1. **Substitution Lemma**

   (a) $M$ is a variable.
   
   i. $M \equiv x$. Then both sides equal $N[y := L]$ since $x \not\equiv y$.
   
   ii. $M \equiv y$. Then both sides equal $L$ since $x \not\in \text{fv}(L)$.
   
   iii. $M \equiv z$ where $z \not\equiv x$ and $z \not\equiv y$. Then both sides equal $z$.

   (b) $M \equiv \lambda z. M_1$. By the variable convention we assume $z \not\equiv x, y$ and $z \not\in (\text{fv}(N) \cup \text{fv}(L))$.

   Then, using the induction hypothesis we have
   
   $$\left(\lambda z. M_1\right)[x := N][y := L] \equiv \lambda z. M_1[x := N][y := L]$$
   
   $$\equiv \lambda z. M_1[y := L][x := N[y := L]]$$
   
   $$\equiv \left(\lambda z. M_1\right)[y := L][x := N[y := L]]$$

   (c) $M \equiv M_1 M_2$. Then, using the induction hypothesis we have
   
   $$\left(M_1 M_2\right)[x := N][y := L] \equiv M_1[x := N][y := L] M_2[x := N][y := L]$$
   
   $$\equiv M_1[y := L][x := N[y := L]] M_2[y := L][x := N[y := L]]$$
   
   $$\equiv \left(M_1 M_2\right)[y := L][x := N[y := L]]$$

   (d) **Part 2 of the question**

   No. Not true in general. Let $M \equiv x$ and $y \in \text{fv}(N)$ then we have
   
   $$M[x := N][y := L] \equiv x[x := N][y := L]$$
   
   $$\equiv N[y := L]$$
   
   $$M[y := L][x := N] \equiv x[y := L][x := N]$$
   
   $$\equiv x[x := N] \equiv N$$

   So, both sides are not equivalent.

2. **Excercise 1.8.9 on page 38.** Let $P(t) = \text{size}(t) \leq 2^{\text{height}(t)}$ where

   - size(nil) = 0, size(leaf) = 1, size(node($t_1, t_2$)) = size($t_1$) + size($t_2$);
   
   - height(nil) = 0, height(leaf) = 0, height(node($t_1, t_2$)) = max(height($t_1$), height($t_2$)) + 1.

   **Base cases:**
(a) nil: height(nil) = 0, size(nil) = 0 since 0 ≤ 2^0 then P(nil) holds.
(b) leaf: height(leaf) = 0, size(leaf) = 1 since 1 ≤ 2^0 then P(leaf) also holds.

Inductive case:
By induction hypothesis we have size(t_1) ≤ 2^{height(t_1)} and size(t_2) ≤ 2^{height(t_2)} for some trees t_1 and t_2. It follows that size(t_1)+size(t_2) ≤ 2^{height(t_1)}+2^{height(t_2)}. Suppose, without loss of generality, that height(t_1) ≤ height(t_2). Then, we have height(t) = height(t_2) + 1 and also
\[
\text{size}(t_1) + \text{size}(t_2) \leq 2^{\text{height}(t_1)} + 2^{\text{height}(t_2)} \implies \\
\text{size}(t) \leq 2^{\text{height}(t_2) + 1}.
\]

3. Excercise 1.8.10 on page 38. If we assign the values n, n' and n'' to the expressions e, e' and e'', respectively, then by induction hypothesis, we have n ≤ n' and n' ≤ n''. Hence, by definition of ≤ over the set of natural numbers we get n ≤ n''. Therefore, since this holds for all possible values of the variables, then the property holds for (trans).

4. Excercise 1.8.15 on page 42.
(a) ≺_1 is well founded, because it has a minimal element ϵ (empty string).
(b) ≺_2 is well founded, because the smallest length can be 0, and it corresponds to ϵ (empty string).
(c) ≺_3 is not well founded, because we do not have a minimum element, instead we have a loop, for example:
\[
... ≺_2 \text{BC} ≺_1 \text{Z} ≺_2 \text{BC} ≺_1 \text{Z} ≺_2 ...
\]

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