Proving differential privacy in Hoare logic

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Abstract—Differential privacy **is a rigorous, worst-case notion of privacy-preserving computation. Informally, a probabilistic program is differentially private if the participation of a single individual in the input database has a limited effect on the program's distribution on outputs. More technically, differential privacy is a quantitative** 2**-safety property that bounds the distance between the output distributions of a probabilistic program on adjacent inputs. Like many** 2**-safety properties, differential privacy lies outside the scope of traditional verification techniques. Existing approaches to enforce privacy are based on intricate, non-conventional type systems, or customized relational logics. These approaches are difficult to implement and often cumbersome to use.**

We present an alternative approach that verifies differential privacy by standard, non-relational reasoning on non-probabilistic programs. Our approach transforms a probabilistic program into a non-probabilistic program which simulates two executions of the original program. We prove that if the target program is correct with respect to a Hoare specification, then the original probabilistic program is differentially private. We provide a variety of examples from the differential privacy literature to demonstrate the utility of our approach. Finally, we compare our approach with existing verification techniques for privacy.

I. INTRODUCTION

Program verification provides a rich array of techniques and tools for analyzing program properties. However, they typically reason about single program executions or trace properties. In contrast, many security properties—such as non-interference in information flow systems—require reasoning about *multiple* program executions. These *hyperproperties* [17] encompass many standard security analyses, and lie outside the scope of standard verification tools—to date, there is no generally applicable method or tool for verifying hyperproperties. Instead, ad hoc enforcement methods based on type systems, customized program logics, and finite state automata analyses have been applied to specific hyperproperties. While these approaches are effective, their design and implementation often require significant effort.

A promising alternative is to reduce verification of a hyperproperty of a program c to verification of a standard property of a transformed program $T(c)$. For instance, *self-composition* [7], [18] is a general method for reducing *2-safety* properties of a program c—which reason about two runs of c—to safety properties of the sequential composition $c; c'$, where c' is a renaming of c. Self-composition is sound, complete, and applies to many programming languages and verification settings. For instance, it has been used to verify information flow properties using standard deductive methods like Hoare logic.

A close relative of self-composition is the *synchronized product* construction [38]. This transformation also produces a program which emulates two executions of the original program, but while self-composition performs the executions in sequence, synchronized products perform the executions in *lockstep*, dramatically simplifying the verification task for certain properties. This transformation is an instance of the more general class of *product transformations*, as studied by Zaks and Pnueli [38], and recently by Barthe et al. [4], [5].

While there has been much research on combining product constructions and deductive verification to reason about 2-safety for deterministic programs, this approach remains largely unexplored for *probabilistic* programs. This is not for lack of interesting use cases—many security notions of probabilistic computation are naturally 2-safety properties.

Verifying differential privacy: In this paper, we consider on one such property: *differential privacy*, which provides strong guarantees for privacy-preserving probabilistic computation. Formally, a probabilistic program c is (ϵ, δ)-differentially private with respect to $\epsilon > 0$, $\delta \geq 0$, and a relation[1] Φ on the initial memories of c if for every two initial memories m_1 and m_2 related by Φ, and every subset A of output memories,

$$\Pr[c, m_1 : A] \leq \exp(\epsilon) \Pr[c, m_2 : A] + \delta.$$

Here $\Pr[c, m : A]$ denotes the probability of the output memory landing in A according to distribution $[\![c]\!] \, m$, where $[\![c]\!]$ maps an initial memory m to a distribution $[\![c]\!] \, m$ of output memories. Since this definition concerns two runs of the same probabilistic program, differential privacy is a probabilistic 2-safety property.

Differentially private algorithms are typically built from two constructions: *private mechanisms*, which add probabilistic noise to their input, and *composition*, which combines differentially private operations into a single one. This compositional behavior makes differential privacy an attractive target for program verification efforts.

Existing methods for proving differential privacy have been based on type systems, automata analyses, and customized program logics. For instance, Fuzz [33], DFuzz [25] and related systems [24] enforce differential privacy using linear type systems. This approach is expressive enough to type many examples, but it is currently limited to *pure* differential privacy

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[1]We are here taking a generalization of Differential Privacy with respect to an arbitrary relation Φ. The usual definition is obtained by considering an *adjacency* relation between databases.

(where $\delta = 0$), and cannot handle more advanced examples. Alternatively, Tschantz et al. [37] consider a verification technique based on I/O automata; again, this approach is limited to pure differential privacy. Finally, CertiPriv [11] and EasyCrypt [6] use an approximate relational Hoare logic for probabilistic programs to verify differential privacy. This approach is very expressive and can accommodate approximate differential privacy (when $\delta \neq 0$), but relies on a custom and complex logic. For instance, ad hoc rules for loops are required for many advanced examples. Finally, a common weakness of all of the above approaches is that their implementation is non-trivial.

Self-products for differential privacy: To avoid these drawbacks, we investigate a new approach where proving (ϵ, δ)-differential privacy of a program c is reduced to proving a safety property of a transformed program $T(c)$. In view of previous work verifying 2-safety properties, a natural choice for T is some notion of product program. However, the transformed programs would then be probabilistic, and there are few tools for deductive verification of probabilistic programs. Targeting a non-probabilistic language is more appealing in this regard, as there are many established tools for deductive verification of non-probabilistic programs. Since the original program is a probabilistic program, a key part of our approach is to remove the probabilistic behavior from the target.

To define the transformation, we proceed in two steps. Starting from a probabilistic program c, we first construct the synchronized product of c with itself. Using the synchronized product instead of self-composition is essential for our second step, in which the probabilistic product program is transformed into a non-probabilistic program.

For this step, we rely on specific features of the differential privacy property. First, we observe that differential privacy bounds the ratio—hereafter called the *privacy cost*—between the probabilities of producing the same output on two executions on nearby databases. Second, we recall that there are two main tools for building differentially private computations: private mechanisms, and composition. Private mechanisms and composition interact with the privacy cost in different ways; we consider each in turn.

A private mechanism run over two different inputs returns two closely related distributions, at the cost of consuming some privacy budget. The privacy cost depends on the distance between the inputs: as the two inputs become farther apart, the privacy cost also grows. One fundamental insight (due to Barthe et al. [10]) we use is that the property of being "closely related" can be understood as being at distance 0 for a suitable notion of distance on distributions.

Composition takes a set of differentially private operations and returns the sequential composition of the operations, which is also differentially private. By a property of differential privacy, the privacy cost of the composition is upper bounded by the sum of the privacy costs of the individual operations.

We build this reasoning directly into our verification system. First, we apply the synchronized product construction. Then, we replace two corresponding calls to a mechanism with a call

to an abstract procedure that returns equal outputs, at the cost of consuming some privacy budget—roughly, being at distance 0 in the probabilistic setting is equivalent to being equal in the non-probabilistic setting. To keep track of the privacy cost, we use ghost variables v_ϵ and v_δ which are incremented after each mechanism is executed, in terms of the distance between their two inputs.

Note that the second step leverages the synchronized product construction: since the two executions are simulated in lockstep, corresponding calls to a mechanism are next to each other in the product program. Since mechanisms are the only probabilistic parts of our source program, our output program is now non-probabilistic.

To illustrate our approach, consider the following simple program c:

$$s; \; x \leftarrow \mathsf{Lap}_\epsilon(e); \; \text{return } x$$

where s is a deterministic computation and Lap is the Laplace mechanism—a probabilistic operator that achieves differential privacy by adding noise to its input. The synchronized product $T(c)$ of the program c is

$$T(s); \; x_1 \leftarrow \mathsf{Lap}_\epsilon(e_1); \; x_2 \leftarrow \mathsf{Lap}_\epsilon(e_2); \; \text{return } (x_1, x_2)$$

where $T(s)$ is the synchronized product of s. Then, we make the program non-probabilistic by replacing the two calls to the Laplace mechanism with a call to an abstract procedure $\mathsf{Lap}^\diamond$, giving the following transformed program $T(c)$.

$$T(s); (x_1, x_2) \leftarrow \mathsf{Lap}^\diamond(e_1, e_2); \text{return } (x_1, x_2)$$

Roughly, the specification of the procedure invocation $\mathsf{Lap}^\diamond$ states that the same value is assigned to x_1 and x_2. Also, as side effect, the variable v_ϵ is updated to increment the privacy cost, which depends on the distance between the inputs (e_1, e_2) to the Laplace mechanism.

Our main result (Theorem 5 in §III) states that once we perform this transformation, we can use plain Hoare logic to complete the verification. More concretely, for the example above, we represent the relation on memories Φ as a predicate $\hat{\Phi}$ on pairs of memories, and prove that c is $(\epsilon, 0)$-differentially private if the following Hoare specification is valid.

$$\vdash T(c) : \hat{\Phi} \wedge v_\epsilon = 0 \implies x_1 = x_2 \wedge v_\epsilon \leq \epsilon_0$$

In the remainder of this article we use the same representation for a relation and its representation as a predicate on memories.

Contributions: The main contribution of the paper (§III) is a program transformation that operates on programs built from sequential, non-probabilistic constructs and differentially private, probabilistic primitives—such as the Laplace and Exponential mechanisms. The transformed program is non-probabilistic, and differential privacy of the original program can be reduced to a safety property of the transformed program. Then we show in §IV that our approach subsumes the core apRHL logic of Barthe et al. [11], in the sense that every algorithm provable with core apRHL is also provable with our approach.

We illustrate the expressiveness of our approach in §V by verifying differential privacy of several probabilistic algorithms, including a recent algorithm that produces synthetic datasets using a combination of the multiplicative weights update rule and the exponential mechanism [29], [28], and the Propose-Test-Release (PTR) framework [21], [36], which achieves approximate differential privacy without relying on output perturbation. Finally, we discuss the example of vertex cover, which is provable apRHL, but cannot be handled directly by our approach.

II. A PRIMER ON DIFFERENTIAL PRIVACY

Let us begin by recalling the basic definitions of differential privacy.

Definition 1: Let $\epsilon, \delta \geq 0$, and let $\Phi \subseteq \mathcal{S} \times \mathcal{S}$ be a relation on $\mathcal{S}$. A randomized algorithm K taking inputs in $\mathcal{S}$ and returning outputs in $\mathcal{R}$ is (ϵ, δ)-*differentially private with respect to* Φ if for every two inputs $s_1, s_2 \in \mathcal{S}$ such that $s_1 \Phi s_2$ and every subset of outputs $A \subseteq \mathcal{R}$,

$$\Pr[K(s_1) : A] \leq \exp(\epsilon) \Pr[K(s_2) : A] + \delta.$$

When $\delta = 0$, we will call this ϵ-*differential privacy*.

Our definition is a variant of the original definition of differential privacy, [22] where input memories are considered to be databases and Φ relates databases that differ in a single individual's data; let us briefly explain the intuition of differential privacy in this setting. Recall that differential privacy aims to conceal the participation of individuals in a study. To distinguish between the participation or non-participation of an individual, we think of two databases D and D' are *adjacent* or *neighboring* if they differ only in the presence or absence of a single record; note that the adjacency relation is necessarily symmetric.

Differential privacy then states that the two distributions output by K on a pair of adjacent databases are close. In the simple case where $\delta = 0$, the definition above requires that the probability of any output changes by at most a $\exp(\epsilon)$ factor when moving from one input to an adjacent input. When $\delta > 0$ these bounds are still valid except with probability δ. In other words, ϵ controls the strength of the privacy bound, and δ is the probability of failure in ensuring the privacy bound.

Building private programs: Let F be a deterministic computation with inputs in $\mathcal{T}$ and outputs in $\mathcal{R}$. Suppose that we want to make the computation of F (ϵ, δ)-differentially private with respect to some relation Φ. A natural way to achieve this goal is to add random noise to the evaluation of F on an input. In general, the noise that we need to add depends not only on the ϵ and δ parameters (which control the strength of the privacy guarantee), but also on the *sensitivity* of F, a quantity that is closely related to Lipschitz continuity for functions.

Definition 2: Assume that F is real-valued, i.e. $\mathcal{R} = \mathbb{R}$, and let $k > 0$. We say that F is k-*sensitive with respect to* Φ if $|F(t_1) - F(t_2)| \leq k$ for all $t_1, t_2 \in \mathcal{T}$ such that $t_1 \Phi t_2$.

A typical mechanism for privately releasing a k-sensitive function is the *Laplace mechanism.*

Theorem 1 ([19]): Suppose $\epsilon > 0$. The *Laplace mechanism* is defined by

$$\mathrm{Lap}_\epsilon(t) = t + v,$$

where v is drawn from the Laplace distribution $\mathcal{L}(1/\epsilon)$, i.e. with probability density function

$$P(v) = \exp(-\epsilon|v|).$$

If F is k-sensitive with respect to Φ, then the probabilistic function that maps t to $\mathrm{Lap}_\epsilon(F(t))$ is $(k\epsilon, 0)$-differentially private with respect to Φ.

Additionally, the Laplace mechanism satisfies a simple accuracy bound.

Lemma 1: Let $\epsilon, \delta > 0$ and let $T = \log(2/\delta)/(2\epsilon)$. Then for every x, $\mathrm{Lap}_\epsilon(x) \in (x - T, x + T)$ with probability at least $1 - \delta$.

Another mechanism that is fundamental for differential privacy is the *Exponential mechanism* [31]. Let $\mathcal{T}$ be the set of inputs, typically thought of as the private information. Let $\mathcal{R}$ be a set of outputs, and consider a function $F : \mathcal{T} \times \mathcal{R} \to \mathbb{R}$, typically called the *score function*. We first extend the definition of sensitivity to this function.

Definition 3: Assume $F : \mathcal{T} \times \mathcal{R} \to \mathbb{R}$ and let $c > 0$. We say that F is k-*sensitive on $\mathcal{T}$ with respect to* Φ if $|F(t_1, r) - F(t_2, r)| \leq k$ for all $t_1, t_2 \in \mathcal{T}$ such that $t_1 \Phi t_2$ and $r \in \mathcal{R}$.

Then, the Exponential mechanism can be used to output an element of $\mathcal{R}$ that approximately maximizes the score function, if the score function is k-sensitive.

Theorem 2 ([31]): Let $\epsilon, c > 0$. Suppose that F is k-sensitive in $\mathcal{T}$ with respect to Φ. The *Exponential mechanism*[2] $\mathrm{Exp}_\epsilon(F, t)$ takes as input $t \in \mathcal{T}$, and returns $r \in \mathcal{R}$ with probability equal to

$$\frac{\exp(\epsilon F(t, r)/2)}{\sum_{r' \in \mathcal{R}} \exp(\epsilon F(t, r')/2)}.$$

This mechanism is $(k\epsilon, 0)$-differentially private with respect to Φ.

A powerful feature of differential privacy is that by composing differentially private mechanisms, we can construct new mechanisms that satisfy differential privacy. However, the privacy guarantee will degrade: more operations on a database will lead to more privacy loss. In light of this composition property, we will often think of the privacy parameters ϵ and δ of a program as *privacy budgets* that are consumed by sub-operations. This is formalized by the following composition theorem.

Theorem 3 ([30]): Let q_1 be a (ϵ_1, δ_1)-differentially private query and let q_2 be a (ϵ_2, δ_2)-differentially private query. Then, their composition $q(t) = (q_1(t), q_2(t))$ is $(\epsilon_1 + \epsilon_2, \delta_1 + \delta_2)$-differentially private.

Finally, differential privacy is closed under *post-processing*—an output of a private algorithm can be

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[2]The Exponential mechanism as first introduced by McSherry and Talwar [31] is parameterized by a prior distribution μ on $\mathcal{R}$. We consider the special case where μ is uniform; this suffices for typical applications.

arbitrarily transformed, so long as this processing does not involve the private database.

Theorem 4: Let q be (ϵ, δ)-differentially private mapping databases to some output range R, and let $f : R \rightarrow R'$ be an arbitrary function. Then, the post-processing $f \circ q$ is also (ϵ, δ)-differentially private.

III. SELF-PRODUCTS

In this section, we formalize the verification of differential-privacy using traditional Hoare logic. We start with some preliminary definitions and the pWHILE programming language, which will serve as our source language. Then, given a *probabilistic* pWHILE program c, we show how to build a *non-probabilistic* program $T(c)$ that simulates two executions of c on different inputs and tracks the privacy cost via two ghost variables v_ϵ and v_δ. We show that the verification of $T(c)$ with respect to a particular Hoare logic specification ensures differential privacy of the original program c.

A. Distributions

We define the set $\mathcal{D}(A)$ of *sub-distributions* over a set A as the set of functions $\mu : A \rightarrow [0, 1]$ with discrete $\mathsf{support}(\mu) = \{x \mid \mu\, x \neq 0\}$, such that $\sum_{x \in A} \mu\, x \leq 1$; when equality holds, μ is a true *distribution*. (We will often refer to sub-distributions as distributions when there is no confusion.) Sub-distributions can be given the structure of a complete partial order: for all $\mu_1, \mu_2 \in \mathcal{D}(A)$,

$$\mu_1 \sqsubseteq \mu_2 \overset{\text{def}}{=} \forall a \in A. \ \mu_1\, a \leq \mu_2\, a.$$

Moreover, sub-distributions can be given the structure of a monad: for any function $g : A \rightarrow \mathcal{D}(B)$ and distribution $\mu : \mathcal{D}(A)$, we define $g^\star \mu : \mathcal{D}(B)$ to be the sub-distribution

$$g^\star \mu\, (b) \overset{\text{def}}{=} \sum_{a \in A} (g\, a\, b)(\mu\, a),$$

for every $b \in B$. Given an element $a \in A$, let $\mathbb{1}_a$ be the probability distribution that assigns all mass to the value a.

We will use a normalization construction $(\cdot)^{\#}$ that takes as input a function $f : B \rightarrow \mathbb{R}^{\geq 0}$ over a discrete set B and returns $(f)^{\#} \in \mathcal{D}(B)$ such that the probability mass of $f^{\#}$ at b is given by

$$(f)^{\#}\, b \overset{\text{def}}{=} \frac{f\, b}{\sum_{b' \in B} f\, b'}.$$

Intuitively, sampling from the distribution $(f)^{\#}$ is equivalent to sampling "with probability proportional to" f.

B. pWHILE Language

pWHILE programs will serve as our source language, and are defined by the following grammar:

$$
\begin{array}{lll}
\mathcal{C} ::= & \mathsf{skip} & \\
& \mid \mathcal{C}; \mathcal{C} & \text{sequencing} \\
& \mid \mathcal{V} \leftarrow \mathcal{E} & \text{deterministic assignment} \\
& \mid \mathcal{V} \xleftarrow{\$} \mathsf{Lap}_\epsilon(\mathcal{E}) & \text{Laplace assignment} \\
& \mid \mathcal{V} \xleftarrow{\$} \mathsf{Exp}_\epsilon(\mathcal{E}, \mathcal{E}) & \text{Exponential assignment} \\
& \mid \mathsf{if}\ \mathcal{E}\ \mathsf{then}\ \mathcal{C}\ \mathsf{else}\ \mathcal{C} & \text{conditional} \\
& \mid \mathsf{while}\ \mathcal{E}\ \mathsf{do}\ \mathcal{C} & \text{while loop} \\
& \mid \mathsf{return}\ \mathcal{E} & \text{return expression}
\end{array}
$$

Here, $\mathcal{V}$ is a set of *variables* and $\mathcal{E}$ is a set of *expressions*. We consider expressions including simply typed lambda terms and basic operations on booleans, lists and integers. (pWHILE is equipped with a standard type system; we omit the typing rules.)

The probabilistic assignments involving $\mathsf{Lap}_\epsilon(\mathcal{E})$ and $\mathsf{Exp}_\epsilon(\mathcal{E}, \mathcal{E})$ internalize the (discrete version of the) mechanisms of Theorem 1 and Theorem 2 respectively. Note that for examples based on the exponential mechanism we allow function types for representing the score functions; alternatively these score functions can be modeled as finite maps if their domain is finite (as will be the case in our examples).

The semantics of a well-typed pWHILE program is defined by its (probabilistic) action on *memories*; we denote the set of memories by $\mathcal{M}$. A program memory $m \in \mathcal{M}$ is a partial assignment of values to variables. Formally, the semantics of a return-free pWHILE program c is a function $[\![c]\!] : \mathcal{M} \rightarrow \mathcal{D}(\mathcal{M})$ mapping a memory $m \in \mathcal{M}$ to a distribution $[\![c]\!]\, m \in \mathcal{D}(\mathcal{M})$, as defined in Fig. 1.

For simplicity, we only consider programs of the form $c; \mathsf{return}\ e$ in the rest of this paper. Then, the semantics of a program $c; \mathsf{return}\ e$ is simply defined as

$$[\![c; \ \mathsf{return}\ e]\!]\, m \overset{\text{def}}{=} \lambda u.\, \mathbb{1}^\star_{[\![e]\!]u}\, ([\![c]\!]\, m).$$

C. Target Language

To define the target language of our transformation, we remove probabilistic assignments and add an assert instruction, giving the following grammar:

$$
\begin{array}{lll}
\mathcal{C} ::= & \mathsf{skip} & \\
& \mid \mathcal{C}; \mathcal{C} & \text{sequencing} \\
& \mid \mathcal{V} \leftarrow \mathcal{E} & \text{deterministic assignment} \\
& \mid \mathsf{assert}\,(\varphi) & \text{assert} \\
& \mid (\mathcal{V}, \mathcal{V}) \leftarrow \mathsf{Lap}^\diamond_\epsilon(\mathcal{E}, \mathcal{E}) & \text{Laplace invocation} \\
& \mid (\mathcal{V}, \mathcal{V}) \leftarrow \mathsf{Exp}^\diamond_\epsilon(\mathcal{E}, \mathcal{E}, \mathcal{E}, \mathcal{E}) & \text{Exponential invocation} \\
& \mid \mathsf{if}\ \mathcal{E}\ \mathsf{then}\ \mathcal{C}\ \mathsf{else}\ \mathcal{C} & \text{conditional} \\
& \mid \mathsf{while}\ \mathcal{E}\ \mathsf{do}\ \mathcal{C} & \text{while loop} \\
& \mid \mathsf{return}\ \mathcal{E} & \text{return expression}
\end{array}
$$

The semantics of this non-deterministic target language is defined in Figure 2 as a function from a memory to a set of memories. The assert (φ) statement checks at runtime whether the predicate φ is valid, and stops the execution if not. In order to distinguish the failure of assert statements from non-terminating while loops, we lift the domain $\mathcal{P}(\mathcal{M})$ with a $\bot$ element: where $\bigcup_{m \in \bot} f\, m$ is defined as $\bot$ for any f. We defer the presentation of the abstract procedures $\mathsf{Lap}^\diamond$ and $\mathsf{Exp}^\diamond$ until the definition of the self-product construction in §III-E.

The enforcement of safety properties over this target language is formalized by a standard Hoare logic, with judgments of the form

$$\vdash c : \Psi \Longrightarrow \Phi.$$

Here the pre- and post-conditions Ψ and Φ are standard *unary* predicates over memories. Hoare logic judgments can be derived using the rules in Fig. 3; by the standard soundness of Hoare logic, the derivability of a judgment $\vdash c : \Psi \Longrightarrow \Phi$

$$\llbracket \mathsf{skip} \rrbracket\, m = \mathbb{1}_m$$

$$\llbracket c_1;\, c_2 \rrbracket\, m = \llbracket c_2 \rrbracket^\star \left(\llbracket c_1 \rrbracket\, m \right)$$

$$\llbracket x \leftarrow e \rrbracket\, m = \mathbb{1}_{m\{\llbracket e \rrbracket_{\mathcal{E}}\, m/x\}}$$

$$\llbracket x \xleftarrow{\$} \mathsf{Lap}_\epsilon(e) \rrbracket\, m = \left(\lambda v.\ \mathbb{1}_{m\{v/x\}} \right)^\star \left(\lambda r.\mathsf{exp}\left(-\frac{\epsilon |r - \llbracket e \rrbracket\, m|}{2} \right) \right)^{\#}$$

$$\llbracket x \xleftarrow{\$} \mathsf{Exp}_\epsilon(s, e) \rrbracket\, m = \left(\lambda v.\ \mathbb{1}_{m\{v/x\}} \right)^\star \left(\lambda r.\mathsf{exp}\left(\frac{\epsilon \llbracket s \rrbracket m(\llbracket e \rrbracket m, r)}{2} \right) \right)^{\#}$$

$$\llbracket \mathsf{if}\ e\ \mathsf{then}\ c_1\ \mathsf{else}\ c_2 \rrbracket\, m = \mathbf{if}\ (\llbracket e \rrbracket_{\mathcal{E}}\, m = \mathsf{true})\ \mathbf{then}\ (\llbracket c_1 \rrbracket\, m)\ \mathbf{else}\ (\llbracket c_2 \rrbracket\, m)$$

$$\llbracket \mathsf{while}\ e\ \mathsf{do}\ c \rrbracket\, m = \bigsqcup w_i\, m$$
$$\mathrm{where} \quad w_0\, m = \bot$$
$$w_{i+1}\, m = \mathbf{if}\ (\llbracket e \rrbracket_{\mathcal{E}}\, m = \mathsf{true})\ \mathbf{then}\ w_i^\star (\llbracket c \rrbracket\, m)\ \mathbf{else}\ \mathsf{unit}\, m$$

Fig. 1: pWHILE semantics

$$\llbracket \mathsf{skip} \rrbracket\, m = \{m\}$$

$$\llbracket c_1;\, c_2 \rrbracket\, m = \bigcup_{m' \in \llbracket c_1 \rrbracket\, m} \llbracket c_2 \rrbracket\, m$$

$$\llbracket x \leftarrow e \rrbracket\, m = \mathbb{1}_{m\{\llbracket e \rrbracket_{\mathcal{E}}\, m/x\}}$$

$$\llbracket \mathsf{assert}\, (\varphi) \rrbracket\, m = \mathbf{if}\ (\llbracket \varphi \rrbracket_{\mathcal{E}}\, m)\ \mathbf{then}\ \{m\}\ \mathbf{else}\ \bot$$

$$\llbracket (x_1, x_2) \leftarrow \mathsf{Lap}_\epsilon^\diamond(e_1, e_2) \rrbracket\, m = \bigcup_v m \{v/x_1\} \{v/x_2\} \{\mathsf{v}_\epsilon + |e_1 - e_2|\epsilon/\mathsf{v}_\epsilon\}$$

$$\llbracket (x_1, x_2) \leftarrow \mathsf{Exp}_\epsilon^\diamond(s_1, e_1, s_2, e_2) \rrbracket\, m = \mathbf{if}\ (\llbracket s_1 = s_2 \rrbracket_{\mathcal{E}}\, m = \mathsf{true})\ \mathbf{then}$$
$$\bigcup_v m \{v/x_1\} \{v/x_2\} \{\mathsf{v}_\epsilon + |e_1 - e_2|\epsilon/\mathsf{v}_\epsilon\}$$
$$\mathbf{else}\ \bot$$

$$\llbracket \mathsf{if}\ e\ \mathsf{then}\ c_1\ \mathsf{else}\ c_2 \rrbracket\, m = \mathbf{if}\ (\llbracket e \rrbracket_{\mathcal{E}}\, m = \mathsf{true})\ \mathbf{then}\ (\llbracket c_1 \rrbracket\, m)\ \mathbf{else}\ (\llbracket c_2 \rrbracket\, m)$$

$$\llbracket \mathsf{while}\ e\ \mathsf{do}\ c \rrbracket\, m = \bigsqcup \hat{w}_i\, m$$
$$\mathrm{where} \quad \hat{w}_0\, m = \emptyset$$
$$\hat{w}_{i+1}\, m = \mathbf{if}\ (\llbracket e \rrbracket_{\mathcal{E}}\, m = \mathsf{true})\ \mathbf{then}\ \bigcup_{m' \in (\llbracket c \rrbracket\, m)} \hat{w}_i\, m'\ \mathbf{else}\ \{m\}$$

Fig. 2: Semantics of the target language

entails the correctness of c with respect to its specification Ψ, Φ.

D. Product Construction

Before we define the product transformation from pWHILE to our target language, let us first review some preliminaries about product programs.

Product programs have been successfully used to verify 2-safety properties like information-flow, program equivalence, and program robustness. As mentioned in in the introduction, a *synchronized* product program can be used to simulate two runs of the same program, interleaving the two executions and often simplifying the verification effort. This technique, however, has been mostly used in the verification of non-probabilistic programs. In the rest of this section we provide a brief introduction to relational verification by product construction and then extend the approach to handle quantitative reasoning over probabilistic programs.

A simple but necessary concept for the product construction is *memory separability*: we say that two programs are *separable* if they manipulate disjoint sets of program variables. In order to achieve separability in the construction of the product of a program with itself, program variables are duplicated and marked with a left ($-_1$) or right ($-_2$) tag. For any program expression e or predicate φ, we let e_i and φ_i stand for the result of renaming every program variable with the tag $-_i$.

Similarly, we say that two memories are *disjoint* when their domains (the sets of variables on which they are defined) are disjoint. Given two disjoint memories m_1 and m_2, we can build a memory $m = m_1 \oplus m_2$ representing their (disjoint) union. In the following, we exploit separability and use predicates to represent *binary relations* over disjoint memories m_1 and m_2. We will suggestively write $m_1 \Phi m_2$ to denote the unary predicate $\Phi(m_1 \oplus m_2)$ over the combined memory $m_1 \oplus m_2$.

Given two deterministic programs c_1 and c_2, a general *product program* $c_1 \times c_2$ is a syntactic construction that merges

$$\overline{\vdash \mathsf{skip} : \Psi \Longrightarrow \Psi} \qquad \overline{\vdash x \leftarrow e : \Phi\,\{e/x\} \Longrightarrow \Phi} \qquad \overline{\vdash \mathsf{assert}\,(\varphi) : \Phi \wedge \varphi \Longrightarrow \Phi}$$

$$\frac{\vdash c_1 : \Psi \Longrightarrow \varphi \qquad \vdash c_2 : \varphi \Longrightarrow \Phi}{\vdash c_1;\, c_2 : \Psi \Longrightarrow \Phi} \qquad \frac{\vdash c_1 : \Psi \wedge b \Longrightarrow \Phi \qquad \vdash c_2 : \Psi \wedge \neg b \Longrightarrow \Phi}{\vdash \mathsf{if}\ b\ \mathsf{then}\ c_1\ \mathsf{else}\ c_2 : \Psi \Longrightarrow \Phi}$$

$$\frac{\Psi \wedge v \leq 0 \Rightarrow \neg b \qquad \vdash c : \Psi \wedge b \wedge v = k \Longrightarrow \Psi \wedge v < k}{\vdash \mathsf{while}\ b\ \mathsf{do}\ c : \Psi \Longrightarrow \Psi \wedge \neg b} \qquad \frac{\vdash c : \Psi' \Longrightarrow \Phi' \qquad \Psi \Rightarrow \Psi' \qquad \Phi' \Rightarrow \Phi}{\vdash c : \Psi \Longrightarrow \Phi}$$

Fig. 3: Hoare logic for non-probabilistic programs

the executions of c_1 and c_2; this construction is required to correctly represent every pair of executions of c_1 and c_2. Traditional program verification techniques can then be used to enforce a relational property over c_1 and c_2.

In self-composition [7], [18], the product construction $c_1 \times c_2$ is defined simply by the sequential composition $c_1;\, c_2$. An inconvenience of self-composition is that the verification of $c_1;\, c_2$ usually requires independent functional reasoning over c_1 and c_2. The synchronized product construction solves this problem by *interleaving* execution of two runs of the same program—by placing corresponding pieces of the two executions of a program close together, synchronized product programs can more easily maintain inductive invariants relating the two runs. Not only does synchronization reduce the verification effort, we will soon see that synchronization is the key feature that enables our verification approach.

E. Building the Product

We embed the quantitative reasoning on probabilistic programs by introducing the special program variables v_ϵ and v_δ, which serve to accumulate the privacy cost. For every statement c, the self-product $\lceil c \rceil$ is formally defined by the rules shown in Fig. 5. In a nutshell, the deterministic fragment of the code is duplicated with appropriate variable renaming with the flags $-_1$ and $-_2$, and the control flow is *fully synchronized*, i.e., the two executions of the same program must take all the same branches—we use the assert statements to enforce this property.

Moreover, for the self-product of a program c to correctly represent two executions of itself, we require that loop guards do not depend on probabilistically sampled values; we assume in the remainder of this work that the programs under verification satisfy this condition. Additionally, the soundness of the method relies on the fact that all verified programs are terminating, which is enforced by the Hoare logic rules in Fig. 3.

The probabilistic constructions are mapped to invocations to the abstract procedures $\mathsf{Lap}^\diamond$ and $\mathsf{Exp}^\diamond$. The semantics of these procedures is non-deterministic, in order to simulate sampling from a probability distribution. We axiomatize these abstract procedures with Hoare specifications: Figure 4 gives the new specifications. Notice that both abstract procedures have a side effect: they increment the privacy budget variable v_ϵ. In Section V-C, we introduce a alternative specification for $\mathsf{Lap}^\diamond$ that also increments the budget variable v_δ.

$$
\begin{aligned}
\lceil \mathsf{skip} \rceil &= \mathsf{skip} \\
\lceil c_1;\, c_2 \rceil &= \lceil c_1 \rceil;\ \lceil c_2 \rceil \\
\lceil x \leftarrow e \rceil &= x_1 \leftarrow e_1;\ x_2 \leftarrow e_2 \\
\lceil x \xleftarrow{\$} \mathsf{Lap}_\epsilon(e) \rceil &= (x_1, x_2) \leftarrow \mathsf{Lap}^\diamond(e_1, e_2) \\
\lceil x \xleftarrow{\$} \mathsf{Exp}_\epsilon(s, e) \rceil &= (x_1, x_2) \leftarrow \mathsf{Exp}^\diamond(s_1, e_1, s_2, e_2) \\
\lceil \mathsf{if}\ b\ \mathsf{then}\ c\ \mathsf{else}\ d \rceil &= \mathsf{assert}\,(b_1 = b_2); \\
& \quad\ \mathsf{if}\ b_1\ \mathsf{then}\ \lceil c \rceil\ \mathsf{else}\ \lceil d \rceil \\
\lceil \mathsf{while}\ b\ \mathsf{do}\ c \rceil &= \mathsf{assert}\,(b_1 = b_2); \\
& \quad\ \mathsf{while}\ b_1\ \mathsf{do} \\
& \qquad\ \lceil c \rceil;\ \mathsf{assert}\,(b_1 = b_2)
\end{aligned}
$$

Fig. 5: Self-product construction

F. An alternative characterization of privacy

For the proof of soundness, we will use an alternative characterization of (ϵ, δ)-differential privacy based on the notion of ϵ-distance. This notion is adapted from the asymmetric notion of distance used by Barthe et al. [11].

Definition 4 (ϵ-distance): The ϵ-distance Δ_ϵ is defined as

$$\Delta_\epsilon(\mu_1, \mu_2) \overset{\text{def}}{=} \max_{S \subseteq A} (\mu_1\, S - \exp(\epsilon)\, \mu_2\, S),$$

where $\mu\, S \overset{\text{def}}{=} \sum_{a \in S} \mu\, a$. We define max over an empty set to be 0, so $\Delta_\epsilon(\mu_1, \mu_2) \geq 0$.

By the definition of ϵ-distance, a probabilistic program c is (ϵ, δ)-differentially private with respect to $\epsilon > 0$, $\delta \geq 0$, and a relation Φ on the initial memories of c if for every two memories m_1 and m_2 related by Φ, we have

$$\Delta_\epsilon(\llbracket c \rrbracket\ m_1, \llbracket c \rrbracket\ m_2) \leq \delta.$$

The proof of our main theorem relies on a *lifting* operator that turns a relation on memories into a relation on *distributions* over memory. Given a relation on memories Φ, and real values ϵ, δ we define the lifted relation on memory distributions $\Phi_{\langle \epsilon, \delta \rangle}$ as follows.

Definition 5: For all memory distributions μ_1, μ_2, $\mu_1\,\Phi_{\langle \epsilon, \delta \rangle}\,\mu_2$ if there exists μ such that:

1) $\pi_i\,\mu \leq \mu_i$,
2) $\forall m_1, m_2.\ \mu(m_1 \oplus m_2) \Rightarrow m_1\,\Phi\,m_2$, and
3) $\Delta_\epsilon(\mu_i, \pi_i\,\mu) \leq \delta$,

where

- $(\pi_1\,\mu)\, m_1 = \sum_{m_2 \in \mathcal{M}} \mu\,(m_1, m_2)$, and
- $(\pi_2\,\mu)\, m_2 = \sum_{m_1 \in \mathcal{M}} \mu\,(m_1, m_2)$.

$$\overline{\vdash (x_1, x_2) \leftarrow \mathsf{Lap}_\epsilon^\diamond(e_1, e_2) : \mathsf{v}_\epsilon = \epsilon_0 \land \mathsf{v}_\delta = \delta_0 \implies x_1 = x_2 \land \mathsf{v}_\epsilon = \epsilon_0 + |e_1 - e_2|\epsilon \land \mathsf{v}_\delta = \delta_0}$$

$$\overline{\vdash (x_1, x_2) \leftarrow \mathsf{Exp}_\epsilon^\diamond(s_1, e_1, s_2, e_2) : s_1 = s_2 \land \mathsf{v}_\epsilon = \epsilon_0 \land \mathsf{v}_\delta = \delta_0 \implies x_1 = x_2 \land \mathsf{v}_\epsilon = \epsilon_0 + \epsilon \max_r |s_1(x_1, r) - s_2(x_2, r)|}$$

Fig. 4: Hoare specification for $\mathsf{Lap}^\diamond$ and $\mathsf{Exp}^\diamond$

Notice that ϵ-distance between distributions is closely related to the lifting of the equality relation, i.e.,

$$\mu_1 =_{\langle \epsilon, \delta \rangle} \mu_2 \iff \Delta_\epsilon(\mu_1, \mu_2) \le \delta. \tag{1}$$

Note that the second equation is precisely the condition on output distributions needed for (ϵ, δ)-differential privacy.

G. Soundness of the self-product technique

We can now state the soundness theorem for our approach. Recall that we consider only programs with a single return statement; we will label this returned value out_1 and out_2 in the first and second runs, respectively.

Theorem 5: If the following Hoare judgment is valid

$$\vdash \lceil c \rceil : \Psi \land \mathsf{v}_\epsilon = 0 \land \mathsf{v}_\delta = 0 \implies \mathsf{out}_1 = \mathsf{out}_2 \land \mathsf{v}_\epsilon \le \epsilon \land \mathsf{v}_\delta \le \delta$$

then c satisfies (ϵ, δ)-differential privacy.

The proof of Theorem 5 follows from the next lemma.

Lemma 2: Let Φ be a relation on memories, and suppose

$$\vdash \lceil c \rceil : \Psi \land \mathsf{v}_\epsilon = 0 \land \mathsf{v}_\delta = 0 \implies \Phi \land \mathsf{v}_\epsilon \le \epsilon \land \mathsf{v}_\delta \le \delta.$$

Then, for all memories m_1, m_2 such that $m_1 \Psi m_2$, we have

$$(\llbracket c \rrbracket m_1) \Phi_{\langle \epsilon, \delta \rangle} (\llbracket c \rrbracket m_2).$$

The lemma is proved by structural induction on c; we provide technical details in the full version of the paper.

IV. COMPARISON WITH apRHL

Now that we have defined our transformation, we compare our approach to a custom logic for verifying privacy. apRHL [11] is a quantitative, probabilistic and relational program logic for reasoning about differential privacy, with judgments of the form[3]

$$\vdash c_1 \sim_{\langle \alpha, \delta \rangle} c_2 : \Psi \implies \Phi,$$

where c_1 and c_2 are probabilistic programs, Ψ and Φ are memory relations, and ϵ, δ are real values. The main result of apRHL states that if $\vdash c_1 \sim_{\langle \epsilon, \delta \rangle} c_2 : \Psi \implies \mathsf{out}_1 = \mathsf{out}_2$ is derivable, where c_1 and c_2 are the result of renaming variables in c to make them separable, then c is (ϵ, δ)-differentially private with respect to the relation Ψ on initial memories.

The original presentation of the apRHL logic [11] is organized in three sets of rules: the first set includes a set of core rules, the second set includes a generalized rule for loops (see Fig. 7), and the third set includes rules for mechanisms

such as the Laplace and Exponential Mechanism. We refer to the fragment consisting of the first and third set of rules as *core* apRHL; its rules are displayed in Fig. 6. Note that the, in contrast with [11], the rule for sequential composition does not have any side condition; this is due to the fact that the rule for random assignments in [11] allows sampling from strict sub-distributions, whereas we only allow sampling using the Laplace and Exponential mechanisms.

The following lemma shows that our approach subsumes core apRHL, in the sense that every probabilistic program c verified (ϵ, δ)-differentially private using core apRHL can be verified using our self-product technique.

Lemma 3: For every probabilistic program c, memory relations Ψ, Φ and real expressions ϵ, δ such that the following core apRHL judgment is derivable

$$\vdash c \sim_{\langle \epsilon, \delta \rangle} c : \Psi \implies \Phi$$

we have

$$\vdash \lceil c \rceil : \Psi \implies \Phi \land \mathsf{v}_\epsilon \le \epsilon \land \mathsf{v}_\delta \le \delta.$$

The proof of this result is straightforward, by induction on the derivation of the apRHL judgement.

The embedding is more expressive than core apRHL in its treatment of loops. This is because privacy consumption in core apRHL is tracked by an accumulator which is part of the judgment itself, independent of the pre-condition and the initial memory. Using self-products, reasoning about the privacy budget is carried out in the Hoare specification and consequently inherits the full expressivity of the Hoare logic. On the other hand, we have not been able to capture the generalized rule for loops from apRHL, which is given in Fig. 7, with self-products. In the following section, we provide a more detailed comparison with apRHL based on examples.

We conclude with a broader perspective on the two formalisms. The primary goal of our approach is to strike a good balance between expressivity and simplicity, including for the latter ease of use and ease of implementation. In contrast to apRHL, which requires a relational verification infrastructure, our approach reuses a very standard verification technology, namely Hoare logic, and can be directly implemented by defining the appropriate program transformation, and using off-the-shelf tools for Hoare logic or even invariant generation. We believe this latter approach is simpler to deploy for programming languages for which verification environments based on Hoare logic are already available.

---

[3] The original apRHL rules are based on a multiplicative privacy budget. We adapt the rules to an additive privacy parameter for consistency with the rest of the article.

$$\frac{}{\vdash x_1 \leftarrow e_1 \sim_{\langle 0,0 \rangle} x_2 \leftarrow e_2 : \Phi\{e_1/x_1\}\{e_2/x_2\} \Longrightarrow \Phi} [\mathsf{assn}]$$

$$\frac{}{\vdash y_1 \xleftarrow{\$} \mathsf{Lap}_\epsilon(e_1) \sim_{\langle |e_1-e_2|\epsilon,0 \rangle} y_2 \xleftarrow{\$} \mathsf{Lap}_\epsilon(e_2) : \mathsf{true} \Longrightarrow y_1 = y_2} [\mathsf{lap}]$$

$$\frac{}{\vdash y_1 \xleftarrow{\$} \mathsf{Exp}_\epsilon(s_1,e_1) \sim_{\langle \epsilon \max_r |s_1(x_1,r)-s_2(x_2,r)|,0 \rangle} y_2 \xleftarrow{\$} \mathsf{Exp}_{\epsilon,s}(s_2,e_2) : s_1 = s_2 \Longrightarrow y_1 = y_2} [\mathsf{exp}]$$

$$\frac{}{\vdash \mathsf{skip} \sim_{\langle 0,0 \rangle} \mathsf{skip} : \Psi \Longrightarrow \Psi} [\mathsf{skip}]$$

$$\frac{\vdash c_1 \sim_{\langle \epsilon,\delta \rangle} c_2 : \Psi \wedge b_1 \Longrightarrow \Phi \quad \vdash d_1 \sim_{\langle \epsilon,\delta \rangle} d_2 : \Psi \wedge \neg b_1 \Longrightarrow \Phi}{\vdash \mathsf{if}\ b_1\ \mathsf{then}\ c_1\ \mathsf{else}\ d_1 \sim_{\langle \epsilon,\delta \rangle} \mathsf{if}\ b_2\ \mathsf{then}\ c_2\ \mathsf{else}\ d_2 : \Psi \wedge b_1 = b_2 \Longrightarrow \Phi} [\mathsf{cond}]$$

$$\frac{\vdash c_1 \sim_{\langle \epsilon,\delta \rangle} c_2 : \Theta \wedge b_1 \wedge k = e \Longrightarrow \Theta \wedge k < e \quad \Theta \wedge n \le e \Longrightarrow \neg b_1 \quad \Theta \Longrightarrow b_1 = b_2}{\vdash \mathsf{while}\ b_1\ \mathsf{do}\ c_1 \sim_{\langle n\epsilon,n\delta \rangle} \mathsf{while}\ b_2\ \mathsf{do}\ c_2 : \Theta \wedge 0 \le e \Longrightarrow \Theta \wedge \neg b_1} [\mathsf{while}]$$

$$\frac{\vdash c_1 \sim_{\langle \epsilon,\delta \rangle} c_2 : \Psi \Longrightarrow \Phi' \quad \vdash c_1' \sim_{\langle \epsilon',\delta' \rangle} c_2' : \Phi' \Longrightarrow \Phi}{\vdash c_1; c_1' \sim_{\langle \epsilon+\epsilon',\delta+\delta' \rangle} c_2; c_2' : \Psi \Longrightarrow \Phi} [\mathsf{seq}]$$

$$\frac{\vdash c_1 \sim_{\langle \epsilon',\delta' \rangle} c_2 : \Psi' \Longrightarrow \Phi' \quad \Psi \Rightarrow \Psi' \quad \Phi' \Rightarrow \Phi \quad \epsilon' \le \epsilon \quad \delta' \le \delta}{\vdash c_1 \sim_{\langle \epsilon,\delta \rangle} c_2 : \Psi \Longrightarrow \Phi} [\mathsf{weak}]$$

Fig. 6: Core proof rules of the approximate relational Hoare logic

$$\frac{\begin{array}{c} \Theta \implies b_1\langle 1 \rangle \equiv b_2\langle 2 \rangle \wedge P\langle 1 \rangle \equiv P\langle 2 \rangle \wedge i\langle 1 \rangle = i\langle 2 \rangle \quad \Theta \wedge n \le i\langle 1 \rangle \implies \neg b_1\langle 1 \rangle \\ \vdash c_1; \mathsf{assert}\,(\neg P) \sim_{\langle \epsilon_j,0 \rangle} c_2; \mathsf{assert}\,(\neg P) : \Theta \wedge (b_1 \wedge i = j \wedge \neg P)\langle 1 \rangle \implies \Theta \wedge i\langle 1 \rangle = j+1 \\ \vdash c_1; \mathsf{assert}\,(P) \sim_{\langle \epsilon,0 \rangle} c_2; \mathsf{assert}\,(P) : \Theta \wedge (b_1 \wedge i = j \wedge \neg P)\langle 1 \rangle \implies \Theta \wedge i\langle 1 \rangle = j+1 \\ \vdash c_1 \sim_{\langle 0,0 \rangle} c_2 : \Theta \wedge (b_1 \wedge i = j \wedge P)\langle 1 \rangle \implies \Theta \wedge (i = j+1 \wedge P)\langle 1 \rangle \end{array}}{\vdash \mathsf{while}\ b_1\ \mathsf{do}\ c_1 \sim_{\langle \epsilon + \sum_{i=0}^{n-1} \epsilon_i, 0 \rangle} \mathsf{while}\ b_2\ \mathsf{do}\ c_2 : \Theta \wedge i\langle 1 \rangle = 0 \implies \Theta \wedge \neg b_1\langle 1 \rangle} [\mathsf{gwhile}]$$

Fig. 7: Generalized rule for loops

V. EXAMPLES

In this section, we apply our method to four examples. The first example (smart sum) is an algorithm for computing statistics; it involves intricate applications of the composition theorem, and is thus an interesting test case. The second example (Iterative Database Construction, or more precisely the Multiplicative Weights Exponential Mechanism) is an algorithm that computes a synthetic database; it combines the Laplace and the Exponential mechanisms, and has not been verified in earlier work using relational logic. The third example (Propose-Test-Release) is an algorithm that only achieves approximate differential privacy (i.e., (ϵ, δ)-differential privacy with $\delta > 0$) using both the privacy and accuracy properties of the Laplace distribution. To best of our knowledge, we provide the first machine-checked proof of this mechanism. Finally, our last example (vertex cover) is an algorithm that achieves differential privacy by carefully adding noise to sampled values; this example can only be verified partially using our method, and illustrates the differences with apRHL.

A. Smart sum

In this example, a database db is a list of real numbers $[r_1, \ldots, r_T]$ and we consider two databases *adjacent* if they are the same length T, at most one entry differs between the two databases, and that entry differs by at most 1.

Suppose we want to release private sums of the first i entries, simultaneously for every $i \in [1 \ldots T]$: that is, given $[r_1, r_2, r_3, r_4, \ldots, r_T]$ we want to privately release

$$\left[r_1, \sum_{i=1}^{2} r_i, \sum_{i=1}^{3} r_i, \sum_{i=1}^{4} r_i, \ldots, \sum_{i=1}^{T} r_i \right].$$

An interesting sophisticated differentially private algorithm for this problem is the *two-level counter* from Chan, et al. [14]; we call this algorithm smartsum.

At a high level, this algorithm groups the input list into blocks of length q, and adds Laplace noise to the sum for each block. More concretely, to compute a running sum from 1 to t with t a multiple of q, we simply add together the first t/q block sums. If t is not a multiple of q, say $t = qs + r$

with $r < q$, we take the first s block sums and add a noised version of each of the r remaining elements.

For an example, suppose we take $q = 3$ and T is a multiple of 3. For brevity, let us use the notation $\mathsf{L}(r)$ to describe the result of the application of Laplace, for a fixed value ϵ to r. Then, the output of smartsum is

$$
\left[\mathsf{L}(r_1), \mathsf{L}(r_1) + \mathsf{L}(r_2), \mathsf{L}\left(\sum_{i=1}^{3} r_i\right), \right.
$$
$$
\left. \mathsf{L}\left(\sum_{i=1}^{3} r_i\right) + \mathsf{L}(r_4), \dots, \sum_{j=0}^{T/3} \mathsf{L}\left(\sum_{i=1}^{3} r_{3j+i}\right) \right].
$$

To informally argue privacy, observe that if we run the Laplace mechanism on each individual entry, there is no privacy cost for the indices where the adjacent databases are the same. So, the privacy analysis for smartsum is straightforward: changing an input element will change exactly two noisy sums—the sum for the block containing i, and the noisy version of i—and each noisy sum that can change requires ϵ privacy budget, since we are using the Laplace mechanism with parameter ϵ. Thus, smartsum is 2ϵ-private.

The full program, together with the transformation into a synchronized product program, is presented in Fig. 8. The formal verification of the 2ϵ-differential privacy follows the argument above. The pre-condition states that the two input databases are adjacent, while the post-condition requires equality on the outputs and bounds the accumulated privacy budget by 2ϵ.

The interesting part for our verification is the while loop. Indeed, this requires a loop invariant to keep track of the privacy budget, which depends on whether the differing entry has been processed or not. Note that this invariant does not fit the core apRHL while rule of Fig. 6: to deal with this example, Barthe et al. [11] use the generalized while rule from Fig. 7. This rule is able to perform a refined analysis depending on a predicate that is preserved across the first iterations, until some critical iteration is reached. In contrast, here we do not require any special verification rule: the standard while rule from Hoare logic suffices.

More precisely, we apply the Hoare while rule with the invariant:

$\mathsf{adjacent}(l_1, l_2) \wedge out_1 = out_2 \wedge next_1 = next_2 \wedge n_1 = n_2 \wedge$
$|c_1 - c_2| \leq 1 \wedge (l_1 \neq l_2 \Rightarrow \mathsf{v}_\epsilon = 0) \wedge$
$(c_1 \neq c_2 \Rightarrow l_1 = l_2 \wedge \mathsf{v}_\epsilon \leq \epsilon) \wedge (l_1 = l_2 \to \mathsf{v}_\epsilon \leq 2\epsilon)$

Notice from the invariant that if the accumulators c_1 and c_2 differ we have $l_1 = l_2$. This corresponds to the fact that the differing entry has been processed and so the remaining database entries coincide. Also, if this is the case then the privacy budget of 2ϵ has been already consumed.

The verification of this invariant proceeds by case analysis. We have three cases: a) the differing entry has not been processed yet and will not be processed in the following iteration, b) the differing entry has not been processed yet but is going to be processed in the next iteration, and c) the

```
next ← 0; n ← 0; c ← 0;
while 0 < length l do
    if length l mod q = 0 then
        x ← Lap ₑ(c + hd l);
        n ← x + n;
        next ← n;
        c ← 0;
        out ← next :: out;
    else
        x ← Lap ₑ(hd l);
        next ← next + x;
        c ← c + hd l;
        out ← next :: out;
    l ← tl l;
return out;
```

(a) Original probabilistic algorithm

```
vₑ ← 0; next₁ ← 0; next₂ ← 0;
n₁ ← 0; n₂ ← 0; c₁ ← 0; c₂ ← 0;
assert ((0 < length l₁) ⇔ (0 < length l₂));
while 0 < length l₁ do
    assert ((length l₁ mod q = 0) ⇔ (length l₂ mod q = 0));
    if length l₁ mod q = 0 then
        (x₁, x₂) ← Lap◇ₑ(c₁ + hd l₁, c₂ + hd l₂));
        n₁ ← x₁ + n₁; n₂ ← x₂ + n₂;
        next₁ ← n₁; next₂ ← n₂;
        c₁ ← 0; c₂ ← 0;
        out₁ ← next₁ :: out₁; out₂ ← next₂ :: out₂;
    else
        (x₁, x₂) ← Lap◇ₑ(hd l₁, hd l₂));
        next₁ ← next₁ + x₁; next₂ ← next₂ + x₂;
        c₁ ← c₁ + hd l₁; c₂ ← c₂ + hd l₂;
        out₁ ← next₁ :: out₁; out₂ ← next₂ :: out₂;
    l₁ ← tl l₁; l₂ ← tl l₂;
return (out₁, out₂);
```

(b) Synchronized non-probabilistic product

Fig. 8: smartsum algorithm

differing entry has already been processed, in which case there is no more privacy budget consumption.

B. Multiplicative Weights Exponential Mechanism

While answering queries on a database with the Laplace mechanism is a simple way to guarantee privacy, the added noise quickly renders the results useless as the number of queries grows. To handle larger collections of queries, there has been much research on sophisticated algorithms based on learning theory.

One such scheme is *Iterative Database Construction (IDC)*, due to Gupta et al. [27]. The basic idea is simple: given a database $\hat{d}$, the algorithm gradually builds a *synthetic database* that approximates the original database. The synthetic database is built over several rounds; after some fixed number of rounds,

the synthetic database is released and used to answer all queries.

The essence of the algorithm is the computation that it performs at each round. Let $\mathcal{Q}$ be a collection of queries that we want to answer and let d^i be the synthetic database computed at round i. During round $i+1$, the algorithm selects a query $q \in \mathcal{Q}$ with high error; that is, a query where the current approximate database d^i and the true database $\hat{d}$ give very different answers. This selection is done in a differentially private way. Next, the algorithm computes a noisy version v of q evaluated on the true database $\hat{d}$. Again, this step must be differentially private. Finally, q, v and the current database d^i approximation are fed into an update algorithm, which generates the next approximation d^{i+1} of the synthetic database (hopefully performing better on q).

The idea is that in many cases, this iterative procedure will provably find an approximation with low error on *all* queries in $\mathcal{Q}$ in a small number of steps. Hence, we can run IDC for a small number of steps, and release the final database approximation as the output. Queries in $\mathcal{Q}$ can then be evaluated on this output for an accurate estimate of the true answer to the query.

IDC is actually a family of algorithms parameterized by an algorithm to privately find a high-error query (called the *private distinguisher*), and the update function (called the *database update algorithm*). For concreteness, let us consider one well-studied instantiation, the *Multiplicative Weights Exponential Mechanism* (MWEM) algorithm originally due to Hardt and Rothblum [29] and experimentally evaluated by Hardt et al. [28].

MWEM uses the exponential mechanism to privately select a query with high error—the quality score of a query q to be maximized is the error of the query, i.e., the absolute difference between q evaluated on the approximate database d^i and q evaluated on the true database $\hat{d}$. The update function applies the multiplicative weights update [3] to adjust the approximation to perform better on the mishandled query. This step is non-private: it does not touch the private data directly. Hence, we do not concern ourselves with the details here, and treat the update step as a black box. (The reader can find further details in Hardt et al. [28].) The full program, together with the transformation into a synchronized product program, is presented in Fig. 9.

We briefly comment on the program. We let d^i denote the i-th iteration of the synthetic database, and $\hat{d}$ denote the true database. Initially the synthetic database d^0 is set to some default value def. Then we define the score function s^i that takes as inputs a database D and a query Q and returns the error of the query Q on the current approximation d^i compared to D. We then apply the exponential mechanism to the true database $\hat{d}$ with the score function s^i, and we call the result q^i. We then evaluate q^i on the real database, and add Laplace noise; we call the result a^i. Finally, we apply the update function to obtain the next iteration d^{i+1} of the synthetic database. Once the number of rounds is exhausted, we return the last computed synthetic databases.

```
i ← 0;
d⁰ ← def;
while i < T do
    sⁱ ← λD Q. |Q(dⁱ) − Q(D)|
    qⁱ ← Expₑ (sⁱ, d̂);
    aⁱ ← Lapₑ (qⁱ d̂);
    dⁱ⁺¹ ← update (dⁱ, aⁱ, qⁱ);
    i ← i + 1;
return dᵀ;
```

(a) Original probabilistic algorithm

```
vₑ ← 0;  i₁ ← 0;  i₂ ← 0;
d₁⁰ ← def; d₂⁰ ← def;
assert (i₁ < T ⇔ i₂ < T);
while i₁ < T do
    s₁ⁱ ← λD Q. |Q(d₁ⁱ) − Q(D)|;
    s₂ⁱ ← λD Q. |Q(d₂ⁱ) − Q(D)|;
    (q₁ⁱ, q₂ⁱ) ← Expₑ◇ (s₁ⁱ, d̂₁, s₂ⁱ, d̂₂);
    (a₁ⁱ, a₂ⁱ) ← Lapₑ◇ (q₁ⁱ(d̂₁), q₂ⁱ(d̂₂));
    d₁ⁱ⁺¹ ← update (d₁ⁱ, a₁ⁱ, q₁ⁱ);
    d₂ⁱ⁺¹ ← update (d₂ⁱ, a₂ⁱ, q₂ⁱ);
    i₁ ← i₁ + 1;
    i₂ ← i₂ + 1;
    assert (i₁ < T ⇔ i₂ < T);
return (d₁ᵀ, d₂ᵀ);
```

(b) Synchronized non-probabilistic product

Fig. 9: MWEM algorithm

For the privacy proof, we assume that all queries in $\mathcal{Q}$ are 1-sensitive. Note that we run T iterations of MWEM; by the composition theorem, it is sufficient to analyze the privacy budget consumed by each iteration. Each iteration, we select a query with the exponential mechanism with privacy parameter ϵ, and we estimate the true answer of this query with the Laplace mechanism, parameter ϵ. By the composition theorem (Theorem 3), the whole algorithm is private with parameter $2 \cdot T \cdot \epsilon = 2T\epsilon$, as desired. The proof can be transcripted directly into Hoare logic using self-products; we take as precondition adjacency of the two databases, and use adjacency to conclude that the sensitivity of the score function s_i is 1 at each iteration.

C. Propose-Test-Release

The examples we have considered so far all rely on the composition theorem. While this is a quite powerful and useful theorem, not all algorithms use composition. In this section, we consider one such example: the *Propose-Test-Release* (PTR) framework [21], [36]. PTR is also an example of an (ϵ, δ)-differentially private mechanism for $\delta > 0$.

The motivation comes from private release of statistics that are sometimes, but not always, very sensitive. For example, suppose our database is an ordered list of numbers between 0 and 1000, and suppose we want to release the median element

```
x ← DistToInstability (q, d);
y ← Lap_ε x;
if (|y| > log(2/δ)/(2ε))
    return (q d);
else
    return (⊥);
```

Fig. 10: PTR algorithm

of the database. This can be highly sensitive: consider the database $[0, 0, 1000]$ with median 0. Adding a record 1000 to the database would lead to a large change in the median (now 500, if we average the two elements closest to the median when the database has even size). However, many other databases have low sensitivities: for $[0, 10, 10, 1000]$, the median will remain unchanged (at 10) no matter what element we add or remove from the database. We may hope that we can privately compute the median in this second case with much less noise than needed for the first case. More generally, the second database is quite *stable*—all adjacent databases have the same median value. In contrast, the first database is *instable*—adjacent databases may have wildly different median values. With this example in mind, we now explain the general PTR framework.

Suppose we want to privately release the result of a query q evaluated on a database d. We assume that databases are taken from a set $\mathcal{D}$ and that there exists a notion of distance Δ on $\mathcal{D}$, such that pairs of input memory related by Φ correspond to databases at distance at most 1 under Δ. First, we estimate the *distance to instability*—that is, the largest distance x such that $q(d) = q(d')$ for all databases d' at distance x or less from d. Since this a 1-sensitive function (moving to a neighboring database can change the distance to instability by at most 1), we can release this distance privately using the Laplace mechanism (say, with parameter ϵ). Call the result y. Now, we compare y to a threshold t (to be specified later). If y is less than the threshold, we output $q(d)$ with no noise. If y is greater than the threshold, we output a default value $\perp$. The program is given in Fig. 10.

The privacy of the algorithm can be informally justified in two parts. First, suppose that instead of outputting $q(d)$ or $\perp$, we simply output which branch the program took. This is ϵ-differentially private: computing y is ϵ-differentially private (via the Laplace mechanism), and the resulting branch is a post-processing of y. Hence, we can assume that the same branch is taken in both executions.

Second, we can conclude that the original program (outputting $q(d)$ or $\perp$) is (ϵ, δ)-differentially private if for any adjacent databases d and d' with $q(d) \neq q(d')$, the first branch is taken with probability at most δ. By properties of the Laplace mechanism, we can set the threshold t large enough so that with probability at least $1 - \delta$, the first branch is only taken if x is strictly positive. In this case we can conclude $q(d) = q(d')$, since $q(d) \neq q(d')$ implies that x is 0 on both executions. So, we can safely release $q(d) = q(d')$ with no

```
{Δ(d₁, d₂) ≤ 1}
v_ε ← 0;
v_δ ← 0;
x₁ ← DistToInstability (q, d₁);
x₂ ← DistToInstability (q, d₂);
{ (q(d₁) = q(d₂) ∨ x₁ = x₂ = 0)
  ∧v_ε = 0 ∧ v_δ = 0                    }
(y₁, y₂) ← Lap_ε^◇(x₁, x₂);
{ (q(d₁) = q(d₂)) ∨ (x₁ = x₂ = 0
  ∧|y₁ − x₁| ≤ log(2/δ)/(2ε))
  ∧y₁ = y₂ ∧ v_ε ≤ ε ∧ v_δ ≤ δ         }
    assert (|y₁| > log(2/δ)/(2ε) ⇔ |y₂| > log(2/δ)/(2ε));
if (|y₁| > log(2/δ)/(2ε))
    { q(d₁) = q(d₂) ∧ v_ε ≤ ε ∧ v_δ ≤ δ }
    return (q(d₁), q(d₂));
    { out₁ = out₂ ∧ v_ε ≤ ε ∧ v_δ ≤ δ }
else
    { v_ε ≤ ε ∧ v_δ ≤ δ }
    return (⊥, ⊥);
    { out₁ = out₂ ∧ v_ε ≤ ε ∧ v_δ ≤ δ }
```

Fig. 11: Proof of Propose-Test-Release

noise. Of course, if the second branch is taken, then it is also safe to release $\perp$ in both runs.

More formally, the proof of (ϵ, δ)-differential privacy for PTR rests on two properties of the Laplace mechanism: the privacy property captured by Theorem 1 and the accuracy property captured by Lemma 1.

Fig. 11 presents the proof of PTR using the synchronized product program—the code is interleaved with some of the pre- and post-conditions. The proof uses the accuracy property of the Laplace mechanism and the properties of the distance to instability that we give as specifications in Fig. 12. For simplicity, we treat distance to instability as an abstract procedure; however, it can be implemented as a loop over all databases, in which case the specification can be proved. The soundness of the accuracy specification for the Laplace mechanism follows by Lemma 1.

D. Vertex cover

A vertex cover for a graph $g = (N, E)$ is a set S of nodes such that for every edge $(t, u) \in E$, either $t \in S$ or $u \in S$. The minimum vertex cover is the problem of finding a vertex cover of a minimum size. Gupta et al. [26] study the problem of privately computing a minimum vertex cover in a setting where the nodes of the graph are public, but its edges are private. Since a vertex cover leaks information about vertices (for instance, any two nodes that are not in the vertex cover are certainly not connected by an edge), their algorithm outputs an enumeration of the nodes of the graph, from which a vertex cover can be recomputed efficiently from the knowledge of the set E. Their algorithm is challenging to verify because rather

$$\vdash (y_1, y_2) \leftarrow \mathsf{Lap}^\diamond_\epsilon(x_1, x_2) : x_1 = x_2 \land \mathsf{v}_\delta = \hat{\delta} \implies y_1 = y_2 \land |y_1 - x_1| \le \log(2/\delta)/(2\epsilon) \land \mathsf{v}_\delta = \hat{\delta} + \delta$$

$$\vdash x_1 \leftarrow \mathsf{DistToInstability}\,(q, d_1); x_2 \leftarrow \mathsf{DistToInstability}\,(q, d_2) : \Delta(d_1, d_2) \le 1 \implies q(d_1) = q(d_2) \lor x_1 = x_2 = 0$$

Fig. 12: Accuracy specification for the Laplace mechanism, and specification for distance to instability.

```
n ← |E|;
out ← [];
while g ≠ ∅ do
    v ← choose_{ε,n}(g);
    out ← v :: out;
    g ← g \ {v};
return out;
```

Fig. 13: Minimum vertex cover

than relying on mechanisms, it achieves privacy by sampling according to a suitable noisy distribution choose. The code of the algorithm is shown in Fig. 13.

We say that two graphs g_1 and g_2 are adjacent if they differ at most in one edge $\langle t, u \rangle$. By defining choose as

$$\Pr\left[v \leftarrow \mathsf{choose}_{\epsilon,n}(g) : v = v'\right] \propto \left(d_{E,V}(v') + \frac{4}{\epsilon}\sqrt{\frac{n}{|E|}}\right)$$

where $g = (E, V)$ and n is a given parameter, one obtains an $(\epsilon, 0)$-differentially private algorithm with respect to the adjacency relation as defined above.

In [11], Barthe *et al* prove differential privacy of vertex cover in apRHL. The proof uses the generalized rule for loops, a code motion rule that allows to swap independent statements, and a rule for dealing with statements of the form $x \xleftarrow{\$} \mu; \mathsf{assert}\,(\phi)$. It also relies on apRHL specifications of choose, that are proven correct in the Coq proof assistant from the definition of choose.

We now consider the formal verification of the vertex cover algorithm using self-products. We first extend the definition of self-product to choose. Then, there are two cases to consider: $g_2 = g_1 \cup \{\langle u, t \rangle\}$ and $g_1 = g_2 \cup \{\langle u, t \rangle\}$. In the first case, we can use the first Hoare specification from Fig. 14. In the second case, we use the second and third specifications from Fig. 14. Using these specifications, it is possible to verify that the self-product of the vertex cover algorithm satisfies the Hoare specification of Theorem 5. However, we have not yet been able to extend the proof of Theorem 5 to deal with the choose self-product.

E. Formal verification of the examples

The examples above (with the exception of vertex cover) have been formally verified. For each example, we have built the corresponding self-product program, and verified this result using the non-probabilistic and non-relational Hoare logic rules available in the EasyCrypt [6] framework. As described above, we have used non-probabilistic axiomatic specifications for the primitives. Apart from the axiomatic specification, and

the code for the program and the self-product construction, the longest Hoare logic verification proof (for MWEM) consists of about 50 lines of code. This demonstrates the simplicity offered by the self-product construction. The code for these examples (and others) is available online [1].

VI. RELATED WORK

Differential privacy, first proposed by Blum et al. [13] and formally defined by Dwork et al. [22], has been an area of intensive research in the last decade. We have touched on a handful of private algorithms, including algorithms for computing running sums [14], [23] (part of a broader literature on streaming privacy), answering large classes of queries [29], [28] (part of a broader literature on learning-theoretic approaches to data privacy), the Propose-Test-Release framework for answering stable queries in a noiseless way [21], [36], and private combinatorial optimization [26]. We refer readers interested in a more comprehensive treatment to the excellent surveys by Dwork [19], [20].

Verifying differential privacy: Several tools have been proposed for providing formal verification of the differential privacy guarantee; we can roughly classify them by the verification approach they use. PINQ [30] provides an encapsulation for LINQ—an SQL-like language embedded in C#—tracking at runtime the privacy budget consumption, and aborting the computation when the budget is exhausted. Airavat [34] combines a similar runtime monitor with access control in a MapReduce framework. While PINQ is restricted to ϵ-differential privacy, Airavat can handle also approximate differential privacy using a runtime monitor for δ.

Another approach is based on linear type systems. Fuzz [33] and DFuzz [25] use a type-based approach for inferring and checking the sensitivity of functional programs. This sensitivity analysis combined with the use of trusted probabilistic primitives provides the differential privacy guarantee. Interestingly, this type-based approach can be combined with type systems for cryptographic protocols to verify differential privacy for distributed protocols [24]. All these systems provide automatic verification of differential privacy. However, they fail to verify all the examples that we can handle, like advanced sum statistics [14] and the Propose-Test-Release framework [21]. Moreover, so far they can address only pure differential privacy, where $\delta = 0$.

Tschantz, et al. [37] consider a verification framework for interactive private programs, where the algorithm can receive new input and produce multiple outputs over a series of steps. They follow an approach similar to ours by verifying the correct use of differentially private primitives. However, their programs are well-modeled by probabilistic I/O-automata,

$$\vdash (v_1, v_2) \leftarrow \mathsf{choose}^\diamond_{\epsilon,n}(g_1, g_2) : g_1 \cup \{\langle u, t \rangle\} = g_2 \wedge \mathtt{v}_\epsilon = \epsilon_0 \implies v_1 = v_2 \wedge \mathtt{v}_\epsilon = \epsilon_0 + \epsilon / \left(2\sqrt{n}\sqrt{|g_1|} \right)$$

$$\vdash (v_1, v_2) \leftarrow \mathsf{choose}^\diamond_{\epsilon,n}(g_1, g_2) : g_1 = g_2 \cup \{\langle u, t \rangle\} \wedge \mathtt{v}_\epsilon = \epsilon_0 \implies (v \neq t \wedge v \neq u) \wedge \mathtt{v}_\epsilon = \epsilon_0$$

$$\vdash (v_1, v_2) \leftarrow \mathsf{choose}^\diamond_{\epsilon,n}(g_1, g_2) : g_1 = g_2 \cup \{\langle u, t \rangle\} \wedge \mathtt{v}_\epsilon = \epsilon_0 \implies (v = t \vee v = u) \wedge \mathtt{v}_\epsilon = \epsilon_0 + \epsilon/4$$

Fig. 14: Hoare specifications for $\mathsf{choose}^\diamond$

and they provide a proof technique based on probabilistic bisimulation. Also, their method is currently limited to pure differential privacy.

Finally, CertiPriv [11] and EasyCrypt [6] use custom relational logics to verify differential privacy. These systems are very expressive: they supports general (ϵ, δ)-differential privacy, they can verify privacy for mechanisms like the Laplace and the Exponential mechanism, and they can capture advanced examples that go beyond mechanisms and composition, like the private vertex cover algorithm of Gupta et al. [26]. The difficulty with their approach is that it relies on a customized and complex logic. Moreover, ad hoc rules for loops are required for many advanced examples.

Verifying 2-safety properties: Beyond differential privacy, there is a large body of literature on verifying 2-safety properties. Our work is most closely related to deductive methods based on program logics; more precisely, approaches that reduce 2-safety of a program c to safety of a program c' built from c. Such approaches include *self-composition* [7], *product programs* [38], and *type-directed product programs* [35]. These approaches are subsumed by work by Barthe et al. [4], [5].

Another alternative is to reason directly on two programs (or two executions of the same program) using relational program logics such as Benton's relational Hoare logic [12], or specialized relational logics, e.g., for information flow [2]. CertiCrypt [9], and EasyCrypt [8], [6], are computer-aided tools that support relational reasoning about probabilistic programs and have been used to prove security of cryptographic constructions and computational differential privacy of protocols. For such applications, reasoning about structurally different programs is essential.

Chaudhuri et al. [15] develop an automated method for analyzing the continuity and the robustness of programs. Robustness is a 2-safety property that is very similar to sensitivity as used in differential privacy. An interesting aspect of their work is that their analysis is able to reason about two *unsynchronized* pairs of executions; that is, pairs of executions that may have different control flow.

Verification of hyperproperties: Developing general verification methods for hyperproperties remains a challenge; however, there have been some recent proposals in this direction (e.g., [16], [32]).

Other work: There is an extensive body of work on deductive verification of non-probabilistic and probabilistic programs, as well as many works that consider product con-

structions of Labeled Transition Systems; summarizing this large literature is beyond the scope of this paper.

VII. Conclusion

We have proposed a program transformation that reduces proving (ϵ, δ)-differential privacy of a probabilistic program to proving a safety property of a deterministic transformed program. The method applies to all standard examples where privacy is achieved through mechanisms and composition theorems; on the other hand, differentially private algorithms based on ad hoc output perturbation, such as the differentially private vertex cover algorithm [26], are more difficult to handle. In particular, they fall outside the scope of Theorem 5 which proves the soundness of our approach. Our method is particularly suited for reasoning about differential privacy, because the transformed program can be analyzed with standard verification tools. Our method can also be extended to reason about probabilistic non-interference, at the cost of targeting an assertion language that supports existential quantification over functions. Directions for further work include extending the scope of Theorem 5 to deal with more complex examples, like vertex cover. On a more practical side, it would be interesting to implement our transformation for a realistic setting, for instance modeling the PINQ language [30].

Acknowledgments

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