NP-hardness

1. Let $G(V, E)$ be a graph. A Hamiltonian cycle of $G$ is a cycle that contains all vertices in $V$ exactly once. The Hamiltonian-cycle problem (HCP) asks whether there exists a Hamiltonian-cycle in a given graph $G$. Is this problem in NP?

2. In the Traveling Salesman Problem (TSP) a salesman wishes to make a tour of $n$ cities. That is, he wants to visit every city exactly once, and finish in the same city as he started. Between any two cities $i$ and $j$ the cost to travel from $i$ to $j$ is $c(i, j) \geq 0$. The traveling salesman problem asks that given cost function $c(i, j)$ for every pair $i, j$ and an integer $k$ does there exist a tour of the cities of total cost at most $k$? Show that TSP is in NP. It is known that the Hamiltonian cycle problem in the previous exercise is NP-complete. Show that TSP is also NP-complete by reducing the HCP to it.


4. Recall what the topological order of a directed acyclic graph is. We showed that a DAG always has a topological order. However, it is well-known that finding a maximum directed acyclic subgraph (MaxDAG problem) of an arbitrary directed graph is NP-hard. Think of the following algorithm: Let $G(V, E)$ be an arbitrary directed graph. Fix the order of nodes in $V$, let this fixed order be $v_1 \leq v_2 \leq \ldots \leq v_n$. We say that an edge $(v_i, v_j)$ is a forward edge if $v_i < v_j$, else it is called a backward edge. Let $E_f$ be the set of forward and $E_b$ be the set of backward edges in $G$. Show that $G_f(V, E_f)$ and $G_b(V, E_b)$ are both DAGs. Without loss of generality let us assume that $|E_f| \geq |E_b|$. Show that in this case the graph $G_f(V, E_f)$ is a 2-approximation to the MaxDAG problem.