Approximation Algorithms for Submodular Multiway Partition

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Abstract— We study algorithms for the SUBMODULAR MULTIWAY PARTITION problem (SUB-MP). An instance of SUB-MP consists of a finite ground set V, a subset $S = \{s_1, s_2, \ldots, s_k\} \subseteq V$ of k elements called terminals, and a non-negative submodular set function $f : 2^V \to \mathbb{R}_+$ on V provided as a value oracle. The goal is to partition V into k sets $A_1, \ldots, A_k$ to minimize $\sum_{i=1}^k f(A_i)$ such that for $1 \le i \le k$, $s_i \in A_i$. SUB-MP generalizes some well-known problems such as the MULTIWAY CUT problem in graphs and hypergraphs, and the NODE-WEIGHED MULTIWAY CUT problem in graphs. SUB-MP for arbitrary submodular functions (instead of just symmetric functions) was considered by Zhao, Nagamochi and Ibaraki [29]. Previous algorithms were based on greedy splitting and divide and conquer strategies. In recent work [5] we proposed a convex-programming relaxation for SUB-MP based on the Lovász-extension of a submodular function and showed its applicability for some special cases. In this paper we obtain the following results for arbitrary submodular functions via this relaxation.

- A 2-approximation for SUB-MP. This improves the $(k-1)$-approximation from [29].
- A $(1.5 - \frac{1}{k})$-approximation for SUB-MP when f is *symmetric*. This improves the $2(1 - \frac{1}{k})$-approximation from [23], [29].

1. INTRODUCTION

In this paper we consider the approximability of the following problem.

SUBMODULAR MULTIWAY PARTITION (SUB-MP). Let $f : 2^V \to \mathbb{R}_+$ be a non-negative submodular set function[1] over a finite ground set V and let $S = \{s_1, s_2, \ldots, s_k\} \subseteq V$ be a set of k terminals. The submodular multiway partition problem is to partition $A_1, \ldots, A_k$ of V to minimize $\sum_{i=1}^k f(A_i)$ such that for $1 \le i \le k$, $s_i \in A_i$. An important special case is when f is symmetric and we refer to it as SYM-SUB-MP.

Motivation and Related Problems: We are motivated to consider SUB-MP for two reasons. First, SUB-MP generalizes several problems that have been well-studied. We discuss them now. Perhaps the most well-known of the special cases is the GRAPH MULTIWAY CUT problem: the input is an undirected edge-weighted graph $G = (V, E)$ and a set $S \subseteq V$ of terminals, and the objective is to remove a minimum weight set of edges to separate the terminals [9]. Although the objective is stated in terms of edge-removals, the problem can also be viewed as a special case of SYM-SUB-MP with the cut-capacity function of G as f. One obtains two interesting and related problems if one generalizes GRAPH MULTIWAY CUT to *hypergraphs*. Let $G = (V, \mathcal{E})$ be an edge-weighted hypergraph.

HYPERGRAPH MULTIWAY CUT is the problem where the goal is to remove a minimum-weight set of hyperedges to disconnect the given set of terminals. HYPERGRAPH MULTIWAY PARTITION problem is the special case of SYM-SUB-MP where f is the hypergraph-cut function: $f(A) = \sum_{e \in \delta(A)} w(e)$ where $w(e)$ is the weight of e and $\delta(A)$ is the set of all hyperedges that intersect A but are not contained in A. The distinction between HYPERGRAPH MULTIWAY CUT and HYPERGRAPH MULTIWAY PARTITION is that in the former a hyperedge incurs a cost only once if the vertices in it are split across terminals while in HYPERGRAPH MULTIWAY PARTITION the cost paid by a hyperedge is the number of non-trivial pieces it is partitioned into. Both problems have applications, in particular for circuit partitioning problems in VLSI design [1]. We wish to draw special attention to HYPERGRAPH MULTIWAY CUT since it is approximation equivalent to the NODE WEIGHTED MULTIWAY CUT problem in graphs where the nodes have weights and the goal is to remove a minimum-weight subset of nodes to disconnect a given set of terminals [13], [14]. An important motivation to consider SUB-MP is that HYPERGRAPH MULTIWAY CUT can be cast as a special case of it [22]; the reduction is simple, yet interesting, and we stress that the resulting function f is *not* symmetric. From the above discussion it follows that NODE WEIGHTED MULTIWAY CUT, via HYPERGRAPH MULTIWAY CUT, can be viewed indirectly as a partition problem with an appropriate submodular function. We believe this is a useful observation. In fact, SUB-MP (and related generalizations) were introduced by Zhao, Nagamochi and Ibaraki [29] partly motivated by the applications to hypergraph cut and partition problems.

A second important motivation to consider SUB-MP and SYM-SUB-MP is the following question. To what extent do current algorithms and techniques for GRAPH MULTIWAY

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[1]A set function $f : 2^V \to \mathbb{R}$ is submodular iff $f(A) + f(B) \ge f(A \cap B) + f(A \cup B)$ for all $A, B \subseteq V$. Moreover, f is symmetric if $f(A) = f(V - A)$ for all $A \subseteq V$.

CUT and NODE WEIGHTED MULTIWAY CUT depend on the fact that the underlying structure is a graph or a hypergraph? Is it the case that submodularity of the cut function the key underlying phenomenon? For GRAPH MULTIWAY CUT Dahlhaus *et al.* [9] gave a simple $2(1-\frac{1}{k})$-approximation via the isolating cut heuristic. Queyranne [23] showed that this same bound can be achieved for SYM-SUB-MP (see also [29]). For GRAPH MULTIWAY CUT, Calinescu, Karloff and Rabani [3], in a breakthrough, obtained a $(1.5-\frac{1}{k})$ approximation via an interesting geometric relaxation. The integrality gap for this relaxation has been subsequently improved to 1.3438 by Karger *et al.* [18]. Once again it is natural to ask if this geometric relaxation is specific to graphs and whether corresponding results exist for SYM-SUB-MP. Further, the current best approximation for SUB-MP is $(k-1)$ [29] and is obtained via a simple greedy splitting algorithm. SUB-MP generalizes NODE WEIGHTED MULTIWAY CUT and the latter has a constant factor approximation [14] but it is a non-trivial LP relaxation based algorithm. Therefore it is reasonable to expect that one needs a mathematical programming relaxation to obtain a constant factor approximation for SUB-MP.

In recent work [5] we developed a convex-programming relaxation for SUB-MP via the Lovász-extension of a submodular function (we discuss this in more detail below). An interesting observation is that this specializes to the CKR-relaxation when we consider GRAPH MULTIWAY CUT. A natural question, that was raised in [5], is whether the convex relaxation can be used to obtain a better than 2 approximation for SYM-SUB-MP. In this paper we answer this question in the positive and also obtain a 2-approximation for SUB-MP improving the known $(k-1)$-approximation [29]. We now describe the convex relaxation.

A convex relaxation via the Lovász extension. The relaxation SUBMP-REL for SUB-MP was introduced in [5]. It is based on the well-known Lovász extension of a submodular function which we describe for completeness. Let V be a finite ground set of size n, and let $f : 2^V \to \mathbb{R}$ be a real-valued set function. By representing a set by its characteristic vector, we can think of f as a function that assigns a value to each vertex of the boolean hypercube $\{0,1\}^n$. The Lovász extension $\hat{f}$ extends f to all of $[0,1]^n$ and is defined as follows[2]:

$$\hat{f}(\mathbf{x}) = \mathop{\mathbb{E}}_{\theta\in[0,1]}\left[f(\mathbf{x}^\theta)\right] = \int_0^1 f(\mathbf{x}^\theta)d\theta$$

where $\mathbf{x}^\theta \in \{0,1\}^n$ for a given vector $\mathbf{x}\in[0,1]^n$ is defined as: $x_i^\theta = 1$ if $x_i \ge \theta$ and $x_i^\theta = 0$ otherwise. Lovász showed that $\hat{f}$ is convex if and only if f is submodular [21].

SUB-MP is defined as a partition problem. We can equivalently interpret it as an *allocation* problem (also a labeling problem) where for each $v \in V$ we decide which of the k terminals it is allocated to. Thus we have non-negative variables $x(v,i)$ for $v\in V$ and $1 \le i \le k$ and an allocation is implied by the constraint $\sum_{i=1}^k x(v,i) = 1$ for each v. Of course a terminal s_i is allocated to itself. The only complexity is in the objective function since f is submodular. However, the Lovász extension gives a direct and simple way to express the objective function. Let $\mathbf{x}_i$ be the vector obtained by restricting $\mathbf{x}$ to the i'th terminal s_i; that is, $\mathbf{x}_i = (x(v_1,i),\ldots,x(v_n,i))$. If $\mathbf{x}_i$ is integral $\hat{f}(\mathbf{x}_i) = f(A_i)$ where A_i is the support of $\mathbf{x}_i$. Therefore we obtain the following relaxation.

SUBMP-REL

$$\begin{array}{llcll}
\min & \displaystyle\sum_{i=1}^k \hat{f}(\mathbf{x}_i) & & & \\
 & \displaystyle\sum_{i=1}^k x(v,i) & = & 1 & \forall v \\
 & x(s_i,i) & = & 1 & \forall i \\
 & x(v,i) & \ge & 0 & \forall v,i
\end{array}$$

The above relaxation can be solved *exactly* in polynomial time[3]. We give algorithms to round an optimum fractional solution to SUBMP-REL and obtain the following two results which also establish corresponding upper bounds on the integrality gap of SUBMP-REL.

Theorem 1.1. *There is a* $(1.5-\frac{1}{k})$*-approximation for* SYM-SUB-MP.

Theorem 1.2. *There is a* 2*-approximation for* SUB-MP.

We give the proofs of Theorem 1.1 and Theorem 1.2 in Section 2 and Section 3 respectively.

Remark 1.3. It is shown in [14] that an α-approximation for NODE WEIGHTED MULTIWAY CUT implies an α-approximation for the VERTEX COVER problem. Therefore, improving the 2-approximation for SUB-MP is infeasible without a corresponding improvement for VERTEX COVER. It is easy to show that the integrality gap of SUBMP-REL is at least $2(1-\frac{1}{k})$ even for instances of HYPERGRAPH MULTIWAY CUT. The best lower bound on the integrality gap of SUBMP-REL for SYM-SUB-MP that we know is $8/(7+\frac{1}{k-1})$, the same as that for GRAPH MULTIWAY CUT shown in [11].

Remark 1.4. Related to SUB-MP and SYM-SUB-MP are k-way partition problems K-WAY SUB-MP and K-WAY SYM-SUB-MP where no terminals are specified but the goal is to partition V into k non-empty sets $A_1,\ldots,A_k$ to minimize $\sum_{i=1}^k f(A_i)$. When k is part of the input these

[2]The standard definition is slightly different but is equivalent to the one we give.

[3]The running time is polynomial in n and $\log\left(\max_{S\subseteq V} f(S)\right)$. See Appendix A in [6] for the details.

problems are NP-Hard but for fixed k, [23] claimed a polynomial time algorithm for K-WAY SYM-SUB-MP while the polynomial-time solvability of K-WAY SUB-MP is open. For fixed k one can reduce K-WAY SUB-MP to SUB-MP by guessing k terminals and this leads to a 2-approximation via Theorem 1.2, improving the previously known ratio of $(k+1-2\sqrt{k-1})$ [22].

Our results build on some basic insights that were outlined in [5] where the special cases of HYPERGRAPH MULTIWAY PARTITION and HYPERGRAPH MULTIWAY CUT were considered (among other results). In [5] a $(1.5 - \frac{1}{k})$-approximation for HYPERGRAPH MULTIWAY PARTITION and a $\min\{2(1-\frac{1}{k}), H_\Delta\}$-approximation for HYPERGRAPH MULTIWAY CUT were given, where Δ is the maximum hyperedge degree and H_i is the i'th harmonic number (improvement over 2 is for $\Delta \leq 3$). Our contribution in this paper is a non-trivial and technical new result on rounding SUBMP-REL (Theorem 1.5 below) that applies to an arbitrary submodular function. The formulation of the statement of the theorem may appear natural in retrospect but is a significant part of the contribution. We now give an overview of the rounding algorithm(s) and the new result. We then discuss and compare to prior work.

1.1. Overview of rounding and the main technical result

Let $\mathbf{x}$ be a fractional allocation and $\sum_i \hat{f}(\mathbf{x}_i)$ the corresponding objective function value. How do we round $\mathbf{x}$ to an integral allocation while approximately preserving the convex objective function? The simple insight in [5] is that we simply follow the definition of the Lovász function and do *θ-rounding*: pick a (random) threshold $\theta \in [0,1]$ and set $x(v_j, i) = 1$ if and only if $x(v_j, i) \geq \theta$. Let $\mathbf{x}^\theta$ be the resulting integer vector. If we pick θ uniformly at random in $[0,1]$ then the expected cost of $\sum_i \mathbb{E}[f(\mathbf{x}_i^\theta)] = \sum_i \hat{f}(\mathbf{x}_i)$. However, the difficulty is that $\mathbf{x}^\theta$ may not correspond to a feasible allocation. Let $A(i,\theta)$ be the support of $\mathbf{x}_i^\theta$, that is, the set of vertices assigned to s_i for a given θ. The reason that $\mathbf{x}^\theta$ may not be a feasible allocation is two-fold. First, a vertex v may be assigned to multiple terminals, that is, the sets $A(i,\theta)$ for $i = 1, \dots, k$ may not be disjoint. Second, the vertices $U(\theta) = V - \cup_{i=1}^k A(i,\theta)$ are unallocated. We let $A(\theta) = \cup_{i=1}^k A(i,\theta)$ be the allocated set.

Our main insight here is that the expected cost of the unallocated set, that is $f(U(\theta))$, can be upper bounded effectively. We can then assign the set $U(\theta)$ to an arbitrary terminal and use sub-additivity of f (since it is submodular and non-negative). Before we formalize this, we discuss how to overcome the overlap in the sets $A(i,\theta)$. If f is symmetric then it is also posi-modular[4] and one can do a simple uncrossing of the sets to make them disjoint without increasing the cost. If f is not symmetric we cannot resort to this trick. In this case we ensure that the sets $A(i,\theta)$ are disjoint by picking θ uniformly in $(1/2, 1]$ rather than $[0,1]$ (we call this half-rounding); the sets $A(i,\theta)$ are disjoint because for any v there can be at most one i such that $x(v,i) > 1/2$. Now the unallocated set and the expected cost of the initial allocation are some what more complex. We analyze both these scenarios using the following theorem which is our main result. The theorem below has a parameter $\delta \in [1/2, 1]$ and this corresponds to rounding where we pick θ uniformly from the interval $(1-\delta, 1]$.

[4]A set function f is posi-modular if $f(A) + f(B) \geq f(A-B) + f(B-A)$ for all sets A and B.

Theorem 1.5. *Let $\mathbf{x}$ be a feasible solution to* SUBMP-REL*. For $\theta \in [0,1]$ let $A(i,\theta) = \{v \mid x(v,i) \geq \theta\}$, $A(\theta) = \cup_{i=1}^k A(i,\theta)$ and $U(\theta) = V - A(\theta)$. For any $\delta \in [1/2, 1]$ the following holds:*

$$\sum_{i=1}^k \int_0^\delta f(A(i,\theta))d\theta \geq \int_0^\delta f(A(\theta))d\theta + \int_0^1 f(U(\theta))d\theta.$$

By setting $\delta = 1$, we obtain the following corollary.

Corollary 1.6.

$$\begin{aligned}\sum_{i=1}^k \hat{f}(\boldsymbol{x}_i) &= \sum_{i=1}^k \int_0^1 f(A(i,\theta)d\theta \\ &\geq \int_0^1 f(A(\theta))d\theta + \int_0^1 f(U(\theta))d\theta.\end{aligned}$$

Theorem 1.5 gives a unified analysis of our algorithms for SYM-SUB-MP and SUB-MP. More precisely, we get Theorem 1.1 and Theorem 1.2 as rather simple corollaries. Corollary 1.6 is sufficient to show that the SYM-SUB-MP algorithm achieves a 1.5-approximation and that the SUB-MP algorithm achieves a 4-approximation. In order to show that the SUB-MP algorithm achieves a 2-approximation, we need the stronger statement of Theorem 1.5.

Remark 1.7. A simpler and more intuitive proof of Theorem 1.5 has been obtained [10]. Additionally, a variant of the algorithm SUBMP-HALF-ROUNDING achieves a $2(1-\frac{1}{k})$ approximation for SUB-MP [10]; this improves the ratio of 2 from this paper.

1.2. Discussion and other related work

We recently considered the MINIMUM SUBMODULAR-COST ALLOCATION problem [5]; this problem contains as special cases SUB-MP and other problems such as uniform metric labeling, non-metric facility location, hub location and variants. It is shown in [5] that a convex programming relaxation via the Lovász extension follows naturally for MINIMUM SUBMODULAR-COST ALLOCATION, and that θ-rounding based algorithms provide a unified way to understand and extend several previous results. The integrality gap of SUBMP-REL for SYM-SUB-MP and SUB-MP were posed as open questions following results for the special cases of HYPERGRAPH MULTIWAY CUT and HYPERGRAPH

MULTIWAY PARTITION. These results subsequently inspired the formulation of Theorem 1.5.

Geometry plays a key role in the formulation, rounding and analysis of the relaxation proposed for GRAPH MULTIWAY CUT by Calinescu, Karloff and Rabani [3]; they obtained a $(1.5 - \frac{1}{k})$ approximation. The subsequent work of Karger *et al.* exploits the geometric aspects further to obtain an improvement in the ratio to 1.3438. If one views GRAPH MULTIWAY CUT as a special case of SYM-SUB-MP then the function f under consideration is the cut function. The cut function f can be decomposed into several simple submodular functions, corresponding to the edges, each of which depends only on two vertices. This allows one to focus on the probability that an edge is cut in the rounding process. Our work in [5] for HYPERGRAPH MULTIWAY PARTITION and HYPERGRAPH MULTIWAY CUT is also in a similar vein since one can visualize and analyze the simple functions that arise from the hypergraph cut function. Our current analysis differs substantially in that we no longer have a local handle on f, and hence the need for Theorem 1.5. It is interesting that the integrality gap of SUBMP-REL is at most $(1.5 - \frac{1}{k})$ for any symmetric function f, matching the bound achieved by [3] for GRAPH MULTIWAY CUT. Our rounding differs from that in [3]; both do θ-rounding but our algorithm uncrosses the sets $A(i, \theta)$ to make them disjoint while CKR-rounding does it by picking a random permutation. One can understand the random permutation as an oblivious uncrossing operation that is particularly suited for submodular functions that depend on only two variables (in this case the edges); it is unclear whether this is suitable for arbitrary symmetric functions.

As we remarked, SYM-SUB-MP and SUB-MP were considered in several papers [23], [29], [22] with HYPERGRAPH MULTIWAY CUT and K-WAY HYPERGRAPH CUT as interesting applications for SUB-MP. These papers primarily relied on greedy methods. It was noted in [22] that HYPERGRAPH MULTIWAY CUT and NODE WEIGHTED MULTIWAY CUT are essentially equivalent problems. Garg, Vazirani and Yannakakis [14] gave a $2(1-\frac{1}{k})$-approximation for NODE WEIGHTED MULTIWAY CUT [14] via a natural distance based LP relaxation; we note that this result is non-trivial and relies on proving the existence of a half-integral optimum fractional solution. It is noted in [5] that SUBMP-REL gives a new and strictly stronger relaxation for NODE WEIGHTED MULTIWAY CUT (via the connection to HYPERGRAPH MULTIWAY CUT). The previous best approximation for SUB-MP was $(k-1)$ [29]. As we already remarked, obtaining a constant factor approximation for SUB-MP without a mathematical programming relaxation like SUBMP-REL is difficult given the lack of combinatorial algorithms for special cases like NODE WEIGHTED MULTIWAY CUT.

Submodular functions play a fundamental role in classical combinatorial optimization. In recent years there have been several new results on approximation algorithms for problems with objective functions that depend on submodular functions. In addition to combinatorial techniques such as greedy and local-search, mathematical programming methods have been particularly important. It is natural to use the Lovász extension for problems involving minimization since the extension is convex; see [17], [15], [5] for instance. For maximization problems involving submodular functions the multilinear extension introduced in [2] has been useful [25], [19], [20], [26], [8].

2. SYMMETRIC SUBMODULAR MULTIWAY PARTITION

We consider the following algorithm to round a feasible solution $\mathbf{x}$ to SUBMP-REL.

SYMSUBMP-ROUNDING
- let $\mathbf{x}$ be a feasible solution to SUBMP-REL
- pick $\theta \in [0, 1]$ uniformly at random
- for $i = 1$ to k
  - $A(i, \theta) \leftarrow \{v \mid x(v, i) \geq \theta\}$
- $A(\theta) \leftarrow \bigcup_{1 \leq i \leq k} A(i, \theta)$
- $U(\theta) \leftarrow V - A(\theta)$
- for $i = 1$ to k
  - $A'_i \leftarrow A(i, \theta)$
- $\langle\langle$*uncross* $A'_1, \ldots, A'_k\rangle\rangle$
- while there exist $i \neq j$ such that $A'_i \cap A'_j \neq \emptyset$
  - if $(f(A'_i) + f(A'_j - A'_i) \leq f(A'_i) + f(A'_j))$
    - $A'_j \leftarrow A'_j - A'_i$
  - else
    - $A'_i \leftarrow A'_i - A'_j$
- return $(A'_1, \cdots, A'_{k-1}, A'_k \cup U(\theta))$

We prove the following theorem.

Theorem 2.1. *For a symmetric submodular function f, the algorithm* **SYMSUBMP-ROUNDING** *outputs a valid multiway partition of expected cost at most* $1.5 \cdot \sum_{i=1}^{k} \hat{f}(\boldsymbol{x}_i)$.

The algorithm does θ-rounding in the interval $[0, 1]$ to obtain (random) sets $A(i, \theta)$ for $i = 1, \ldots, k$. Let $\text{OPT}_{\text{FRAC}} = \sum_{i=1}^{k} \hat{f}(\mathbf{x}_i)$ (the notation is with the implicit understanding that $\mathbf{x}$ is an optimal fractional solution). Note that $\mathbb{E}[f(A(i, \theta))] = \hat{f}(\mathbf{x}_i)$ and hence $\sum_{i=1}^{k} \mathbb{E}[f(A(i, \theta))] = \sum_{i=1}^{k} \hat{f}(\mathbf{x}_i) = \text{OPT}_{\text{FRAC}}$. The lemma below shows that the uncrossing operation does not increase the cost. This was used in the context of multiway cuts previously [24], [5]; we include the proof for completeness.

Lemma 2.2 ([5]). *Let $A'_1, \ldots, A'_k$ denote the sets after uncrossing the sets $A(1, \theta), \ldots, A(k, \theta)$. If f is a symmetric submodular function then $\cup_{i=1}^{k} A'_i = \cup_{i=1}^{k} A(i, \theta)$ and*

$$\sum_{i=1}^{k} f(A'_i) \leq \sum_{i=1}^{k} f(A(i, \theta)).$$

Proof: In each uncrossing step we replace A'_i and A'_j either by A'_i and $A'_j - A'_i$ or by $A'_i - A'_j$ and A'_j. Since f is submodular and symmetric, f is posi-modular; that is, for any two sets X and Y, $f(X)+f(Y) \geq f(X-Y)+f(Y-X)$. Therefore, for any two sets X and Y, $\min\{f(X-Y)+f(Y), f(X)+f(Y-X)\}$ is at most $f(X)+f(Y)$. Thus it follows by induction that $\sum_{i=1}^{k} f(A_i') \leq \sum_{i=1}^{k} f(A(i,\theta))$ and $\cup_{i=1}^{k} A_i' = \cup_{i=1}^{k} A(i,\theta)$. ■

Lemma 2.3. *If f is a symmetric submodular function,*

$$\mathop{\mathbb{E}}_{\theta \in [0,1]}[f(U(\theta))] \leq \frac{1}{2}\text{OPT}_{\text{FRAC}}.$$

Proof: By setting $\delta = 1$ in Theorem 1.5, we get

$$\text{OPT}_{\text{FRAC}} \geq \int_0^1 f(V - U(\theta))d\theta + \int_0^1 f(U(\theta))d\theta.$$

Since f is symmetric, $f(V-U(\theta)) = f(U(\theta)$ for all θ and hence,

$$\text{OPT}_{\text{FRAC}} \geq 2\int_0^1 f(U(\theta))d\theta = 2 \mathop{\mathbb{E}}_{\theta \in [0,1]}[f(U(\theta))].$$

■

The random partition returned by the algorithm is $(A_1', \ldots, A_{k-1}', A_k' \cup U(\theta))$. A non-negative submodular function is sub-additive, hence $f(A_k' \cup U(\theta)) \leq f(A_k') + f(U(\theta))$. The expected cost of the partition is

$$\begin{aligned}
&\sum_{i=1}^{k-1} \mathbb{E}[f(A_i')] + \mathbb{E}[f(A_k' \cup U(\theta))] \\
&\quad \leq \sum_{i=1}^{k} \mathbb{E}[f(A_i')] + \mathbb{E}[f(U(\theta))] \\
&\quad \leq \sum_{i=1}^{k} \mathbb{E}[f(A(i,\theta))] + \mathbb{E}[f(U(\theta))] \quad \text{(Using Lemma 2.2)} \\
&\quad \leq \text{OPT}_{\text{FRAC}} + \frac{1}{2}\text{OPT}_{\text{FRAC}} \quad \text{(Using Lemma 2.3)} \\
&\quad = 1.5\text{OPT}_{\text{FRAC}}.
\end{aligned}$$

This finishes the proof of Theorem 2.1. It is not hard to verify that the algorithm runs in polynomial time. One can easily derandomize the algorithm as follows. The only randomness is in the choice of θ. As θ ranges in the interval $[0,1]$, the collection of sets $\{A(i,\theta) \mid 1 \leq i \leq k\}$ changes only when θ crosses some $x(v_j, i)$ value. Thus there are at most nk such distinct values. We can try each of them as a choice for θ and pick the least cost partition obtained among all the choices.

Achieving a $(1.5 - \frac{1}{k})$-aproximation: We can improve the approximation to $1.5 - \frac{1}{k}$ as follows. We relabel the terminals so that $k = \arg\max_{1 \leq i \leq k} \hat{f}(\mathbf{x}_i)$. We perform θ-rounding with respect to the first $k-1$ terminals in order to get the sets $A(i,\theta)$ for each $i \neq k$, and we let $U(\theta) = V - \cup_{1 \leq i \leq k-1} A(i,\theta)$. We uncross the sets $\{A(i,\theta) \mid 1 \leq i < k\}$ to get $k-1$ disjoint sets A_i', and we return $(A_1', \cdots, A_{k-1}', U(\theta))$. We can prove a variant of Theorem 1.5 that shows that the expected cost of $U(\theta)$ is at most $\text{OPT}_{\text{FRAC}}/2$, even when $U(\theta)$ is the set of all vertices that are unallocated when we perform θ-rounding with respect to only the first k' terminals, for any $k' \leq k$. The proof of this extension, though closely based on the proof of Theorem 1.5, is notationally and technically messy, and we omit it in this version of the paper. The total expected cost of the sets $A_1', \cdots, A_{k-1}'$ is at most $(1 - \frac{1}{k})\text{OPT}_{\text{FRAC}}$ (since we saved on $\hat{f}(\mathbf{x}_k)$), and the expected cost of $U(\theta)$ is at most $\text{OPT}_{\text{FRAC}}/2$.

3. Submodular Multiway Partition

In this section we consider Sub-MP when f is an arbitrary non-negative submodular function. We choose $\theta \in (1/2, 1]$ to ensure that the sets $\{A(i,\theta) \mid 1 \leq i \leq k\}$ are disjoint.

SubMP-Half-Rounding
let $\mathbf{x}$ be a feasible solution to SubMP-Rel
pick $\theta \in (1/2, 1]$ uniformly at random
for $i = 1$ to k
  $A(i,\theta) \leftarrow \{v \mid x(v,i) \geq \theta\}$
$A(\theta) \leftarrow \bigcup_{1 \leq i \leq k} A(i,\theta)$
$U(\theta) \leftarrow V - A(\theta)$
return $(A(1,\theta), \cdots, A(k-1,\theta), A(k,\theta) \cup U(\theta))$

Proof of Theorem 1.2: In the following, we show that SubMP-Half-Rounding achieves a 2-approximation for Sub-MP. As before, let $\text{OPT}_{\text{FRAC}} = \sum_{i=1}^{k} \hat{f}(\mathbf{x}_i)$. Since f is sub-additive, the expected cost of the partition returned by SubMP-Half-Rounding is

$$\begin{aligned}
&\mathop{\mathbb{E}}_{\theta \in (1/2,1]}\left[\sum_{i=1}^{k-1} f(A(i,\theta)) + f(A(k,\theta) \cup U(\theta))\right] \\
&\leq \mathop{\mathbb{E}}_{\theta \in (1/2,1]}\left[\sum_{i=1}^{k} f(A(i,\theta)) + f(U(\theta))\right] \quad (f \text{ is sub-additive}) \\
&= 2\left(\sum_{i=1}^{k}\int_{1/2}^{1} f(A(i,\theta))d\theta + \int_{1/2}^{1} f(U(\theta))d\theta\right) \\
&= 2\left(\text{OPT}_{\text{FRAC}} - \sum_{i=1}^{k}\int_{0}^{1/2} f(A(i,\theta))d\theta + \int_{1/2}^{1} f(U(\theta))d\theta\right).
\end{aligned}$$

To show that the expected cost is at most $2\text{OPT}_{\text{FRAC}}$ it suffices to show that $\sum_{i=1}^{k}\int_0^{1/2} f(A(i,\theta))d\theta \geq \int_{1/2}^{1} f(U(\theta))d\theta$. Setting $\delta = 1/2$ in Theorem 1.5 we obtain

$$\begin{aligned}
&\sum_{i=1}^{k}\int_{0}^{1/2} f(A(i,\theta))d\theta \\
&\qquad \geq \int_0^{1/2} f(V - U(\theta))d\theta + \int_0^1 f(U(\theta))d\theta \\
&\qquad \geq \int_{1/2}^{1} f(U(\theta))d\theta \qquad (f \text{ is non-negative}).
\end{aligned}$$

Thus SUBMP-HALF-ROUNDING achieves a randomized 2-approximation for SUB-MP. The algorithm can be derandomized in the same fashion as the one for symmetric functions. ∎

Improving the factor of 2: As we remarked earlier the VERTEX COVER problem can be reduced in an approximation preserving fashion to SUB-MP, and hence it is unlikely that the factor of 2 for SUB-MP can be improved. The approximation ratio (and also the integrality gap) has been improved recently to $2(1-\frac{1}{k})$ [10].

4. PROOF OF MAIN THEOREM

In this section we prove Theorem 1.5, our main technical result. We recall some relevant definitions. Let $\mathbf{x}$ be a solution to SUBMP-REL. We are interested in analyzing θ-rounding when θ is chosen uniformly at random from an interval $[1-\delta, 1]$ for some $\delta \geq 0$. In this section we use the terminology of labels instead of terminals. For a label i let $A(i,\theta) = \{v \in V \mid x(v,i) \geq \theta\}$ be the set of all vertices that are assigned/allocated to i for some fixed θ. Note that for distinct labels i, i' the sets $A(i,\theta)$ and $A(i',\theta)$ may not be disjoint if $\theta \leq 1/2$, although they are disjoint if $\theta > 1/2$. Let $A(\theta) = \bigcup_{1\leq i\leq k} A(i,\theta)$ be the set of all vertices that are allocated to the terminals when θ is the chosen threshold. We let $U(\theta) = V - A(\theta)$ denote the set of unallocated vertices. With this notation in place we restate Theorem 1.5.

Theorem 4.1. *For any $\delta \in [1/2, 1]$, we have*

$$\sum_{i=1}^{k}\int_0^{\delta} f(A(i,\theta))d\theta \geq (k\delta - \delta - 1)f(\emptyset) + \int_0^{\delta} f(A(\theta))d\theta + \int_0^{1} f(U(\theta))d\theta.$$

The proof of the above theorem is somewhat long and technical. At a high-level it is based on induction on the number of vertices relative to a particular ordering that we discuss now. In the following, we use i to index over the labels, and we use j to index over the vertices. For vertex v_j, let $\alpha_j = \max_i x(v_j, i)$ be the maximum amount to which $\mathbf{x}$ assigns v_j to a label. We renumber the vertices so that $0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq 1$. For notational convenience we let $\alpha_0 = 0$ and $\alpha_{n+1} = 1$. Further, for each vertex v_j we let ℓ_j be a label such that $\alpha_j = x(v_j, \ell_j)$; note that ℓ_j is not necessarily unique unless $\alpha_j > 1/2$.[5]

We observe that in θ-rounding, v_j is allocated to a terminal (that is, $v_j \in A(\theta)$) if and only if $\theta \leq \alpha_j$; otherwise $v_j \in U(\theta)$ and thus it is unallocated. It follows from our ordering that $U(\theta) = \{v_1, v_2, \ldots, v_{j-1}\}$ iff $\theta \in (\alpha_{j-1}, \alpha_j]$ and in this case $A(\theta) = V - U(\theta) = \{v_j, \ldots, v_n\}$. Thus, prefixes of the ordering given by the α values are the only interesting sets to consider when analyzing the rounding process from the point of view of allocated and unallocated vertices. To help with notation, for $1 \leq j \leq n$ we let $V_j = \{v_1, v_2, \ldots, v_j\}$ and $V_0 = \emptyset$. The following proposition captures this discussion.

Proposition 4.2. *Let $\alpha_0 = 0$ and let j be any index such that $1 \leq j \leq n$. For any $\theta \in (\alpha_{j-1}, \alpha_j]$, $A(\theta) = V - V_{j-1}$ and $U(\theta) = V_{j-1}$.*

It helps to rewrite the expected cost of $f(A(\theta))$ and $f(U(\theta))$ under θ-rounding in a more convenient form given below.

Proposition 4.3. *Let $r \in [0,1]$, and let h be the largest value of j such that $\alpha_j \leq r$. We have*

$$\int_0^r f(A(\theta))d\theta = \sum_{j=1}^{h} \alpha_j (f(V - V_{j-1}) - f(V - V_j)) + rf(V - V_h)$$

and

$$\int_0^r f(U(\theta))d\theta = \sum_{j=1}^{h} \alpha_j (f(V_{j-1}) - f(V_j)) + rf(V_h).$$

Proof: Recall from Proposition 4.2 that $A(\theta) = V - V_{j-1}$ when $\theta \in (\alpha_{j-1}, \alpha_j]$. Therefore,

$$\int_0^r f(A(\theta))d\theta = \sum_{j=1}^{h}\int_{\alpha_{j-1}}^{\alpha_j} f(A(\theta))d\theta + \int_{\alpha_h}^{r} f(A(\theta)d\theta$$

$$= \sum_{j=1}^{h}(\alpha_j - \alpha_{j-1})f(V - V_{j-1}) + (r - \alpha_h)f(V - V_h)$$

$$= rf(V - V_h) + \sum_{j=1}^{h}\alpha_j(f(V - V_{j-1}) - f(V - V_j)).$$

The second identity follows from a very similar argument. ∎

The inductive approach: Recall that numbering the vertices in increasing order of their α values ensures that $U(\theta)$ is V_j for some $0 \leq j \leq n$. Let $\mathbf{x}_j$ be the restriction of $\mathbf{x}$ to V_j. Note that $\mathbf{x}_j$ gives a feasible allocation of V_j to the k labels although it does not necessarily correspond to a multiway partition with respect to the original terminals. Also, note that the function f when restricted to V_j is still submodular but may not be symmetric even if f is. In order to argue about $\mathbf{x}_j$ we introduce additional notation. Let $A_j(i,\theta) = A(i,\theta) \cap V_j$, $A_j(\theta) = A(\theta) \cap V_j$, and $U_j(\theta) = U(\theta) \cap V_j$. In other words, $A_j(\theta)$ and $U_j(\theta)$ are the allocated and unallocated sets if we did θ-rounding with respect to $\mathbf{x}_j$ that is defined over V_j.

Let $\rho_j = \sum_{i=1}^{k}\int_0^{\delta} f(A_j(i,\theta))d\theta$; we have $\rho_0 = k\delta f(\emptyset)$. Note that the left hand side of the inequality in Theorem 1.5 is $\rho_n = \sum_{i=1}^{k}\int_0^{\delta} f(A_n(i,\theta))d\theta$, since $A_n(i,\theta) = A(i,\theta)$.

[5] An alert reader may notice that we do not distinguish between terminals and non-terminals. In fact the theorem statement does not rely on the fact that terminals are assigned fully to their respective labels. The only place we use the fact that $x(s_i, i) = 1$ for each i is to show that θ-rounding based algorithms produce a valid multiway partition with respect to the terminals.

To understand ρ_n we consider the quantity $\rho_j - \rho_{j-1}$ which is easier since ρ_j and ρ_{j-1} differ only in v_j. Recall that ℓ_j is a label such that $\alpha_j = \max_{i=1}^{k} x(v_j, i)$. The importance of ℓ_j is that if v_j is allocated then it is allocated to ℓ_j (and possibly to other labels as well). We express $\rho_j - \rho_{j-1}$ as the sum of two quantities with the term for ℓ_j separated out.

Proposition 4.4.

$$\rho_j - \rho_{j-1} = \int_0^{\delta} \left(f(A_j(\ell_j,\theta)) - f(A_{j-1}(\ell_j,\theta)\right) d\theta \\ + \sum_{i \neq \ell_j} \int_0^{\delta} \left(f(A_j(i,\theta)) - f(A_{j-1}(i,\theta)\right) d\theta.$$

We prove the following two key lemmas by using of submodularity of f appropriately.

Lemma 4.5. *For any $\delta \in [1/2, 1]$ and for any j such that $1 \leq j \leq n$,*

$$\sum_{i \neq \ell_j} \int_0^{\delta} (f(A_j(i,\theta)) - f(A_{j-1}(i,\theta))d\theta \\ \geq f(V_j) - f(V_{j-1}) + \alpha_j (f(V_{j-1}) - f(V_j)).$$

Summing the left hand side in the above lemma over all j and applying Proposition 4.3 with $r = 1$ we obtain:

Corollary 4.6.

$$\sum_{j=1}^{n} \left(\sum_{i \neq \ell_j} \int_0^{\delta} (f(A_j(i,\theta)) - f(A_{j-1}(i,\theta))d\theta \right) \\ \geq \int_0^1 f(U(\theta))d\theta - f(\emptyset).$$

Our second key lemma below is the more involved one. Unlike the first lemma above we do not have a clean and easy expression for a single term $\int_0^{\delta} \left(f(A_j(\ell_j,\theta)) - f(A_{j-1}(\ell_j,\theta)\right) d\theta$ but the sum over all j gives a nice telescoping sum that results in the bound below.

Lemma 4.7. *Let $\delta \in [0, 1]$ and let h be the largest value of j such that $\alpha_j \leq \delta$.*

$$\sum_{j=1}^{n} \left(\int_0^{\delta} (f(A_j(\ell_j,\theta)) - f(A_{j-1}(\ell_j,\theta))d\theta \right) \\ \geq \int_0^{\delta} f(A(\theta))d\theta - \delta f(\emptyset).$$

The proofs of the above lemmas are given in Sections 4.1 and 4.2 respectively. We now finish the proof of Theorem 1.5 assuming the above two lemmas.

Proof of Theorem 1.5: Let h be the largest value of j such that $\alpha_j \leq \delta$. From Proposition 4.4 we have

$$\begin{aligned}
\sum_{i=1}^{k} \int_0^{\delta} f(A(i,\theta))d\theta &= \rho_n = \rho_0 + \sum_{j=1}^{n} (\rho_j - \rho_{j-1}) \\
&= \rho_0 + \sum_{j=1}^{n} \left(\int_0^{\delta} (f(A_j(\ell_j,\theta)) - f(A_{j-1}(\ell_j,\theta))d\theta \right) \\
&\quad + \sum_{j=1}^{n} \left(\sum_{i \neq \ell_j} \int_0^{\delta} (f(A_j(i,\theta)) - f(A_{j-1}(i,\theta))d\theta \right) \\
&\geq \rho_0 + \int_0^{\delta} f(A(\theta))d\theta - \delta f(\emptyset) \\
&\quad + \sum_{j=1}^{n} \left(\sum_{i \neq \ell_j} \int_0^{\delta} (f(A_j(i,\theta)) - f(A_{j-1}(i,\theta))d\theta \right) \\
&\qquad \text{(Use Lemma 4.7)} \\
&\geq \rho_0 + \int_0^{\delta} f(A(\theta))d\theta - \delta f(\emptyset) + \int_0^1 f(U(\theta))d\theta - f(\emptyset) \\
&\qquad \text{(Use Corollary 4.6)} \\
&\geq (k\delta - \delta - 1) f(\emptyset) + \int_0^{\delta} f(A(\theta))d\theta + \int_0^1 f(U(\theta))d\theta.
\end{aligned}$$

We used $\rho_0 = \delta k f(\emptyset)$ in the final inequality. ■

4.1. Proof of Lemma 4.5

Recall that the lemma states that for $\delta \in [1/2, 1]$ and for any j,

$$\sum_{i \neq \ell_j} \int_0^{\delta} (f(A_j(i,\theta)) - f(A_{j-1}(i,\theta))d\theta \\ \geq f(V_j) - f(V_{j-1}) + \alpha_j (f(V_{j-1}) - f(V_j)).$$

Proof of Lemma 4.5: Fix j and label i. We have

$$\int_0^{\delta} (f(A_j(i,\theta)) - f(A_{j-1}(i,\theta))d\theta \\ = \int_0^{\min(\delta, x(v_j,i))} (f(A_j(i,\theta)) - f(A_{j-1}(i,\theta))d\theta$$

since $A_j(i,\theta) = A_{j-1}(i,\theta)$ when θ is in the interval $(\min(\delta, x(v_j,i)), \delta]$ (or the interval is empty). When $\theta \leq x(v_j, i)$ we have $A_j(i,\theta) = A_{j-1}(i,\theta) + v_j$. Since f is submodular and $A_{j-1}(i,\theta) \subseteq V_{j-1}$, it follows that, for any $\theta \leq x(v_j, i)$, we have

$$\begin{aligned}
&f(A_j(i,\theta)) - f(A_{j-1}(i,\theta)) \\
&\quad = f(A_{j-1}(i,\theta) + v_j) - f(A_{j-1}(i,\theta)) \\
&\quad \geq f(V_{j-1} + v_j) - f(V_{j-1}) \\
&\quad = f(V_j) - f(V_{j-1}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_0^{\delta}(f(A_j(i,\theta)) - f(A_{j-1}(i,\theta))d\theta \\
&\quad = \int_0^{\min(\delta,x(v_j,i))}(f(A_j(i,\theta)) - f(A_{j-1}(i,\theta))d\theta \\
&\quad \geq \int_0^{\min(\delta,x(v_j,i))}(f(V_j) - f(V_{j-1}))d\theta \\
&\quad = \min(\delta,x(v_j,i))(f(V_j) - f(V_{j-1})).
\end{aligned}$$

Note that, for any $i \neq \ell_j$, $x(v_j,i) \leq \delta$: if $\alpha_j \leq \delta$, the claim follows, since $x(v_j,i) \leq \alpha_j$; otherwise, since $\delta \geq 1/2$ and $\sum_i x(v_j,i) = 1$, it follows that $x(v_j,i) \leq \delta$ for all $i \neq \ell_j$. Therefore, by using the previous bound,

$$\begin{aligned}
&\sum_{i\neq\ell_j}\int_0^{\delta}(f(A_j(i,\theta) - f(A_{j-1}(i,\theta))d\theta \\
&\quad \geq \sum_{i\neq\ell_j}\min(\delta,x(v_j,i))(f(V_j) - f(V_{j-1})) \\
&\quad = \sum_{i\neq\ell_j}x(v_j,i)(f(V_j) - f(V_{j-1})) \\
&\quad = (1 - x(v_j,\ell_j))(f(V_j) - f(V_{j-1})) \\
&\quad = f(V_j) - f(V_{j-1}) + \alpha_j(f(V_{j-1}) - f(V_j)).
\end{aligned}$$

■

4.2. Proof of Lemma 4.7

We recall the statement of the lemma. Let $\delta \in [0,1]$ and let h be the largest value of j such that $\alpha_j \leq \delta$. We want to show that

$$\begin{aligned}
&\sum_{j=1}^{n}\left(\int_0^{\delta}(f(A_j(\ell_j,\theta)) - f(A_{j-1}(\ell_j,\theta))d\theta\right) \\
&\qquad \geq \int_0^{\delta}f(A(\theta))d\theta - \delta f(\emptyset).
\end{aligned}$$

Our goal is to obtain a suitable expression that is upper bounded by the quantity $\int_0^{\delta}(f(A_j(\ell_j,\theta)) - f(A_{j-1}(\ell_j,\theta))d\theta$. It turns out that this expression has several terms and when we sum over all j they telescope to give us the desired bound.

We begin by simplifying $\int_0^{\delta}(f(A_j(\ell_j,\theta)) - f(A_{j-1}(\ell_j,\theta))d\theta$ by applying submodularity. The following proposition follows from the fact that f is submodular and $A_j(\ell_j,\theta) \subseteq A_j(\theta)$.

Proposition 4.8. *For any j such that $1 \leq j \leq n$,*

$$\begin{aligned}
&\int_0^{\delta}\left(f(A_j(\ell_j,\theta)) - f(A_{j-1}(\ell_j,\theta))\right)d\theta \\
&\quad \geq \int_0^{\min(\alpha_j,\delta)}\left(f(A_{j-1}(\theta) + v_j) - f(A_{j-1}(\theta))\right)d\theta.
\end{aligned}$$

Proof: If $\theta \in [0,\min(\delta,\alpha_j)]$ we have $A_j(\ell_j,\theta) = A_{j-1}(\ell_j,\theta) + v_j$. If $\theta \in (\min(\delta,\alpha_j),\delta]$ then $A_j(\ell_j,\theta) = A_{j-1}(\ell_j,\theta)$. Therefore

$$\begin{aligned}
&\int_0^{\delta}\left(f(A_j(\ell_j,\theta)) - f(A_{j-1}(\ell_j,\theta))\right)d\theta \\
&\quad = \int_0^{\min(\delta,\alpha_j)}\left(f(A_j(\ell_j,\theta)) - f(A_{j-1}(\ell_j,\theta))\right)d\theta \\
&\quad = \int_0^{\min(\delta,\alpha_j)}\left(f(A_{j-1}(\ell_j,\theta) + v_j) - f(A_{j-1}(\ell_j,\theta))\right)d\theta
\end{aligned}$$

Since f is submodular and $A_{j-1}(\ell_j,\theta) \subseteq A_{j-1}(\theta)$, it follows that, for any $\theta \leq \alpha_j$,

$$f(A_j(\ell_j,\theta)) - f(A_{j-1}(\ell_j,\theta)) \geq f(A_{j-1}(\theta) + v_j) - f(A_{j-1}(\theta))$$

and the proposition follows. ■

Let $\Delta_j = \int_0^{\alpha_j}\left(f(A_{j-1}(\theta) + v_j) - f(A_{j-1}(\theta))\right)d\theta$, and $\Lambda_j = \int_{\delta}^{\alpha_j}\left(f(A_{j-1}(\theta) + v_j) - f(A_{j-1}(\theta))\right)d\theta$. Note that the right hand side of the inequality in Proposition 4.8 is equal to Δ_j if $j \leq h$, and it is equal to $\Delta_j - \Lambda_j$ otherwise. Proposition 4.9 and Proposition 4.11 express Δ_j and Λ_j in a more convenient form.

Let $V_{j',j} = \{v_{j'}, v_{j'+1}, \cdots, v_j\}$ for all j' and j such that $j' \leq j$; let $V_{j',j} = \emptyset$ for all j' and j such that $j' > j$.

Proposition 4.9.

$$\Delta_j = \sum_{j'=1}^{n}(\alpha_{j'} - \alpha_{j'-1})(f(V_{j',j}) - f(V_{j',j-1})).$$

Proof: It follows Proposition 4.2 that, if $\theta \in (\alpha_{j'-1},\alpha_{j'}]$, $A(\theta) = V_{j',n}$ and $A_j(\theta) = V_{j',j}$. Therefore

$$\begin{aligned}
\Delta_j &= \int_0^{\alpha_j}(f(A_{j-1}(\theta) + v_j) - f(A_{j-1}(\theta))d\theta \\
&= \sum_{j'=1}^{j}\int_{\alpha_{j'-1}}^{\alpha_{j'}}\left(f\left(A_{j-1}(\theta) + v_j\right) - f\left(A_{j-1}(\theta)\right)\right)d\theta \\
&= \sum_{j'=1}^{j}(\alpha_{j'} - \alpha_{j'-1})(f(V_{j',j}) - f(V_{j',j-1})) \\
&= \sum_{j'=1}^{n}(\alpha_{j'} - \alpha_{j'-1})(f(V_{j',j}) - f(V_{j',j-1})).
\end{aligned}$$

The last line follows from the fact that, if $j' > j$, $V_{j',j} = V_{j',j-1} = \emptyset$. ■

The corollary below follows by simple algebraic manipulation; see Appendix A in [4] for a proof.

Corollary 4.10.

$$\sum_{j=1}^{n}\Delta_j = \sum_{j=1}^{n}\alpha_j(f(V - V_{j-1}) - f(V - V_j)).$$

We now consider Γ_j.

Proposition 4.11. *For all $j > h$ where h is the largest index such that $\alpha_h \leq \delta$,*

$$\Lambda_j = (\alpha_{h+1} - \delta)(f(V_{h+1,j}) - f(V_{h,j-1})) + \sum_{j'=h+2}^{n} (\alpha_{j'} - \alpha_{j'-1})(f(V_{j',j}) - f(V_{j',j-1})).$$

Proof: For notational convenience let $\beta_h = \delta$ and $\beta_j = \alpha_j$ for all $j > h$. It follows Proposition 4.2 that, if $\theta \in (\beta_{j'-1}, \beta_{j'}]$, $A(\theta) = V_{j',n}$ and $A_j(\theta) = V_{j',j}$. Therefore

$$\begin{aligned}
&\int_{\delta}^{\alpha_j} (f(A_{j-1}(\theta) + v_j) - f(A_{j-1}(\theta)))d\theta \\
&= \sum_{j'=h+1}^{j} \int_{\beta_{j'-1}}^{\beta_{j'}} (f(A_{j-1}(\theta) + v_j) - f(A_{j-1}(\theta)))d\theta \\
&= \sum_{j'=h+1}^{j} (\beta_{j'} - \beta_{j'-1})(f(V_{j',j}) - f(V_{j',j-1})) \\
&= \sum_{j'=h+1}^{n} (\beta_{j'} - \beta_{j'-1})(f(V_{j',j}) - f(V_{j',j-1})).
\end{aligned}$$

The last line follows from the fact that, if $j' > j$, $V_{j',j} = V_{j',j-1} = \emptyset$. The lemma follows by noting that

$$\begin{aligned}
\sum_{j'=h+1}^{n} &(\beta_{j'} - \beta_{j'-1})(f(V_{j',j}) - f(V_{j',j-1})) \\
&= (\alpha_{h+1} - \delta)(f(V_{h+1,j}) - f(V_{h,j-1})) \\
&\quad + \sum_{j'=h+2}^{n} (\alpha_{j'} - \alpha_{j'-1})(f(V_{j',j}) - f(V_{j',j-1})).
\end{aligned}$$

■

The corollary below follows by simple algebraic manipulation; see Appendix A in [4] for a proof.

Corollary 4.12.

$$\sum_{j=h+1}^{n} \Lambda_j = \sum_{j=h+1}^{n} \alpha_j(f(V - V_{j-1}) - f(V - V_j)) - \delta(f(V - V_h) - f(\emptyset)).$$

Now we finish the proof.

Proof of Lemma 4.7: We apply Proposition 4.8 in the first inequality below, and then Corollary 4.10 and Corollary 4.12 to derive the third line from the second.

$$\begin{aligned}
&\sum_{j=1}^{n} \int_{0}^{\delta} (f(A_j(\ell_j, \theta)) - f(A_{j-1}(\ell_j, \theta)))\, d\theta \\
&\geq \sum_{j=1}^{n} \int_{0}^{\min(\alpha_j, \delta)} (f(A_{j-1}(\theta) + v_j) - f(A_{j-1}(\theta)))\, d\theta \\
&= \sum_{j=1}^{n} \Delta_j - \sum_{j=h+1}^{n} \Lambda_j \\
&= \sum_{j=1}^{h} \alpha_j(f(V - V_{j-1}) - f(V - V_j)) \\
&\qquad + \delta(f(V - V_h) - f(\emptyset)) \\
&= \int_{0}^{\delta} f(A(\theta))d\theta - \delta f(\emptyset).
\end{aligned}$$

The last equality follows from Proposition 4.3. ■

5. Conclusions and Open Problems

The main open question is whether the integrality gap of SubMP-Rel for Sym-Sub-MP is stricly smaller than the bound of $(1.5 - \frac{1}{k})$ we showed in this paper. Karger *et al.* [18] rely extensively on the geometry of the simplex to obtain a bound of 1.3438 for Graph Multiway Cut via the relaxation from [3]. However, we mention that the rounding algorithms used in [18] have natural analogues for rounding SubMP-Rel but analyzing them appears challenging for an arbitrary symmetric submodular function.

Zhao, Nagamochi and Ibaraki [29] considered a common generalization of Sub-MP and k-way Sub-MP where we are given a set S of terminals with $|S| \geq k$ and the goal is to partition V into k sets $A_1, \ldots, A_k$ such that each A_i contains at least one terminal and $\sum_{i=1}^{k} f(A_i)$ is minimized. Note that when $|S| = k$ we get Sub-MP and when $S = V$ we get k-way Sub-MP. The advantage of the greedy splitting algorithms developed in [29] is that they extend to these more general problems. However, unlike the case of Sub-MP, there does not appear to be an easy way to write a mathematical programming relaxation for this more general problem; see [7] for a relaxation in the case of graphs. An interesting open problem here is whether the k-way cut problem in graphs admits an approximation better than $2(1 - \frac{1}{k})$.

Related to the above questions is the complexity of k-way Sub-MP when k is a fixed constant. For Sym-Sub-MP a polynomial-time algorithm was claimed in [23] although no formal proof has been published; this generalizes the polynomial-time algorithm for graph k-cut problem first developed by Goldschmidt and Hochbaum [16]. There has been particular interest in the special case of k-way Sub-MP, namely, the hypergraph k-cut problem [28], [22], [12]. It is an open problem whether the hypergraph k-cut problem has a polynomial time algorithm for $k = 4$.

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