Chapter 6

The Mean Variance Frontier

Contents

We now discuss a classical topic in finance, mean variance analysis. This leads to ways of accounting for the riskiness of cashflows.

Mean variance analysis concerns investors choices between portfolios of risky assets, and how an investor chooses portfolio weights. Let r_p be a portfolio return. We assume that investors preferences over portfolios p satisfy a mean variance utility representation, $u(p) = u(E[r_p], \sigma(r_p))$, with utility increasing in expected return ($\partial u / \partial E[r_p] > 0$) and decreasing in variance ($\partial u / \partial \text{var}(r_p) < 0$). In this part we consider the representation of the *portfolio opportunity set* of such decision makers. There are a number of useful properties of this opportunity set which follows purely from the mathematical formulation of the optimization problem. It is these properties we focus on here.

6.1 Setup

We assume there exists $n \geq 2$ risky securities, with expected returns $\mathbf{e}$

$$\mathbf{e} = \begin{bmatrix} E[r_1] \\ E[r_2] \\ \vdots \\ E[r_n] \end{bmatrix}$$

and covariance matrix $\mathbf{V}$:

$$\mathbf{V} = \begin{bmatrix} \sigma(r_1,r_1) & \sigma(r_1,r_2) & \dots & \\ \sigma(r_2,r_1) & \sigma(r_2,r_2) & \dots & \\ \vdots & & & \\ \sigma(r_n,r_1) & & \dots & \sigma(r_n,r_n) \end{bmatrix}$$

The covariance matrix $\mathbf{V}$ is assumed to be invertible.

A *portfolio* p is defined by a set of weights $\mathbf{w}$ invested in the risky assets.

$$\mathbf{w} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{bmatrix},$$

where w_i is the fraction of the investors wealth invested in asset i. Note that the weights sum to one. The expected return on a portfolio is calculated as

$$E[r_p] = \mathbf{w}'\mathbf{e}$$

and the variance of the portfolio is

$$\sigma^2(r_p) = \mathbf{w}'\mathbf{V}\mathbf{w}$$

Example

An investor can invest in three assets with expected returns and variances as specified in the following table.

t Asset	$E[r]$	$\sigma^2(r)$
1	10%	0.20
2	11.5%	0.10
3	8%	0.15

The three assets are independent (uncorrelated).

1. Determine the expected return and standard deviation of an equally weighted portfolio of the three assets

Matlab program:

```
e=[0.1 0.11 0.08]
V=[ 0.2 0 0; 0 0.1 0 ; 0 0 0.15]
w=1/3*[1 1 1]
er= e*w'
sigma=sqrt(w*V*w')
```

Output from Matlab program:

```
e =
 0.100000 0.110000 0.080000
V =
 0.20000 0.00000 0.00000
 0.00000 0.10000 0.00000
 0.00000 0.00000 0.15000
w =
 0.33333 0.33333 0.33333
er = 0.096667
sigma = 0.22361
```

6.2 The minimum variance frontier

A portfolio is a *frontier* portfolio if it minimizes the variance for a given expected return, that is, a frontier portfolio p solves

$$\mathbf{w}_p = \arg\min_{\mathbf{w}} \frac{1}{2}\mathbf{w}'\mathbf{V}\mathbf{w}$$

subject to:

$$\mathbf{w}'\mathbf{e} = E[\tilde{r}_p]$$

$$\mathbf{w}'\mathbf{1} = 1$$

The set of all frontier portfolios is called the *minimum variance frontier*.

6.3 Calculation of frontier portfolios

Proposition 1 *If the matrix $\mathbf{V}$ is full rank, and there are no restrictions on shortsales, the weights $\mathbf{w}_p$ for a frontier portfolio p with mean $E[\tilde{r}_p]$ can be found as*

$$\mathbf{w}_p = \mathbf{g} + \mathbf{h}E[r_p]$$

where

$$\mathbf{g} = \frac{1}{D}\left(B\mathbf{1}' - A\mathbf{e}'\right)\mathbf{V}^{-1}$$

$$\mathbf{h} = \frac{1}{D}\left(C\mathbf{e}' - A\mathbf{1}'\right)\mathbf{V}^{-1}$$

$$A = \mathbf{1}'\mathbf{V}^{-1}\mathbf{e}$$

$$B = \mathbf{e}'\mathbf{V}^{-1}\mathbf{e}$$

$$C = \mathbf{1}'\mathbf{V}^{-1}\mathbf{1}$$

$$\mathbf{A} = \begin{bmatrix} B & A \\ A & C \end{bmatrix}$$

$$D = BC - A^2 = |\mathbf{A}|$$

Proof

Any minimum variance portfolio solves the program

$$\mathbf{w}_p = \arg\min_{\mathbf{w}} \frac{1}{2}\mathbf{w}'\mathbf{V}\mathbf{w}$$

subject to

$$\mathbf{w}'\mathbf{e} = E[\tilde{r}_p]$$

$$\mathbf{w}'\mathbf{1} = 1$$

Set up the Lagrangian corresponding to this problem

$$L(\mathbf{w}, \lambda, \gamma|\mathbf{e}, \mathbf{V}) = \frac{1}{2}\mathbf{w}'\mathbf{V}\mathbf{w} - \lambda\left(E[\tilde{r}_p] - \mathbf{w}'\mathbf{e}\right) - \gamma(1 - \mathbf{w}'\mathbf{1})$$

Differentiate

$$\frac{\partial L}{\partial \mathbf{w}} = \mathbf{w}'\mathbf{V} - \lambda \mathbf{e}' - \gamma \mathbf{1}' = 0$$

$$\frac{\partial L}{\partial \lambda} = E[r_p] - \mathbf{w}'\mathbf{e} = 0$$

$$\frac{\partial L}{\partial \gamma} = 1 - \mathbf{w}'\mathbf{1} = 0$$

Rewrite conditions above as (note that this requires the invertibility of $\mathbf{V}$.

$$\mathbf{w}' = \lambda \mathbf{e}' V^{-1} - \gamma \mathbf{1}'\mathbf{V}^{-1} \tag{6.1}$$

$$\mathbf{w}'\mathbf{e} = E[\tilde{r}_p] \tag{6.2}$$

$$\mathbf{w}'\mathbf{1} = 1 \tag{6.3}$$

Post-multiply equation (6.1) with $\mathbf{e}$ and recognise the expression for $E[r_p]$ in the second equation

$$\mathbf{w}'\mathbf{e} = E[r_p] = \lambda \mathbf{e}'\mathbf{V}^{-1}\mathbf{e} + \gamma \mathbf{1}'\mathbf{V}^{-1}\mathbf{e}$$

Similarly post-multiply equation (6.1) with $\mathbf{1}$ and recognise the expression for 1 in the third equation

$$\mathbf{w}'\mathbf{1} = 1 = \lambda \mathbf{e}'\mathbf{V}^{-1}\mathbf{1} + \gamma \mathbf{1}'\mathbf{V}^{-1}\mathbf{1}$$

With the definitions of A, B, C and D above, this becomes the following system of equations

$$\left\{ \begin{array}{lcl} E[r_p] & = & \lambda B + \gamma A \\ 1 & = & \lambda A + \gamma C \end{array} \right\}$$

Solving for λ and γ, get

$$\gamma = \frac{B - AE[r_p]}{D}$$

$$\lambda = \frac{CE[r_p] - A}{D}$$

Plug in expressions for λ and γ into equation (6.1) above, and get

$$\mathbf{w}' = \frac{1}{D}\left(B\mathbf{1}' - A\mathbf{e}'\right)\mathbf{V}^{-1} + \frac{1}{D}\left(C\mathbf{e}' - A\mathbf{1}'\right)\mathbf{V}^{-1}E[r_p] = \mathbf{g} + \mathbf{h}E[r_p]$$

The portfolio defined by weights $\mathbf{g}$ is a portfolio with expected return 0. The portfolio defined by weights $(\mathbf{g} + \mathbf{h})$ is a portfolio with expected return 1. This implies the useful property that $\mathbf{g}\mathbf{1}' = 1$, and $\mathbf{h}\mathbf{1}' = 0$.

Example

An investor can invest in three assets with expected returns and variances as specified in the following table.

t Asset	$E[r]$	$\sigma^2(r)$
1	10%	0.20
2	11.5%	0.10
3	8%	0.15

The three assets are independent (uncorrelated).

1. What are the weights of the minimum variance portfolio with mean 9%?

Matlab program:

```
e=[0.1 0.11 0.08]'
V=[ 0.2 0 0; 0 0.1 0 ; 0 0 0.15]
r=0.09
n = length(e)
a = ones(1,n)*inv(V)*e
b = e'*inv(V)*e
c = ones(1,n)*inv(V)*ones(n,1)
A = [b a;a c]
d = det(A)
g = 1/d*(b*ones(1,n) - a*e')*inv(V)
h = h = 1/d*(c*e' - a*ones(1,n))*inv(V)
w=g+h*r
```

Output from Matlab program:

```
e =
 0.100000
 0.110000
 0.080000
V =
 0.20000 0.00000 0.00000
 0.00000 0.10000 0.00000
 0.00000 0.00000 0.15000
r = 0.090000
n = 3
a = 2.1333
b = 0.21367
c = 21.667
A =
 0.21367 2.13333
 2.13333 21.66667
d = 0.078333
g =
 0.021277 -2.680851 3.659574
h =
 2.1277 31.9149 -34.0426
w =
 0.21277 0.19149 0.59574
```

This calculation is put into a Matlab function in Matlab **Code 6.1**.

```
function w = min_variance_portfolio(e,V,r)
   n = length(e);
   a = ones(1,n)*inv(V)*e;
   b = e'*inv(V)*e;
   c = ones(1,n)*inv(V)*ones(n,1);
   A = [b a;a c];
   d = det(A);
   g = 1/d*(b*ones(1,n) - a*e')*inv(V);
   h = h = 1/d*(c*e' - a*ones(1,n))*inv(V);
   w=g+h*r;
end
```

Matlab **Code 6.1**: Calculation of minimum variance portfolio for given return

Calculating the weights of the minimum variance portfolio given an interest rate

$$\mathbf{w}_p = \mathbf{g} + \mathbf{h}E[r_p]$$

where

$$\mathbf{g} = \frac{1}{D}\left(B\mathbf{1}' - A\mathbf{e}'\right)\mathbf{V}^{-1}$$

$$\mathbf{h} = \frac{1}{D}\left(C\mathbf{e}' - A\mathbf{1}'\right)\mathbf{V}^{-1}$$

$$A = \mathbf{1}'\mathbf{V}^{-1}\mathbf{e}$$

$$B = \mathbf{e}'\mathbf{V}^{-1}\mathbf{e}$$

$$C = \mathbf{1}'\mathbf{V}^{-1}\mathbf{1}$$

$$D = BC - A^2 =$$

Notation: r_p desired portfolio returns.$\mathbf{V}$: covariance matrix of asset returns. $\mathbf{e}$: vector of expected asset returns.

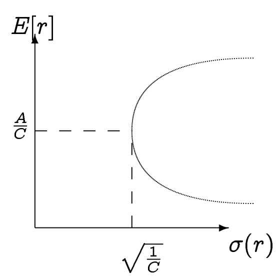
6.4 The global minimum variance portfolio

The portfolio that minimizes variance regardless of expected return is called the *global minimum variance portfolio*. Let mvp be the global minimum variance portfolio.

Proposition 2 (Global Minimum Variance Portfolio) *The global minimum variance portfolio has weights*

$$\mathbf{w}'_{mvp} = \left(\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}\right)^{-1}\mathbf{1}'\mathbf{V}^{-1} = \frac{1}{C}\mathbf{1}'\mathbf{V}^{-1},$$

expected return $E[r_{mvp}] = \frac{A}{C}$ *and variance* $var(r_{mvp}) = \frac{1}{C}$.



6.5 Efficient portfolios

Portfolios on the minimum variance frontier with expected returns higher than or equal to $E[r_{mvp}]$ are called *efficient* portfolios.