algorithm for Balanced Cut

IDEA BEHIND OUR ALGORITHM:

Replace eigenvector in recursive eigenvector algorithm with

Heat-Kernel random walk H_G^τ for $\tau = \log n/\gamma$

Consider the embedding $\{v_i\}$ given by H_G^τ :



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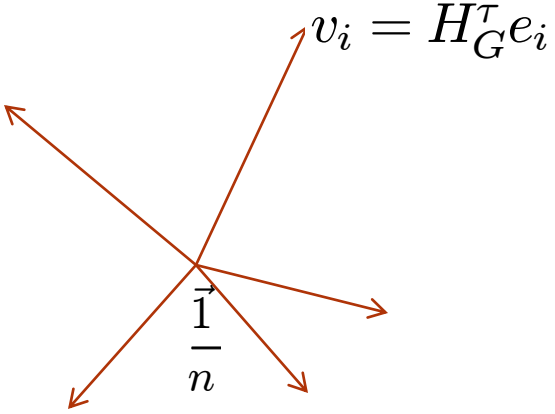

Stationary distribution is uniform as G is regular

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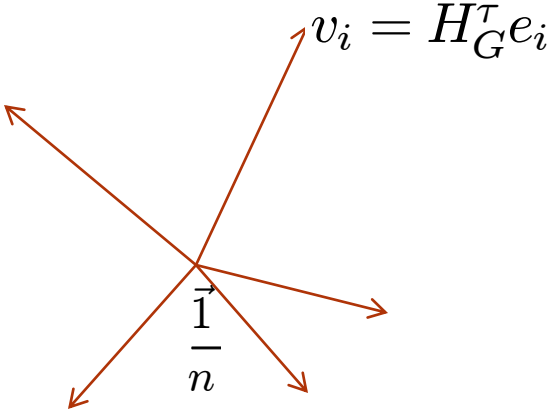


Stationary distribution is uniform as G is regular

MIXING:

Define the total deviation from stationary for a set $S \subseteq V$ for walk H_G^τ

$$\Psi(H_G^\tau, S) = \sum_{i \in S} ||v_i - \vec{1}/n||^2$$

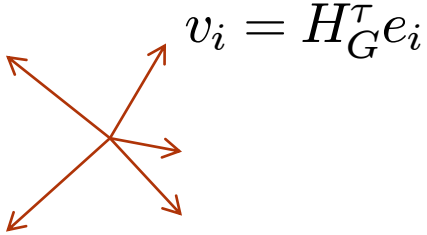
FUNDAMENTAL QUANTITY TO UNDERSTAND CUTS IN G

Our Algorithm: Case Analysis

Recall:

$$\tau = \log n/\gamma \qquad \Psi(H_G^\tau, S) = \sum_{i \in S} ||H_G^\tau e_i - \vec{1}/n||^2$$

CASE 1: Random walks have **mixed**



ALL VECTORS ARE SHORT

$$\Psi(H_G^\tau, V) \cdot \frac{1}{\text{poly}(n)}$$

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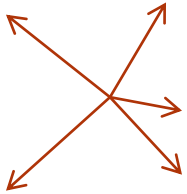
$$v_i = H_G^\tau e_i$$



ALL VECTORS ARE SHORT

$$\Psi(H_G^\tau, V) \cdot \frac{1}{\text{poly}(n)}$$

By definition of τ

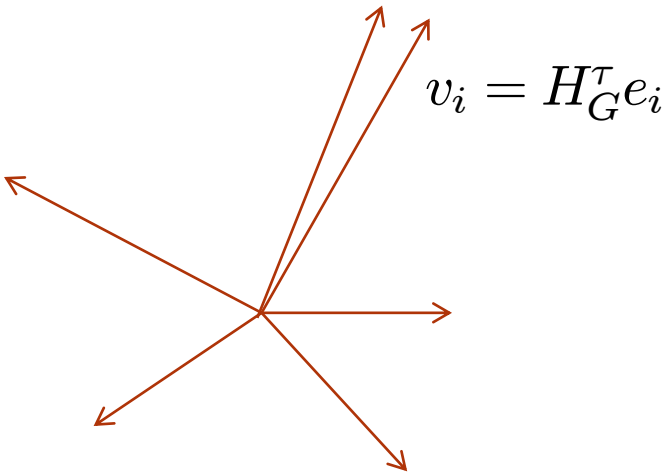
$$\lambda_2 \geq - \ (\gamma)$$

$$\phi_G \geq - \ (\gamma)$$

Our Algorithm

$\tau = \log n/\gamma$

$\Psi(H_G^\tau, S) = \sum_{i \in S} ||H_G^\tau e_i - \vec{1}/n||^2$



CASE 2: Random walks have **not mixed**

$$\Psi(H_G^\tau, V) > \frac{1}{\text{poly}(n)}$$

We can either find an $\Omega(b)$-balanced cut with conductance $O(\sqrt{\gamma})$

Our Algorithm

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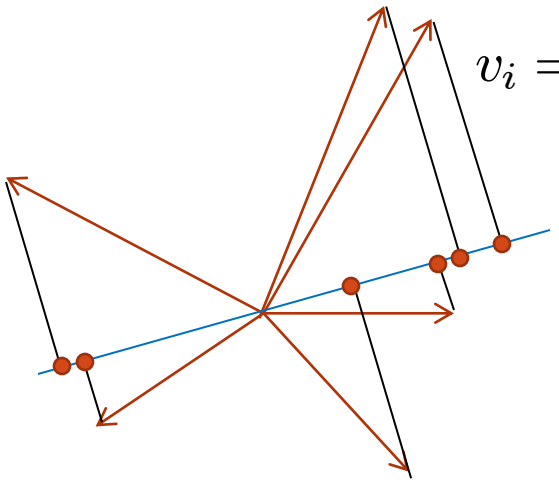

$v_i = H_G^\tau e_i$

RANDOM PROJECTION
+
SWEEP CUT

CASE 2: Random walks have **not mixed**

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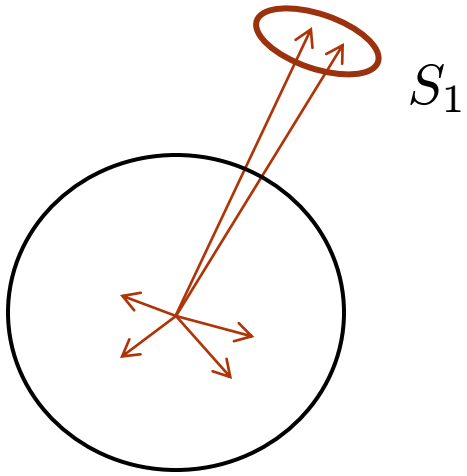


BALL ROUNDING

CASE 2: Random walks have **not mixed**

$$\Psi(H_G^\tau, V) > \frac{1}{\text{poly}(n)}$$

We can either find an $\Omega(b)$-balanced cut with conductance · $O(\sqrt{\gamma})$

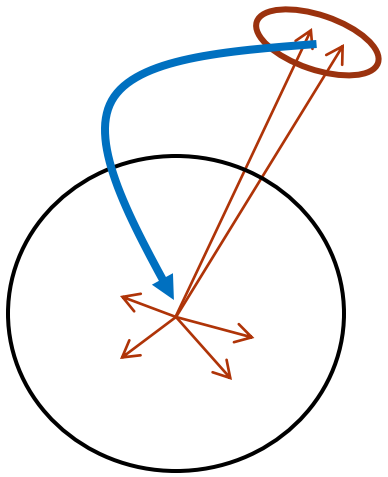
OR a ball cut yields S_1 such that $\phi(S_1)$ · $O(\sqrt{\phi})$ and

$$\Psi(H_G^\tau, S_1) \geq \tfrac{1}{2}\Psi(H_G^\tau, V).$$

Our Algorithm: Iteration

$\tau = \log n/\gamma$

$\Psi(H_G^\tau, S) = \sum_{i \in S} ||H_G^\tau e_i - \vec{1}/n||^2$



CASE 2: We found an unbalanced cut S_1 with $\phi(S_1) \cdot O(\sqrt{\phi})$ and

$$\Psi(H_G^\tau, S_1) \geq \tfrac{1}{2}\Psi(H_G^\tau, V).$$

Modify G=$G^{(1)}$ by **adding edges** across $(S_1, \bar{S_1})$ to construct $G^{(2)}$.

Analogous to removing unbalanced cut S_1
in Recursive Eigenvector algorithm

Our Algorithm: Modifying G

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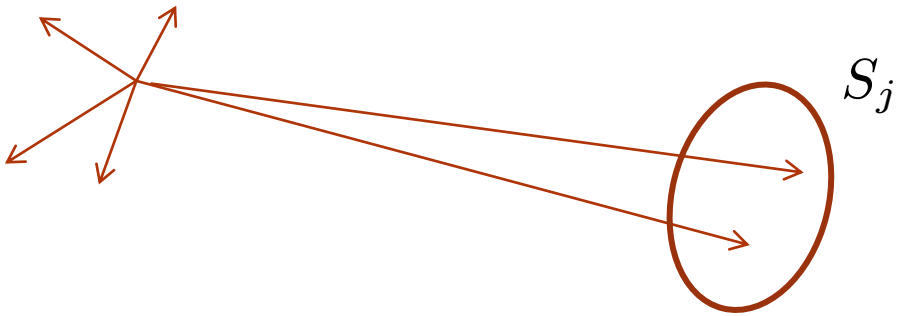

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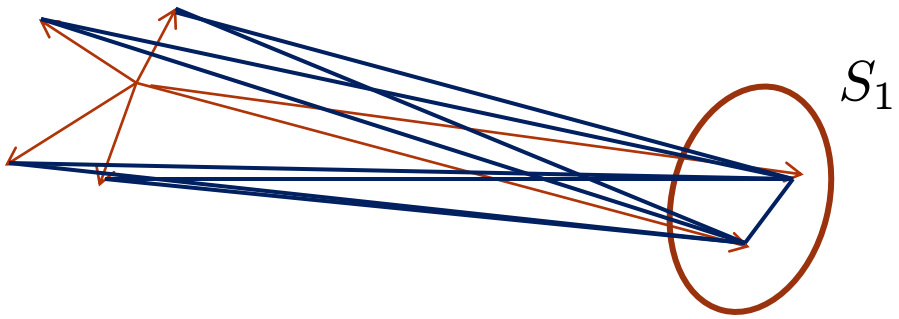

$$G^{(t+1)} = G^{(t)} + \gamma \sum_{i \in S_t} \mathrm{Star}_i$$

where Star_i is the star graph rooted at vertex i.

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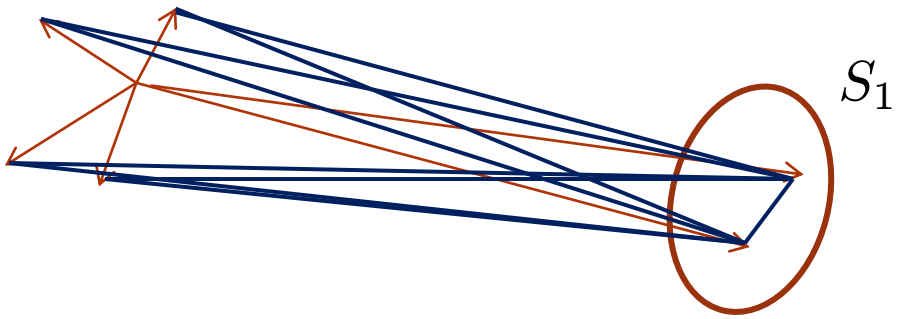

$$G^{(t+1)} = G^{(t)} + \gamma \sum_{i \in S_t} \mathrm{Star}_i$$

where Star_i is the star graph rooted at vertex i.

The random walk can now escape S_1 more easily.

Our Algorithm: Iteration

$\tau = \log n/\gamma$

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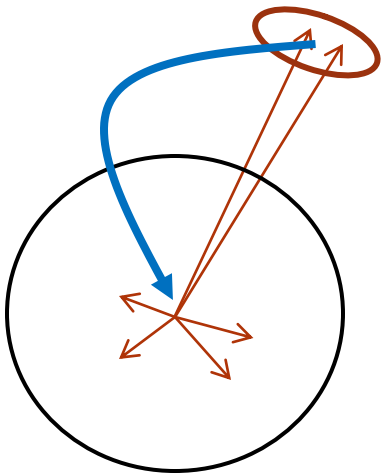

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Modify G=$G^{(1)}$ by **adding edges** across $(S_1, \bar{S}_1)$ to construct $G^{(2)}$.

POTENTIAL REDUCTION:

$$\Psi(H^\tau_{G^{(t+1)}}, V) \cdot \Psi(H^\tau_{G^{(t)}}, V) - \tfrac{1}{2}\Psi(H^\tau_{G^{(t)}}, S_t) \cdot \tfrac{3}{4}\Psi(H^\tau_{G^{(t)}}, V)$$

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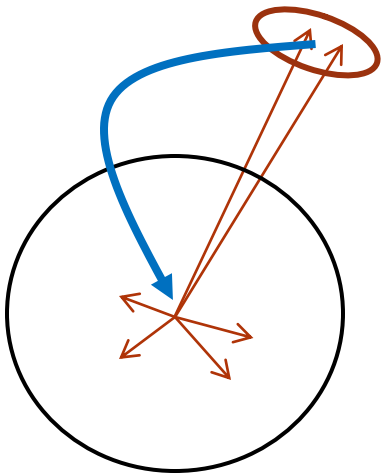

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CRUCIAL APPLICATION OF STABILITY OF RANDOM WALK

Summary and Potential Analysis

IN SUMMARY:

At every step t of the recursion, we either

1. Produce a $\Omega(b)$-balanced cut of the required conductance, OR

Potential Reduction

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At every step t of the recursion, we either

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$$\Psi(H_G^\tau, V) \cdot \frac{1}{\text{poly}(n)}$$, OR

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3. Find an unbalanced cut S_t of the required conductance,
such that for the graph $G^{(t+1)}$, modified to have increased edges from S_t,

$$\boxed{\Psi(H^{\tau}_{G^{(t+1)}}, V) \cdot \frac{3}{4}\Psi(H^{\tau}_{G^{(t)}}, V)}$$

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$$\Psi(H_{G^{(t+1)}}^\tau, V) \cdot \frac{3}{4}\Psi(H_{G^{(t)}}^\tau, V)$$

After $T{=}O(\log n)$ iterations, if no low-conductance $\Omega(b)$-balanced cut is found:

$$\Psi(H_{G^{(T)}}^\tau, V) \cdot \frac{1}{\mathrm{poly}(n)}$$

From this guarantee, using the definition of $G^{(T)}$, we derive an SDP-certificate that no b-balanced cut of conductance $O(\gamma)$ exists in G.

NB: Only O(log n) iterations to remove unbalanced cuts.

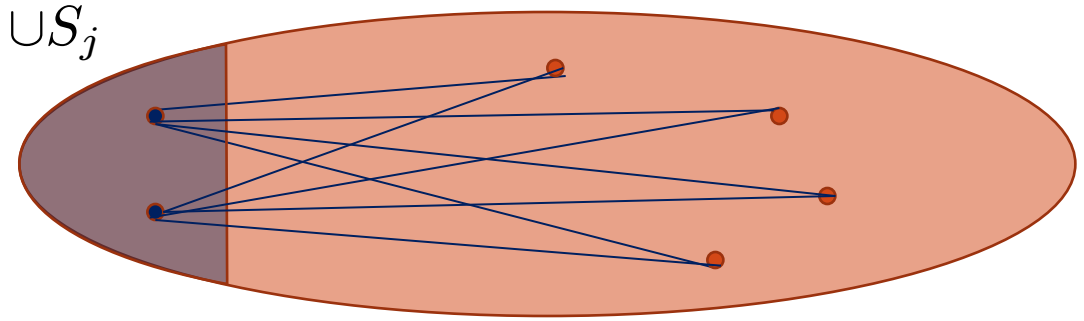
Heat-Kernel and Certificates

- If no balanced cut of conductance is found, our potential analysis yields:

$$\Psi(H^{\tau}_{G^{(T)}}, V) \cdot \frac{1}{\mathrm{poly}(n)} \longrightarrow L + \gamma \sum_{j=1}^{T-1} \sum_{i \in S_j} L(\mathrm{Star}_i) \succeq \gamma L(K_V)$$

Modified graph has $\lambda_2 \geq \gamma$

CLAIM: This is a certificate that no balanced cut of conductance < γ existed in G.



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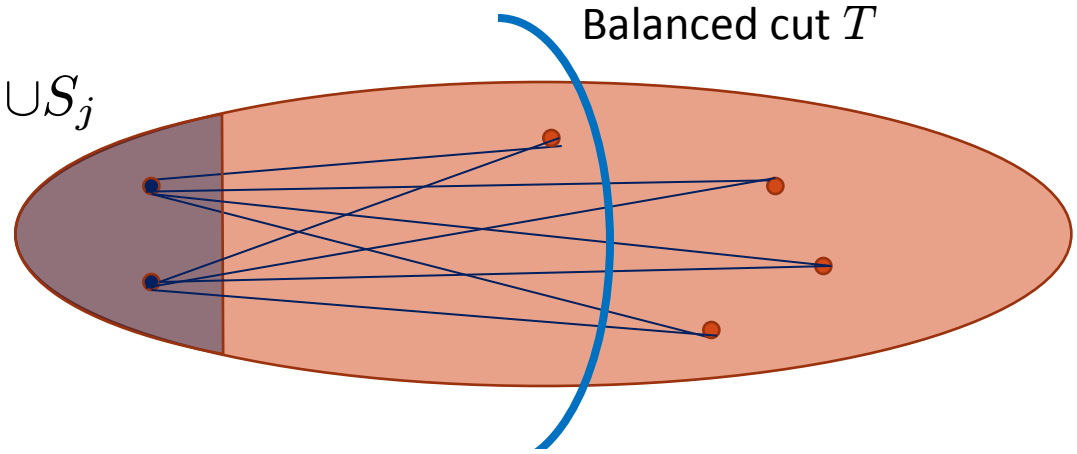


$$\phi(T) \geq \gamma - \gamma \frac{|\cup S_j|}{|T|}$$

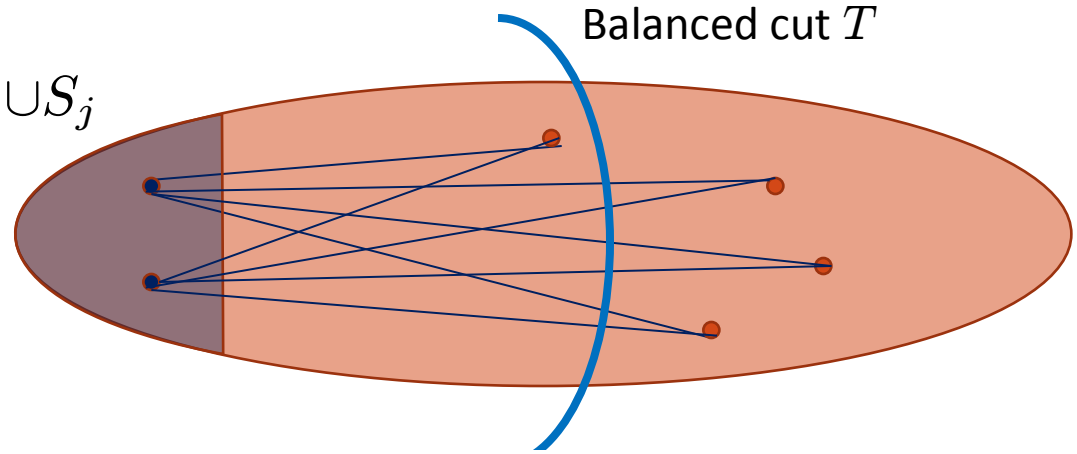
Heat-Kernel and Certificates

- If no balanced cut of conductance is found, our potential analysis yields:

$$\Psi(H^{\tau}_{G^{(T)}}, V) \cdot \frac{1}{\mathrm{poly}(n)} \longrightarrow L + \gamma \sum_{j=1}^{T-1} \sum_{i \in S_j} L(\mathrm{Star}_i) \succeq \gamma L(K_V)$$

Modified graph has $\lambda_2 \geq \gamma$

CLAIM: This is a certificate that no balanced cut of conductance < γ existed in G.



$$\phi(T) \geq \gamma - \gamma \frac{|\cup S_j|}{|T|} \geq \gamma - \gamma \frac{b/2}{b} \geq \gamma/2$$

Comparison with Recursive Eigenvector

RECURSIVE EIGENVECTOR:

We can only bound number of iterations by volume of graph removed.

$\Omega(n)$ iterations possible.

OUR ALGORITHM:

Use variance of random walk as potential.

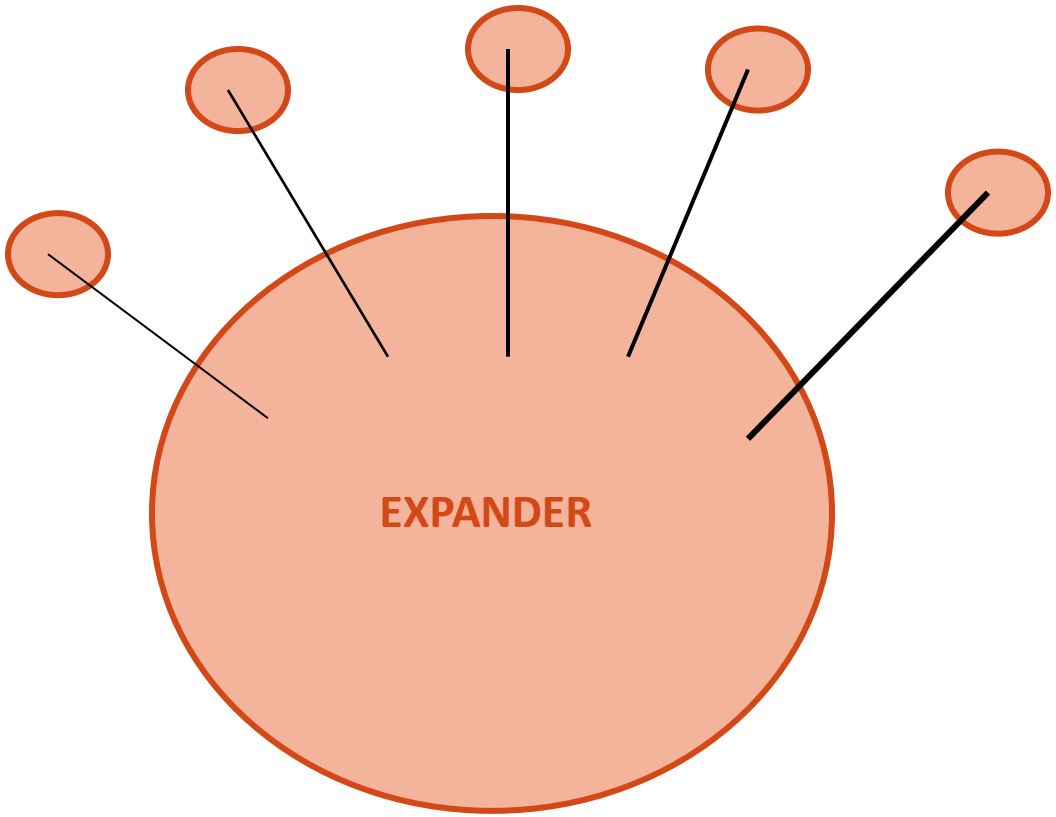
Only $O(\log n)$ iterations necessary.

$$\Psi(P, V) = \sum_{i \in V} ||Pe_i - \vec{1}/n||^2$$

RIGHT SPECTRAL NOTION OF POTENTIAL

Comparison with Recursive Eigenvector

Recall worst-case example for Recursive Eigenvector:



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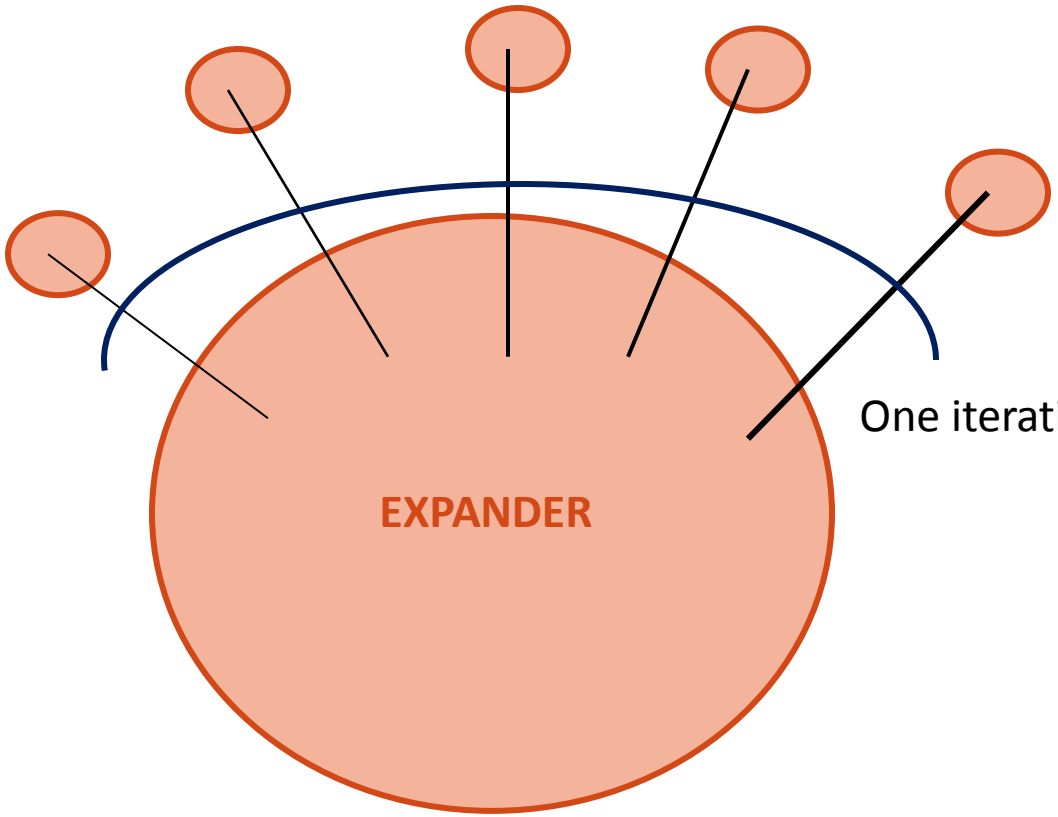

One iteration of our algorithm

Running Time

- Our Algorithm runs in $O(\log n)$ iterations.

- In one iteration, we perform some simple computation (projection, sweep cut) on the vector embedding $H^{\tau}_{G^{(t)}}$.
  This takes time $\tilde{O}(md)$, where d is the dimension of the embedding.

- Can use Johnson-Lindenstrauss to obtain $d = O(\log n)$.

- Hence, we only need to compute $O(\log^2 n)$ matrix-vector products

$$H^{\tau}_{G^{(t)}} u$$

- We show how to perform one such product in time $\tilde{O}(m)$ for all τ.

- OBSTACLE:

  τ, the mean number of steps in the Heat-Kernel random walk, is $\Omega(n^2)$ for path.

PART 2
Matrix Exponential Computation

Computation of Matrix Exponential

GOAL: For symmetric diagonally-dominant A, with sparsity m, compute vector v such that

$$||e^{-A}u - v|| \cdot \frac{1}{\text{poly(n)}}$$

in time $\tilde{O}(m)$.

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$$A = \frac{\log n}{\gamma} L$$

$$||A|| = \tilde{O}(\frac{1}{\gamma})$$

$\longrightarrow$ Running Time: $\tilde{O}\left(\frac{m}{\sqrt{\gamma}}\right)$

Review of Lanczos Method

IDEA: Perform k matrix-vector multiplications

$$\{u, Au, A^2u, A^3u, A^4u, \ldots, A^ku\}$$

Review of Lanczos Method

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$$R_k = \text{Span}\{u, Au, A^2u, A^3u, A^4u, \ldots, A^ku\}$$

k-Krylov Subspace

Compute orthonormal basis Q_k for R_k, consider A restricted to R_k

$$T_k = Q_k A Q_k^T$$

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True if f is close to a polynomial of degree < k

Lanczos Method and Approximation

CLAIM: There exists a polynomial p such that

$$\sup_{x \in (0, ||A||)} |p(x) - e^{-x}| \cdot \frac{1}{\text{poly}(n)}$$

and p has degree $\tilde{O}(\sqrt{||A||})$

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OPTIMAL for this approach

Lanczos Method II: Inverse Iteration

IDEA: Apply Lanczos method to

$$B = (I + \frac{A}{k})^{-1}$$

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CLAIM: $\boxed{k = O(\log n)}$ matrix-vector multiplication by B

RUNNING TIME: Multiplication by B is SDD linear system

$\tilde{O}(m)$ Time per multiplication → $\tilde{O}(m)$ Total running time

Lanczos Method II: Conclusion

$$||e^{-A}u - v|| \cdot \quad \delta$$

GENERAL RESULT:

- For general SDD matrix A, running time is

$$\tilde{O}((m+n) \cdot \log(2 + ||A||) \cdot \text{polylog}(1/\delta))$$

CONTRIBUTION: Analysis has a number of obstacles

- Linear solver is approximate: error propagation in Lanczos
- Loss of symmetry

Conclusion

NOVEL ALGORITHMIC CONTRIBUTIONS

- Balanced-Cut Algorithm using Random Walks in time $\tilde{O}(m)$
- Computing the Heat-Kernel vectors in time $\tilde{O}(m)$

MAIN IDEA

Random walks provide a very useful
stable analogue of the graph eigenvector
via regularization

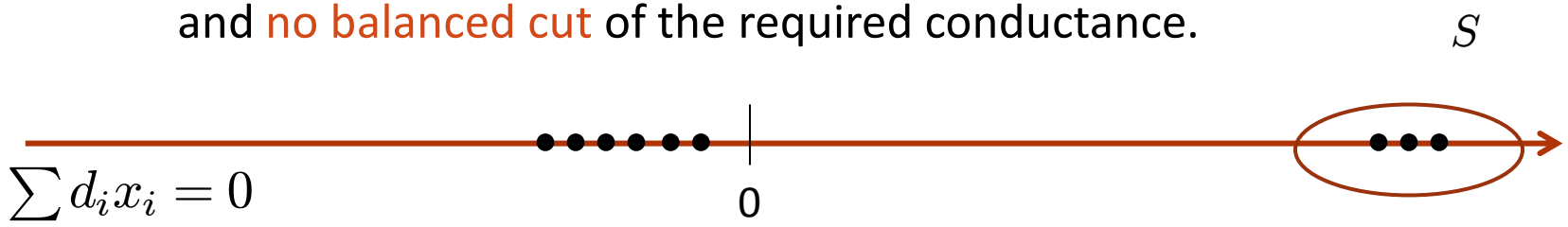
OPEN QUESTION

More applications of this idea?
Applications beyond design of fast algorithms?

A Different Interpretation

THEOREM:

Suppose eigenvector x yields an unbalanced cut S of low conductance and no balanced cut of the required conductance.



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$$\sum_{i \in S} d_i x_i^2 \geq \frac{1}{2} \sum_{i \in V} d_i x_i^2.$$

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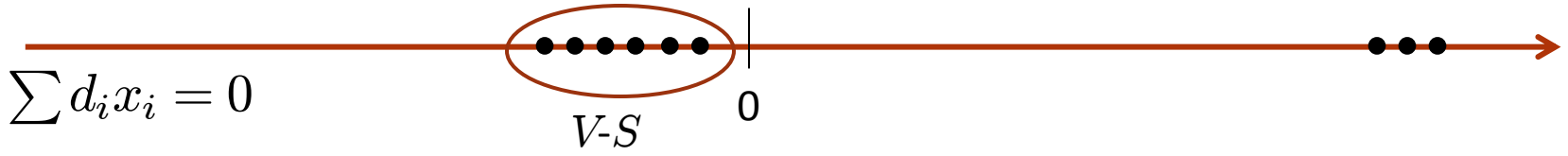

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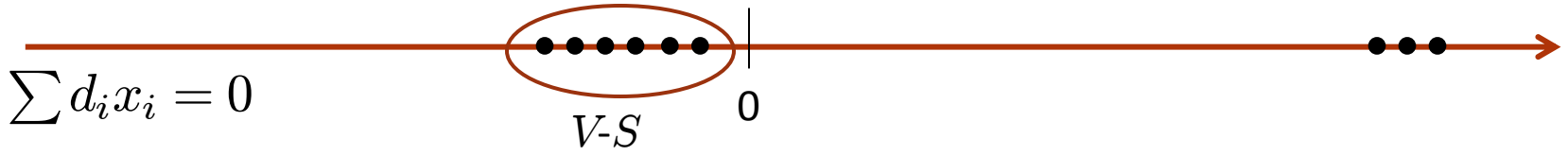

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ANSWER: **NO**. x may contain little or no information about G on V–S.

Next eigenvector may be only infinitesimally larger.

CONCLUSION: To make significant progress, we need an analogue of the eigenvector that captures sparse

Theorems for Our Algorithm

THEOREM 1: (WALKS HAVE NOT MIXED)

$$\Psi(P^{(t)}, V) > \frac{1}{\text{poly}(n)}$$

Can find cut of conductance $O(\sqrt{\gamma})$

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$P^{(t)} = e^{-\tau Q^{(t)}}$ $\quad \tau = \log n/\gamma$ $\quad \Psi(P, V) = \sum_{i \in V} ||Pe_i - \vec{1}/n||^2$

Use the definition of τ. The spectrum of $P^{(t)}$ implies that

$$\sum_{ij \in E} \underbrace{||P^{(t)}e_i - P^{(t)}e_j||^2}_{\text{EDGE LENGTH}} \cdot \quad O(\gamma) \cdot \underbrace{\Psi(P^{(t)}, V)}_{\text{TOTAL VARIANCE}}$$

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Hence, by a random projection of the embedding $\{P\, e_i\}$, followed by a sweep cut, we can recover the required cut.

SDP ROUNDING TECHNIQUE

Theorems for Our Algorithm

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This is a certificate that no $\Omega(1)$-balanced cut of conductance $O(\gamma)$ exists, as evaluating the quadratic form for a vector representing a balanced cut U yields

$$\phi(U) \geq -(\gamma) - \frac{\text{vol}(S)}{\text{vol}(U)} O(\gamma) \geq -(\gamma)$$

as long as S is sufficiently unbalanced.

SDP Interpretation

$$\mathbb{E}_{\{i,j\}\in E_G} \quad ||v_i - v_j||^2 \cdot \gamma, \qquad \text{SHORT EDGES}$$

$$\mathbb{E}_{\{i,j\}\in V\times V} \quad ||v_i - v_j||^2 = \frac{1}{2m}, \qquad \text{FIXED VARIANCE}$$

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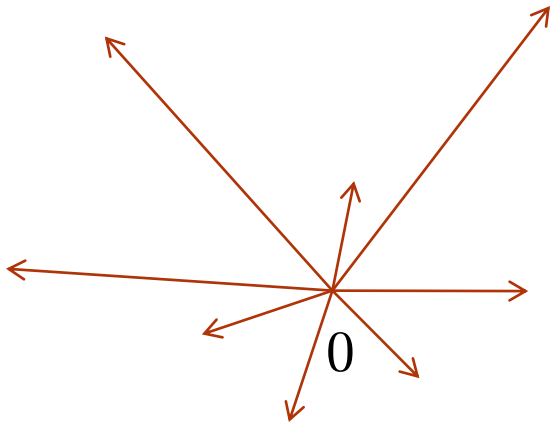

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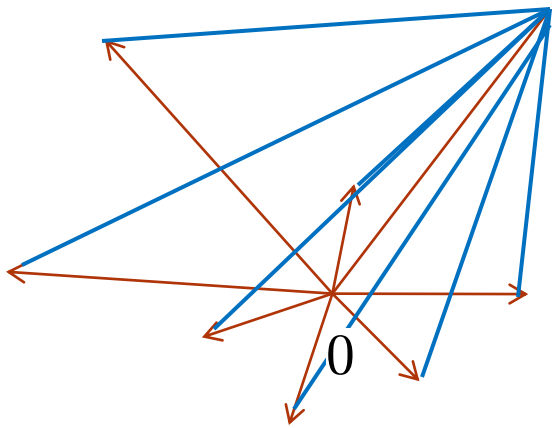

SHORT RADIUS