Asymptotic properties of functions and recurrences

The exercises in this and all following problem sets follow the chapters of Cormen, Leiserson, Rivest and Stein: Introduction to Algorithms, third edition, from now on refered to as clrs. Many of the problems are taken directly from the book. When applicable the source in the book will be referenced.

The problem sets that we discuss in class can be downloaded at cs-people.bu.edu/edori/CS330/.

Note that the exercises provided here are only for practice purpose. You are not required to solve any of these problems and your solutions will not be graded. Reading the pages of clrs referenced here is not enough to be successful in the course. We will usually focus on one or two concepts that are especially important or difficult to understand. Assigned reading material is always what is assigned during the lectures on Tuesday and Thursday.

Asymptotic properties of functions

Read clrs chapter 3. pages 43-52. for definitions on $\Theta$, $O$, $\Omega$, $o$ and $\omega$ notation.

1. Find an example for functions $f(n)$ and $g(n)$, such that $f(n) = O(g(n))$ and $f(n) \neq \Theta(g(n))$. Can you give an example where $f(n) = \Theta(g(n))$ but $f(n) \neq O(g(n))$?

Give an example where $f(n) = o(g(n))$, give another example where $f(n) = O(g(n))$ but $f(n) \neq o(g(n))$. Now find an example where $f(n) = \omega(g(n))$, and an example where $f(n) = \Omega(g(n))$ but $f(n) \neq \omega(g(n))$.

2. Group the following functions into equivalence classes based on their asymptotic growth with regard to (a.) $\Theta$ (b.) $O$ (c.) $\Omega$. Now order the functions in non-decreasing order of their $\Theta$, $O$ and $\Omega$ values. What can you say about the relationship between the equivalence classes and the orderings?

$5n^3 + n^2$, $n^2 + 2$, $\frac{3}{n}$, $2n + \frac{1}{n}$, $3n + n^2 + \frac{4}{5}n - 2$

$5n^3 + \frac{3}{n}$, $n$, $5n^3 + \frac{3}{n^2}$, $n + 2n^2$, 125

3. clrs p.62, prob. 3-2 and clrs p. 61, prob. 3-3. a. Make sure you are familiar with all of the basic math-related concepts mentioned in clrs pages 51-58.

4. prove theorem 3.1 (clrs p. 48): For two functions $f(n)$ and $g(n)$ we have $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$. 
If you had trouble figuring out this proof, then also solve clrs p. 52. prob. 3.1-1. Also look at clrs p. 62, prob. 3-4.

5. (clrs p. 53. prob. 3.1-4) Is \(2^{n+1} = O(2^n)\)? Is \(2^{2n} = O(2^n)\)?

6. (clrs p. 53. prob. 3.1-6) Prove that the running time of an algorithm is \(\Theta(g(n))\) if and only if its worst-case running time is \(O(g(n))\) and its best case running time is \(\Omega(g(n))\).

Recurrences: divide-and-conquer method

Read chapter 4. in clrs, especially sections 4.3, 4.4 and 4.5 to learn the three standard methods for solving recurrences.

1. Solve the following recurrences with the substitution method:
   
   \[T(n) \leq T(n - 1) + n\]
   \[T(n) \leq n + 2T\left(\frac{n}{2}\right)\]
   \[T(n) \leq T(n - 1) + \lg n\]
   \[T(n) \leq 2T(n - 1) + n\]
   \[T(n) \leq 3T\left(\frac{n}{2}\right)\]

   If you cannot guess the asymptotic upper bound for the recurrences, then use the recursion-tree method to do so. Proceed with the substitution method, using this guess.

2. Write the recurrence describing the insertion-sort algorithm (the algorithm is described in clrs, section 2.1).

3. Write the recurrence describing the merge-sort algorithm (the algorithm is described in clrs, section 2.3.1.).

4. For \(T(n)\) defined by the recurrence \(T(n) \leq 2T\left(\frac{n}{2}\right) + n\) we know that \(T(n) = O(n \lg n)\). Knowing this, solve the recurrence \(R(n) = 2R(\sqrt{n}) + \lg n\), by changing variables. (Solution to this problem is given in clrs p. 86.).

5. Apply the master theorem to the recurrences \(T(n) = 9T\left(\frac{n}{3}\right) + n\), \(T(n) = T\left(\frac{2n}{3}\right) + 1\) and \(T(n) = 3T\left(\frac{n}{4}\right) + n \lg n\). Can you apply the master theorem to \(T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n)\)? Apply the master theorem to solve the recurrence \(T(n) = 2T\left(\frac{n}{2}\right) + \Theta(1)\). (All solutions to this problem are described in clrs p. 95-96.)

6. Many-many basic practice problems: clrs exercises 4.3-1 . . . 4.3-9, 4.4-1 . . . 4.4-9, 4.5-1