What is a Model for a Semantically Linear λ-calculus?

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This paper is about a categorical approach to model a simple term calculus, named $\mathcal{S}\ell\lambda$-calculus. This is the core calculus underlying the programming language $\mathcal{S}\ell$PCF that have been conceived in order to program only linear functions between Coherence Spaces.

In this work, we introduce the notion of $\mathcal{S}\ell\lambda$-category, which is able to describe a large class of sound models of $\mathcal{S}\ell\lambda$-calculus. $\mathcal{S}\ell\lambda$-category extends in the natural way Benton, Bierman, Hyland and de Paiva's Linear Category.

We will define two interpretations of $\mathcal{S}\ell\lambda$-calculus types and terms into objects and morphisms of $\mathcal{S}\ell\lambda$-categories: the first one is a generalization of the translation given in [18] but lacks in completeness; the second one uses the comonadic properties of ! to recover completeness.

Finally, we show that this notion is general enough to catch interesting models in Scott Domains and Coherence Spaces.

1 Introduction

In this paper we investigate a categorical approach to give a model to the $\mathcal{S}\ell\lambda$-calculus - acronym for $\mathcal{S}$emantically ℓinear λ-calculus - that is a simple term calculus based on λ-calculus. More specifically, the $\mathcal{S}\ell\lambda$-calculus extends and refines simply typed λ-calculus by imposing a restrictive discipline on the usage of certain kinds of variables, as well as by adding some programming features like numerals, conditional and fix-point operators. The resulting calculus is expressive enough to program all the first-order computable functions and some simple higher-order functions.

Semantically Linear λ-calculus was introduced in [18] (with an additional operator called `which?`, that is not present here) as the term rewriting system on which the programming language $\mathcal{S}\ell$PCF is based [7, 18]. The language $\mathcal{S}\ell$PCF has been designed starting from a syntactical restriction of PCF, inspired by the semantics of Girard's Linear Logic [10], with the aim of obtaining a language where only linear functions between Coherence Spaces can be programmed. In [18] a *concrete* model of $\mathcal{S}\ell$PCF (and consequently of $\mathcal{S}\ell\lambda$-calculus) in the category **Coh** of Coherence Spaces and Linear Functions have been proposed. Moreover, in the same work a token definability result and a corresponding closed full abstraction results for such a model have been proved. More recently, in [8] $\mathcal{S}\ell$PCF have been extended in order to prove a finite clique definability result from which a standard full abstraction result follows.

The aim of the present paper is to give an *abstract* description of the possible models of the $\mathcal{S}\ell\lambda$-calculus. The broader interest in such a study is to highlight the properties that a mathematical structure must satisfy in order to describe, by means of its equational theory, the operational theory induced by the reduction rules of the calculus. For the sake of generality, we give this abstract description by means of tools from category theory. The choice of

the categorical approach allows us to reason abstractly about the properties the $\mathcal{S}\ell\lambda$-calculus enjoys without restricting our attention to concrete models limited to particular mathematical structures.

The categorical approach in describing models of linear logic has a well established tradition [1, 5, 16]. For instance, the category **Coh**, as well as many other categories, is a well known concrete instance of Benton, Bierman, Hyland and de Paiva's Linear Categories, introduced in [1] to provide an abstract description of models of Intuitionistic Linear Logic. Linear Categories are symmetric monoidal closed categories equipped with a symmetric monoidal comonad ! used to interpret the exponential modality and satisfying some additional properties needed to correctly interpret intuitionistic linear logic [1]. The idea is by starting with a symmetric monoidal closed category to impose enough conditions on the comonad in order to make its induced Kleisli category a Cartesian Closed Category with exponential object $A \Rightarrow B = !A \multimap B$. The original construction does not require this, but it would actually be the case, if the monoidal closed category is also cartesian.

In the present paper, we introduce the notion of $\mathcal{S}\ell\lambda$-category. This extends in a natural way the definition of Linear Category, in order to be able to interpret all programming constructs of the $\mathcal{S}\ell\lambda$-calculus. In particular, we ask that this category admits a morphism ℓif acting as a "conditional" and a morphism fix acting as a "fix-point operator". The latter turns out to be the expected decomposition of a fix-point morphism in a Cartesian Closed Category. Furthermore, to interpret ground values, we require the existence of a distinguished object N with the usual zero, successor and predecessor morphisms satisfying the expected equations. However, since variables of ground type can be freely contracted and weakened, we need to ask that all numeral morphisms behaves properly with respect to the comonad !. For this reason we ask the existence of a !-coalgebra $p : N \to !N$ which is also comonoidal and moreover we ask that all numeral built by zero and successor are coalgebraic.

The notion of natural number object in a symmetric monoidal closed category is not new and it was introduced by Paré and Román in [19]. Based on this definition Mackie, Román and Abramsky introduced an internal language for autonomous categories with natural number objects in [15]. The main similarity between the definitions of natural number object given in [19, 15] and our definition is the requirement of comonoidality of the natural number object; moreover their definition does not take into consideration the relationship between the natural number object and the exponential comonad !; in fact there, only a strictly linear language without exponential was analyzed. More detail on this matter can be found in Section 3.1. In particular we give some sufficient condition on a Linear Category with natural number object to be a $\mathcal{S}\ell\lambda$-category (see Theorem 3).

We define two interpretation of $\mathcal{S}\ell\lambda$-calculus types and terms into objects and morphisms of $\mathcal{S}\ell\lambda$-category. The first one is a generalization of the translation given in [18], which at the beginning has been developed specifically for Coherence Spaces. Even though it is possible to prove that this interpretation is sound w.r.t. the operational semantics of the calculus, the translation is not complete. To recover completeness, we introduce a second interpretation, which makes explicit use of the comonadic properties of !. The completeness is proved by relating the $\mathcal{S}\ell\lambda$-calculus with the Linear Term Calculus introduced by Benton et al. [1].

Moreover, this abstract definition of model for the $\mathcal{S}\ell\lambda$-calculus allows us to analyze in a modular way many different concrete examples. In particular, we build a non-trivial model of the $\mathcal{S}\ell\lambda$-calculus in the category **StrictBcdom** of Scott Domains and strict continuous functions. We also study models of the $\mathcal{S}\ell\lambda$-calculus in the category **Coh** of Coherence Spaces and linear

stable functions and in the category **LinBcdom** of Scott Domains and linear functions. More specifically, this implies that the model we defined in [18] is equivalent to a particular instance of $\mathcal{S}\ell\lambda$-category, in the category **Coh**.

Plan of the paper. Section 2 recalls the definition of $\mathcal{S}\ell\lambda$-calculus. Section 3 gives the categorical basis for the rest of the paper. Section 4 defines the notion of $\mathcal{S}\ell\lambda$-category. Section 5 introduces the first interpretation and proves its soundness and the lack of completeness. Section 6 presents the second interpretation and it proves its soundness and completeness. Section 7 gives some concrete examples of models of $\mathcal{S}\ell\lambda$-calculus in the setting of Scott Domains and Coherence Spaces.

2 Semantically Linear λ-calculus

The *Semantically Linear λ-calculus*, named $\mathcal{S}\ell\lambda$-calculus, is a typed term rewriting system on which the programming language $\mathcal{S}\ell$PCF is based [7, 18]. This is obtained by considering a restriction of the usual simply typed λ-calculus and by extending it by means of constants for natural numbers, conditional and fix-point operations. Truth-values are encoded in the $\mathcal{S}\ell\lambda$-calculus as integers where zero encodes "true" while any other numeral stands for "false".

Definition 1. *The set of* $\mathcal{S}\ell\lambda$-types *is defined by the following grammar:*

$$\sigma,\tau ::= \iota \mid (\sigma \multimap \tau)$$

where ι is the only atomic *type (i.e. natural numbers), $\multimap$ is the only* type constructor *and $\sigma,\tau,\ldots$ are meta-variables ranging over types.*

As customary $\multimap$ associates to right. Hence $\sigma_1 \multimap \sigma_2 \multimap \sigma_3$ is an abbreviation for $\sigma_1 \multimap (\sigma_2 \multimap \sigma_3)$. It is easy to see that all types τ have the shape $\tau_1 \multimap \ldots \multimap \tau_n \multimap \iota$, for some type $\tau_1,\ldots,\tau_n$ where $n \geq 0$.

Definition 2. *Let* $\mathrm{Var}^\sigma, \mathcal{S}\mathrm{Var}^\sigma$ *be enumerable disjoint sets of variables of type σ. The set of* ground *variables is* Var^ι, *the set of* linear *variables is* $\ell\mathrm{Var} = \bigcup_{\sigma,\tau} \mathrm{Var}^{\sigma\multimap\tau}$, *the set of* stable *variables is* $\mathcal{S}\mathrm{Var} = \bigcup_\sigma \mathcal{S}\mathrm{Var}^\sigma$ *and the whole set of variables is* $\mathrm{Var} = \mathrm{Var}^\iota \cup \ell\mathrm{Var} \cup \mathcal{S}\mathrm{Var}$.

Letters $\mathtt{x}^\sigma$ range over variables in Var^σ, letters $\mathtt{y}^\iota, \mathtt{z}^\iota, \ldots$ range over variables in Var^ι, letters $\mathtt{f}^{\sigma\multimap\tau}, \mathtt{g}^{\sigma\multimap\tau}, \ldots$ range over variables in $\ell\mathrm{Var}$, while $F^\sigma, F_1{}^\sigma, F_2{}^\sigma, \ldots$ range over stable variables, namely variables in $\mathcal{S}\mathrm{Var}^\sigma$. Last, $\varkappa$ will denote any kind of variables. Latin letters $\mathtt{M},\mathtt{N},\mathtt{L},\ldots$ range over terms.

Definition 3. *The set of* $\mathcal{S}\ell\lambda$-terms, *denoted* $\mathcal{S}\ell\Lambda$, *is defined by the following grammar:*

$$\mathtt{M} \;::=\; \varkappa^\tau \mid \underline{\mathtt{0}} \mid \mathtt{succ} \mid \mathtt{pred} \mid \ell\mathtt{if}\,\mathtt{M}\,\mathtt{M}\,\mathtt{M} \mid (\mathtt{M}\mathtt{M}) \mid (\lambda\mathtt{x}^\sigma.\mathtt{M}) \mid \mu F.\mathtt{M}$$

Free variables of terms are defined as expected. A term $\mathtt{M}$ is *closed* if and only if $\mathrm{FV}(\mathtt{M}) = \emptyset$, otherwise $\mathtt{M}$ is *open*. Terms are considered up to α-equivalence, namely a bound variable can be renamed provided that no free variable is captured. Moreover, the expected capture-free substitutions are denoted $\mathtt{M}[\underline{\mathtt{n}}/\mathtt{y}]$, $\mathtt{M}[\mathtt{N}/\mathtt{f}]$ and $\mathtt{M}[\mathtt{N}/F]$.

A **basis** Γ is a finite list of variables in Var. We denote with Γ^* (resp Γ^ι) a basis Γ containing variables in $\mathcal{S}\mathrm{Var}$ (resp. in Var^ι). We will denote with Γ,Δ the concatenation of two basis and with $\Gamma \cap \Delta$ the intersection of two basis, defined in the expected way.

$$\frac{}{\vdash \underline{0} : \iota}\ (z) \qquad \frac{}{\vdash \mathtt{succ} : \iota \multimap \iota}\ (s) \qquad \frac{}{\vdash \mathtt{pred} : \iota \multimap \iota}\ (p) \qquad \frac{\Gamma \cap \Delta = \emptyset \quad \Gamma \vdash \mathtt{M} : \iota \quad \Delta \vdash \mathtt{L} : \iota \quad \Delta \vdash \mathtt{R} : \iota}{\Gamma, \Delta \vdash \ell\mathtt{if}\, \mathtt{M}\, \mathtt{L}\, \mathtt{R} : \iota}\ (\ell\mathtt{if})$$

$$\frac{\Gamma, \varkappa_2^{\sigma_2}, \varkappa_1^{\sigma_1}, \Delta \vdash \mathtt{M} : \tau}{\Gamma, \varkappa_1^{\sigma_1}, \varkappa_2^{\sigma_2}, \Delta \vdash \mathtt{M} : \tau}\ (ex) \qquad \frac{}{\mathtt{x}^{\iota} \vdash \mathtt{x} : \iota}\ (gv) \qquad \frac{\Gamma \vdash \mathtt{M} : \tau}{\Gamma, \mathtt{x}^{\iota} \vdash \mathtt{M} : \tau}\ (gw) \qquad \frac{\Gamma, \mathtt{x}_1{}^{\iota}, \mathtt{x}_2{}^{\iota} \vdash \mathtt{M} : \tau}{\Gamma, \mathtt{x}^{\iota} \vdash \mathtt{M}[\mathtt{x}/\mathtt{x}_1, \mathtt{x}_2] : \tau}\ (gc)$$

$$\frac{}{\mathtt{f}^{\sigma \multimap \tau} \vdash \mathtt{f} : \sigma \multimap \tau}\ (hv) \qquad \frac{\Gamma \cap \Delta = \emptyset \quad \Gamma \vdash \mathtt{M} : \sigma \multimap \tau \quad \Delta \vdash \mathtt{N} : \sigma}{\Gamma, \Delta \vdash \mathtt{MN} : \tau}\ (ap) \qquad \frac{\Gamma, \mathtt{x}^{\sigma} \vdash \mathtt{M} : \tau}{\Gamma \vdash \lambda \mathtt{x}^{\sigma}.\mathtt{M} : \sigma \multimap \tau}\ (\lambda)$$

$$\frac{}{F^{\sigma} \vdash F : \sigma}\ (sv) \qquad \frac{\Gamma, F_1^{\sigma}, F_2^{\sigma} \vdash \mathtt{M} : \tau}{\Gamma, F^{\sigma} \vdash \mathtt{M}[F/F_1, F_2] : \tau}\ (sc) \qquad \frac{\Gamma \vdash \mathtt{M} : \tau}{\Gamma, F^{\sigma} \vdash \mathtt{M} : \tau}\ (sw) \qquad \frac{\Gamma^{\iota}, \Delta^{*}, F^{\sigma} \vdash \mathtt{M} : \sigma}{\Gamma^{\iota}, \Delta^{*} \vdash \mu F.\mathtt{M} : \sigma}\ (\mu)$$

Table 1: Type assignment system for $\mathcal{S}\ell\lambda$-calculus

Definition 4. *Typed* terms *of $\mathcal{S}\ell\lambda$-calculus are defined by using a type assignment proving judgements of the shape $\Gamma \vdash \mathtt{M} : \sigma$, in Table 1.*

Typing rules deserve some explanation. Note that only linear variables are subject to syntactical constraints. Except for the $\ell\mathtt{if}$ construction typed by an additive rule doing an implicit contraction, linear variables cannot be contracted or weakened. Ground and stable variables belong to distinct kinds only for sake of simplicity, their free use implies that $\mathcal{S}\ell\lambda$-calculus is not syntactically linear (in the sense of [18]).

Note that the constraints on the use of variables are preserved by the substitutions.

Lemma 1 (Substitution)**.** *Let $\mathtt{M}, \mathtt{N} \in \mathcal{S}\ell\Lambda$.*

1. *If $\Gamma, \mathtt{x}^{\sigma \multimap \mu} \vdash \mathtt{M} : \tau$ and $\Delta \vdash \mathtt{N} : \sigma \multimap \mu$ with $\ell\mathrm{FV}(\Gamma) \cap \ell\mathrm{FV}(\Delta) = \emptyset$ then $\Gamma, \Delta \vdash \mathtt{M}[\mathtt{N}/\mathtt{x}^{\sigma}] : \tau$.*
2. *If $\Gamma, F^{\sigma} \vdash \mathtt{M} : \tau$ and $\Delta \vdash \mathtt{N} : \sigma$ with $\ell\mathrm{FV}(\Delta) = \emptyset$ then $\Gamma, \Delta \vdash \mathtt{M}[\mathtt{N}/F^{\sigma}] : \tau$.*
3. *$\Gamma, \varkappa_1^{\sigma}, \ldots, \varkappa_n^{\sigma} \vdash \mathtt{M} : \tau$ and $\Delta \vdash \mathtt{N} : \sigma$ with $\ell\mathrm{FV}(\Delta) = \emptyset$ then $\Gamma, \Delta \vdash \mathtt{M}[\mathtt{N}/\varkappa_1^{\sigma}, \ldots, \mathtt{N}/\varkappa_n^{\sigma}] : \tau$.*

Proof. Easy, by induction on terms. □

We will write $\underline{\mathtt{n}}$ for $\mathtt{succ}\,(\cdots(\mathtt{succ}\,\underline{\mathtt{0}})\cdots)$ where succ is applied n-times to $\underline{\mathtt{0}}$. Moreover, we denote by $\mathcal{P} = \{\mathtt{M} \in \mathcal{S}\ell\Lambda \mid \vdash \mathtt{M} : \iota\}$ the set of *programs* and by $\mathcal{N} = \{\underline{\mathtt{0}}, \ldots, \underline{\mathtt{n}}, \ldots\}$ the set of *numerals*. The above substitution lemma makes sure that the following is a binary relation between typed $\mathcal{S}\ell\Lambda$-terms.

Definition 5. *We denote $\rightsquigarrow$ the firing (without any context-closure) of one of the following rules:*

$$(\lambda \mathtt{f}^{\sigma \multimap \tau}.\mathtt{M})\mathtt{N} \rightsquigarrow_{\beta} \mathtt{M}[\mathtt{N}/\mathtt{f}] \qquad (\lambda \mathtt{z}^{\iota}.\mathtt{M})\underline{\mathtt{n}} \rightsquigarrow_{\iota} \mathtt{M}[\underline{\mathtt{n}}/\mathtt{z}] \qquad \mu F.\mathtt{M} \rightsquigarrow_{\Upsilon} \mathtt{M}[\mu F.\mathtt{M}/F]$$
$$\mathtt{pred}\,(\mathtt{succ}\,\underline{\mathtt{n}}) \rightsquigarrow_{\delta} \underline{\mathtt{n}} \qquad \ell\mathtt{if}\,\underline{\mathtt{0}}\,\mathtt{L}\,\mathtt{R} \rightsquigarrow_{\delta} \mathtt{L} \qquad \ell\mathtt{if}\,\underline{\mathtt{n+1}}\,\mathtt{L}\,\mathtt{R} \rightsquigarrow_{\delta} \mathtt{R}$$

We call redex each term or sub-term having the shape of a left-hand side of rules defined above. We denote $\to_{\mathcal{S}\ell}$ the contextual closure of $\rightsquigarrow$. Moreover, we denote $\to^{*}_{\mathcal{S}\ell}$ and $=_{\mathcal{S}\ell}$ respectively, the reflexive and transitive closure of $\to_{\mathcal{S}\ell}$ and the reflexive, symmetric and transitive closure of $\to_{\mathcal{S}\ell}$.

We remark that $\rightsquigarrow_{\beta}$ formalises a call-by-name parameter passing. On the other hand, $\rightsquigarrow_{\iota}$ formalises a call-by-value parameter passing, namely the reduction can fire only when the argument is a numeral. As done in [3], it is easy to prove properties as subject-reduction, postposition of δ-rules in a sequence of reductions, the confluence and a standardisation theorem.

It will be convenient also to consider the $\mathcal{S}\ell\lambda$-calculus with its operational semantics.

$$\frac{}{\underline{0}\Downarrow\underline{0}}\ (0) \qquad \frac{\mathtt{M}\Downarrow\underline{\mathtt{n}}}{\mathtt{succ}\,\mathtt{M}\Downarrow\mathtt{succ}\,\underline{\mathtt{n}}}\ (\mathrm{s}) \qquad \frac{\mathtt{M}\Downarrow\mathtt{succ}\,\underline{\mathtt{n}}}{\mathtt{pred}\,\mathtt{M}\Downarrow\underline{\mathtt{n}}}\ (\mathrm{p})$$

$$\frac{\mathtt{M}\Downarrow\underline{0}\quad \mathtt{L}\Downarrow\underline{\mathtt{m}}}{\ell\mathtt{if}\,\mathtt{M}\,\mathtt{L}\,\mathtt{R}\Downarrow\underline{\mathtt{m}}}\ (\mathrm{ifl}) \qquad \frac{\mathtt{M}\Downarrow\mathtt{succ}\,\underline{\mathtt{n}}\quad \mathtt{R}\Downarrow\underline{\mathtt{m}}}{\ell\mathtt{if}\,\mathtt{M}\,\mathtt{L}\,\mathtt{R}\Downarrow\underline{\mathtt{m}}}\ (\mathrm{ifr}) \qquad \frac{\mathtt{M}[\mathtt{N}/\mathtt{f}]\mathtt{P}_1\cdots\mathtt{P}_\mathtt{i}\Downarrow\underline{\mathtt{n}}}{(\lambda\mathtt{f}^{\sigma\multimap\tau}.\mathtt{M})\mathtt{N}\mathtt{P}_1\cdots\mathtt{P}_\mathtt{i}\Downarrow\underline{\mathtt{n}}}\ (\lambda^{\multimap})$$

$$\frac{\mathtt{M}[\mu F.\mathtt{M}/F]\mathtt{P}_1\cdots\mathtt{P}_\mathtt{i}\Downarrow\underline{\mathtt{n}}}{(\mu F.\mathtt{M})\mathtt{P}_1\cdots\mathtt{P}_\mathtt{i}\Downarrow\underline{\mathtt{n}}}\ (\mu) \qquad \frac{\mathtt{N}\Downarrow\underline{\mathtt{m}}\quad \mathtt{M}[\underline{\mathtt{m}}/\mathtt{x}]\mathtt{P}_1\cdots\mathtt{P}_\mathtt{i}\Downarrow\underline{\mathtt{n}}}{(\lambda\mathtt{x}^{\iota}.\mathtt{M})\mathtt{N}\mathtt{P}_1\cdots\mathtt{P}_\mathtt{i}\Downarrow\underline{\mathtt{n}}}\ (\lambda^{\iota})$$

Table 2: Structural operational semantics for the $\mathcal{S}\ell\lambda$-calculus

Definition 6. *The* evaluation relation $\Downarrow\subseteq\mathcal{P}\times\mathcal{N}$ *is the smallest relation inductively satisfying the rules of Table 2. If there exists a numeral* $\underline{\mathtt{n}}$ *such that* $\mathtt{M}\Downarrow\underline{\mathtt{n}}$ *then we say that* $\mathtt{M}$ converges, *and we write* $\mathtt{M}\Downarrow$, *otherwise we say that it* diverges, *and we write* $\mathtt{M}\Uparrow$.

Proposition 1. *If* $\mathtt{M}\in\mathcal{P}$ *and* $\mathtt{M}=_{\mathscr{E}}\underline{\mathtt{n}}$, *then* $\mathtt{M}\Downarrow\underline{\mathtt{n}}$.

Moreover, it is convenient to consider an observational equivalence induced by the operational semantics of the $\mathcal{S}\ell\lambda$-calculus.

Definition 7. *Let* $[\sigma]$ *be a* special constant *of type* σ. *The set of* σ-context Ctx_σ *is generated by the following grammar:*

$$\mathtt{C}[\sigma] \quad ::= \quad [\sigma] \mid \varkappa^{\tau} \mid F^{\tau} \mid \underline{\mathtt{0}} \mid \mathtt{succ} \mid \mathtt{pred} \mid \ell\mathtt{if}\,\mathtt{C}[\sigma]\,\mathtt{C}[\sigma]\,\mathtt{C}[\sigma] \mid (\mathtt{C}[\sigma]\mathtt{C}[\sigma]) \mid (\lambda\mathtt{x}^{\sigma}.\mathtt{C}[\sigma]) \mid \mu F.\mathtt{C}[\sigma]$$

$\mathtt{C}[\mathtt{N}^{\sigma}]$ *denotes the result obtained by replacing all the occurrences of* $[\sigma]$ *in the context* $\mathtt{C}[\sigma]$ *by the term* $\mathtt{N}^{\sigma}$ *and by allowing the capture of its free variables.*

It should be noted that $\mathtt{N}^{\sigma}\in\mathcal{S}\ell\Lambda$ and $\mathtt{C}[\sigma]\in\mathrm{Ctx}_\sigma$ doesn't imply that $\mathtt{C}[\mathtt{N}^{\sigma}]\in\mathcal{S}\ell\Lambda$. For instance all contexts in $\ell\mathtt{if}$ need to be typed ι. The operational equivalence we consider involves substitution of closed terms to stable variables.

Definition 8 (Operational Equivalence)**.** *Let* $\mathtt{M},\mathtt{N}\in\mathcal{S}\ell\Lambda$, *such that* $\Gamma\vdash\mathtt{M}:\sigma$ *and* $\Gamma\vdash\mathtt{N}:\sigma$ *with* $\mathcal{S}\mathrm{FV}(\mathtt{M}),\mathcal{S}\mathrm{FV}(\mathtt{N})\subseteq\{F_1^{\sigma_1},\ldots,F_n^{\sigma_n}\}$. *Then:*

1. $\mathtt{M}\lesssim_\sigma\mathtt{N}$ *whenever, for all closed term* $\mathtt{P}_1^{\sigma_1},\ldots,\mathtt{P}_\mathtt{n}^{\sigma_\mathtt{n}}$ *and for all* $\mathtt{C}[\sigma]$ *such that* $\mathtt{C}[\mathtt{M}[\mathtt{P}_1/F_1,\ldots,\mathtt{P}_\mathtt{n}/F_\mathtt{n}]]$, $\mathtt{C}[\mathtt{N}[\mathtt{P}_1/F_1,\ldots,\mathtt{P}_\mathtt{n}/F_\mathtt{n}]]\in\mathcal{P}$, *if* $\mathtt{C}[\mathtt{M}[\mathtt{P}_1/F_1,\ldots,\mathtt{P}_\mathtt{n}/F_\mathtt{n}]]\Downarrow\underline{\mathtt{n}}$ *then* $\mathtt{C}[\mathtt{N}[\mathtt{P}_1/F_1,\ldots,\mathtt{P}_\mathtt{n}/F_\mathtt{n}]]\Downarrow\underline{\mathtt{n}}$.
2. $\mathtt{M}\sim_\sigma\mathtt{N}$ *if and only if* $\mathtt{M}\lesssim_\sigma\mathtt{N}$ *and* $\mathtt{N}\lesssim_\sigma\mathtt{M}$.

It is easy to verify that $\lesssim_\sigma$ is a preorders while $\sim_\sigma$ is a congruences.

3 The categorical picture

In this section, we introduce step by step the components that we will need in the next section to define the categorical models of $\mathcal{S}\ell\lambda$-calculus. The starting point is the classical definitions of monoidal category which we recall below.

Definition 9 (Monoidal category)**.** *A* monoidal category *consists of a category* $\mathbb{C}$, *a bifunctor* $\otimes:\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ *called* tensor product , *an object* $\mathbf{1}\in Obj(\mathbb{C})$ *and three natural isomorphisms* α,λ,ϱ *such that:*

- $\alpha_{A,B,C} : A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$ *is natural for all* $A, B, C \in Obj(\mathbb{C})$, *and the pentagonal diagram*

$$\begin{array}{ccccc} A\otimes(B\otimes(C\otimes D)) & \xrightarrow{\alpha_{A,B,C\otimes D}} & (A\otimes B)\otimes(C\otimes D) & \xrightarrow{\alpha_{A\otimes B,C,D}} & ((A\otimes B)\otimes C)\otimes D \\ \downarrow{\scriptstyle id_A\otimes\alpha_{B,C,D}} & & & & \uparrow{\scriptstyle \alpha_{A,B,C}\otimes id_D} \\ A\otimes((B\otimes C)\otimes D) & & \xrightarrow{\alpha_{A,B\otimes C,D}} & & (A\otimes(B\otimes C))\otimes D \end{array}$$

  commutes for all $A, B, C, D \in Obj(\mathbb{C})$

- $\lambda_A : \mathbf{1} \otimes A \cong A$ *and* $\varrho_A : A \otimes \mathbf{1} \cong A$ *are natural for all* $A \in Obj(\mathbb{A})$, *and the triangular diagram*

$$\begin{array}{ccc} A\otimes(\mathbf{1}\otimes C) & \xrightarrow{\alpha_{A,\mathbf{1},C}} & (A\otimes\mathbf{1})\otimes C \\ & {\scriptstyle id_A\otimes\lambda_C}\searrow \quad \swarrow{\scriptstyle \varrho_A\otimes id_C} & \\ & A\otimes C & \end{array}$$

  commutes for all $A, C \in Obj(\mathbb{C})$ *and also*

$$\lambda_{\mathbf{1}} = \varrho_{\mathbf{1}} : \mathbf{1} \otimes \mathbf{1} \to \mathbf{1}$$

Definition 10 (Symmetric Monoidal Category)**.** *A monoidal category* $\mathbb{C}$ *is said to be* symmetric *when it is equipped with an isomorphism*

$$\gamma_{A,B} : A \otimes B \cong B \otimes A$$

natural in $A, B \in Obj(\mathbb{C})$, *such that the diagrams*

$$\gamma_{A,B} \circ \gamma_{B,A} = id_{A\otimes B} \qquad \varrho_B = \lambda_B \circ \gamma_{B,\mathbf{1}} : B \otimes \mathbf{1} \cong B$$

$$\begin{array}{ccccc} A\otimes(B\otimes C) & \xrightarrow{\alpha_{A,B,C}} & (A\otimes B)\otimes C & \xrightarrow{\gamma_{A\otimes B,C}} & C\otimes(A\otimes B) \\ \downarrow{\scriptstyle id_A\otimes\gamma_{B,C}} & & & & \downarrow{\scriptstyle \alpha_{C,A,B}} \\ A\otimes(C\otimes B) & \xrightarrow{\alpha_{A,C,B}} & (A\otimes C)\otimes B & \xrightarrow{\gamma_{A,C}\otimes id_B} & (C\otimes A)\otimes B \end{array}$$

all commute.

Definition 11 (Symmetric Monoidal Closed Category)**.** *A symmetric category* $\mathbb{C}$ *is* closed *when for all* $B \in Obj(\mathbb{C})$ *the functor* $- \otimes B : \mathbb{C} \to \mathbb{C}$ *has a specified right adjoint* $B \multimap - : \mathbb{C} \to \mathbb{C}$. *Thus, for every* A, C *there is an object* $B \multimap C$ *and a natural isomorphism*

$$\mathbb{C}(A \otimes B, C) \cong \mathbb{C}(A, B \multimap C)$$

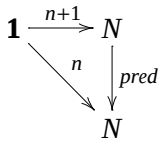
Let $\mathbb{C}$ be a symmetric monoidal closed category. We denote with $\mathrm{curry}(-) : \mathbb{C}(C \otimes A, B) \to \mathbb{C}(C, A \multimap B)$ the isomorphism induced by the canonical adjunction. When $C = A \multimap B$, we denote with $\mathrm{eval} : A \multimap B \otimes A \to B$ the (unique!) morphism such that $\mathrm{curry}(\mathrm{eval}) = id_{A\multimap B}$.

In order to interpret all the constants of our language, we now need to extend monoidal categories.

3.1 Numerals

First of all, we need a canonical object to interpret ground type ι and opportune morphisms to interpret zero, successor and predecessor. The following definition is an adaptation to monoidal categories of the definition of "simple object of numerals" given in [11].

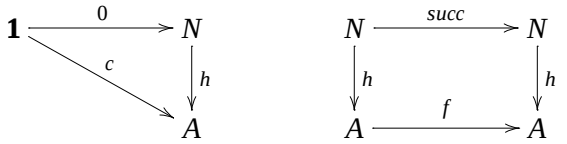
Definition 12 (Monoidal Object of Numerals)**.** *Let $\mathbb{C}$ be a symmetric monoidal category. Let N be an object equipped with two morphisms $0 : \mathbf{1} \to N$ and $succ : N \to N$. A* numeral *$n : \mathbf{1} \to N$ is defined inductively as the map $0 : \mathbf{1} \to N$ for the base case, and as the map $succ \circ n : \mathbf{1} \to N$ for the case $n+1$. N is said to be a* monoidal object of numerals *when it is also equipped with a morphism $pred : N \to N$ such that the following diagram commutes*



The definition above is very weak. It is in fact not required that given two numerals $m : \mathbf{1} \to N$ and $n : \mathbf{1} \to N$ with $n \neq m$ (viewed as numbers), they are distinct morphisms in $\mathbb{C}$. Furthermore, the definition given above does not allow to represent neither recursive nor primitive recursive functions in $\mathbb{C}$. An analogous situation is also present in the definition of simple object of numerals given in [11].

The following definition has been introduced in [19]. It extends Lawvere's notion of natural number object [13], which was specifically defined for cartesian categories, to any monoidal category.

Definition 13 (Natural number object [19, 15])**.** *Let $\mathbb{C}$ be a symmetric monoidal category. By a* natural number object *in $\mathbb{C}$ we mean an object N and two morphisms $0 : \mathbf{1} \to N$ and $succ : N \to N$ such that, given any pair of morphisms $c : \mathbf{1} \to A$ and $f : A \to A$ there is a unique $h : N \to A$ making the following diagrams commute.*

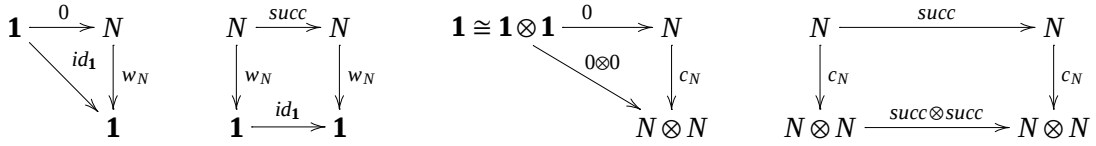


In [19], Paré and Román show that in any symmetric monoidal category $\mathbb{C}$ with a natural number object, the theory of primitive recursive functions can be developed. This is done by considering the category of commutative comonoids in $\mathbb{C}$, denoted with $CC(\mathbb{C})$, which is cartesian (this fact was first observed by Fox [6], who showed that $CC : \mathbf{Mon} \to \mathbf{Cart}$ is right adjoint to the forgetful functor $\mathbf{Cart} \to \mathbf{Mon}$) and where the theory of natural number objects is well developed. In detail, if $\langle C, d_C, e_C \rangle$ and $\langle D, d_D, e_D \rangle$ are two commutative comonoids, then its cartesian product is given by $\langle C \otimes D, d_C \otimes d_D, e_C \otimes e_D \rangle$, while the pairing and the projections are defined as

$$\begin{array}{rll} \pi_1 & \text{is the composite of} & C \otimes D \overset{id_C \otimes e_D}{\to} C \otimes \mathbf{1} \overset{\rho}{\to} C \\ \pi_2 & \text{is the composite of} & C \otimes D \overset{e_C \otimes id_D}{\to} \mathbf{1} \otimes D \overset{\lambda}{\to} D \\ \langle f, g \rangle & \text{is the composite of} & E \overset{d_E}{\to} E \otimes E \overset{f \otimes g}{\to} C \otimes D \end{array}$$

for $f : E \to C$ and $g : E \to D$. The terminal object is $\mathbf{1}$.

More specifically, in [19] it is shown that if N is a natural number object, then it is a commutative comonoid, by taking the morphisms $w_N : N \to \mathbf{1}$ and $c_N : N \to N \otimes N$ to be the unique morphisms making the following diagrams commute

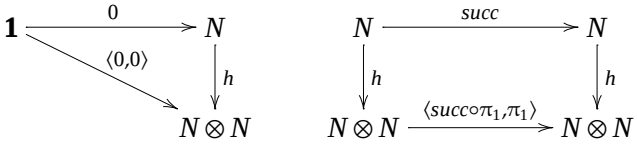


Observe that by definition, $0 : \mathbf{1} \to N$ and $succ : N \to N$ are both comonoid morphisms. Thus, all numerals are comonoid morphisms, and all primitive recursive functions can be represented, in the same way as they were represented in a Cartesian Category [21]. Observe again that the above definition of *natural number object* does not require that given two numerals $n : \mathbf{1} \to N$ and $m : \mathbf{1} \to N$ with $n \neq m$ (viewed as numbers) are distinct morphisms in $\mathbb{C}$.

The following proposition is a corollary of the above statement.

Proposition 2. *Let $\mathbb{C}$ be a symmetric monoidal closed category with a natural number object N. Then N is a monoidal object of numerals.*

Proof. Let $h : N \to N \otimes N$ be the unique morphism making the following diagrams commute (the pairing and projections in the category of commutative comonoid of $\mathbb{C}$ are defined above).



Thus, a choice for $pred : N \to N$ could be the following

$$pred \text{ is the composite of } N \xrightarrow{h} N \otimes N \xrightarrow{\pi_2} N$$

We prove by induction on n that $n = pred \circ n + 1$. We remind that $CC(\mathbb{C})$ is a cartesian category. For the base case, we have

$$\begin{aligned} pred \circ 1 &= \pi_2 \circ h \circ succ \circ 0 \\ &= \pi_2 \circ \langle succ \circ \pi_1, \pi_1 \rangle \circ h \circ 0 \\ &= \pi_2 \circ \langle succ \circ \pi_1, \pi_1 \rangle \circ \langle 0, 0 \rangle \\ &= \pi_2 \circ \langle 1, 0 \rangle \\ &= 0 \end{aligned}$$

For the inductive case, we have

$$\begin{aligned} pred \circ n+2 &= \pi_2 \circ h \circ succ \circ n+1 \\ &= \pi_2 \circ \langle succ \circ \pi_1, \pi_1 \rangle \circ h \circ succ \circ n \\ &= \pi_2 \circ \langle succ \circ \pi_1, \pi_1 \rangle \circ \langle succ \circ \pi_1, \pi_1 \rangle \circ h \circ n \\ &= \pi_2 \circ \langle succ \circ succ \circ \pi_1, succ \circ \pi_1 \rangle \circ h \circ n \\ &= succ \circ \pi_1 \circ h \circ n \\ &= succ \circ \pi_2 \circ \langle succ \circ \pi_1, \pi_1 \rangle \circ h \circ n \\ &= succ \circ pred \circ n+1 \\ &= n+1 \end{aligned}$$

where in the last line we use inductive hypothesis. □

Definition 14. *Let* $\mathbb{C}$ *be a symmetric monoidal category.*

- $\mathbb{C}$ *is* well pointed *when given* $f,g : A \to B$, $f \neq g$ *implies that there is* $a : \mathbf{1} \to A$ *such that* $f \circ a \neq g \circ a$.
- *If* $\mathbb{C}$ *admits a natural number object* N, *we say that* $\mathbb{C}$ *is* N-pointed *when given* $f,g : N \to A$, $f \neq g$ *implies that there is a numeral* $n : \mathbf{1} \to N$ *such that* $f \circ n \neq g \circ n$
- $\mathbb{C}$ *is* N-well pointed *when it is both* N*-pointed and well pointed.*

We stress that many known instances of symmetric monoidal closed categories with a natural number object N are N-pointed. For instance the category **Coh** of Coherence Space and Linear Function, the category **LinBcdom** of Scott Domains and Linear Functions or the category **StrictBcdom** of Scott Domains and Strict Continuous Functions, that will be presented in Section 7, are both N-pointed, by setting N to be the flat domain of natural numbers.

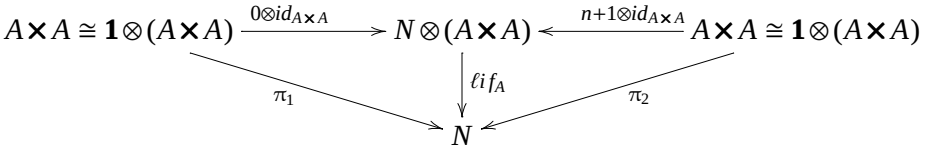
Theorem 1. *Let* $\mathbb{C}$ *be a* N*-well pointed symmetric monoidal category. Suppose that there are two numeral* $n,m : \mathbf{1} \to N$ *such that* $m \neq n$ *viewed as numbers but* $n = m$ *as morphisms. Then* $\mathbb{C}$ *is equivalent to the one-object and one-morphism category.*

Proof. By proposition 2 and since the category is N-pointed, we have that $pred \circ succ = id_N$. By this fact, given two numerals $n,m : \mathbf{1} \to N$, if $succ \circ n = succ \circ m$ then it is easy to see that $n = m$ (viewed as morphisms). So suppose that $0 = succ \circ n$ for some numeral $n : \mathbf{1} \to N$; then $n = (pred \circ succ) \circ n = pred \circ (succ \circ n) = pred \circ 0 = 0$, so $0 = succ \circ 0$. Let $f : X \to X$ and $x : \mathbf{1} \to X$ be two arbitrary morphisms. Since N is a natural number object, there exists a unique $h : N \to X$ such that $h \circ 0 = x$ and $f \circ h = h \circ succ$. So $f \circ x = f \circ (h \circ 0) = (f \circ h) \circ 0 = (h \circ succ) \circ 0 = h \circ (succ \circ 0) = h \circ 0 = x$. But being $x : \mathbf{1} \to X$ arbitrary, we conclude that $f = id_X$ because the category is well pointed. So the proof is done. □

3.2 Conditional operator

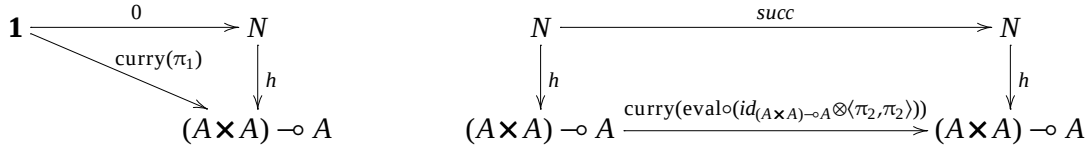
The second ingredient we need is an object to interpret the conditional, this can be easily obtained using the following definition.

Definition 15 (Conditional operator)**.** *Let $\mathbb{C}$ be a symmetric monoidal category, which is also cartesian. We say that $\mathbb{C}$ admits a* conditional operator *if for all objects A there is a morphism $\ell i f_A : N \otimes (A \times A) \to A$ such that the following diagram commutes.*



where $A \times A$ is the cartesian product of A with itself.

Suppose that $\mathbb{C}$ is a symmetric monoidal closed category which is also cartesian and suppose it admits a natural number object. Then it admits also a conditional operator. To see this, define $h : N \to (A \times A) \multimap A$ to be the unique morphism making the following diagrams commute.



Observe that, given a numeral $n : \mathbf{1} \to N$ different from 0, we have that $h \circ n = \mathrm{curry}(\pi_2)$. It can be obtained by induction on n. The case $n = 0$ is vacuous: suppose $n = succ \circ m$, then

$$h \circ succ \circ m \quad = \quad \mathrm{curry}(\mathrm{eval} \circ (id_{(A \times A) \multimap A} \otimes \langle \pi_2, \pi_2 \rangle)) \circ h \circ m$$

If $m = 0$, then

$$\begin{aligned} h \circ succ \circ 0 &= \mathrm{curry}(\mathrm{eval} \circ (id_{(A \times A) \multimap A} \otimes \langle \pi_2, \pi_2 \rangle)) \circ \mathrm{curry}(\pi_1) \\ &= \mathrm{curry}(\mathrm{eval} \circ (\mathrm{curry}(\pi_1) \otimes \langle \pi_2, \pi_2 \rangle)) \\ &= \mathrm{curry}(\pi_1 \circ \langle \pi_2, \pi_2 \rangle) \\ &= \mathrm{curry}(\pi_2) \end{aligned}$$

On the other hand if $m \neq 0$, then

$$\begin{aligned} h \circ succ \circ m &= \mathrm{curry}(\mathrm{eval} \circ (id_{(A \times A) \multimap A} \otimes \langle \pi_2, \pi_2 \rangle)) \circ \mathrm{curry}(\pi_2) \\ &= \mathrm{curry}(\mathrm{eval} \circ (\mathrm{curry}(\pi_2) \otimes \langle \pi_2, \pi_2 \rangle)) \\ &= \mathrm{curry}(\mathrm{eval} \circ (\mathrm{curry}(\pi_2) \otimes \langle \pi_2, \pi_2 \rangle)) \\ &= \mathrm{curry}(\pi_2 \circ \langle \pi_2, \pi_2 \rangle) \\ &= \mathrm{curry}(\pi_2) \end{aligned}$$

Define $\ell i f : N \otimes (A \times A) \to A$ to be the unique morphism such that $\mathrm{curry}(\ell i f) = h$, where h is defined as above: it is not difficult to see that this is a conditional morphism.

Corollary 1. *Let $\mathbb{C}$ be a symmetric monoidal closed category which is also cartesian. Suppose that it admits a natural number object; then it admits a conditional operator.*

3.3 Fix-point operator

The third ingredient we need is an object allowing us to interpret the fix-point operations. This can be interpreted by using the standard *fix-point operator* in a cartesian closed category.

A *Cartesian Closed Category* (CCC for short) is a symmetric monoidal closed category, where the tensor product is cartesian. If $\mathbb{C}$ is a CCC, we will denote with $\Rightarrow$ the right adjoint of the cartesian product and with $\Delta_A : A \to A \times A$ the diagonal morphism. The notion of fix-point operator for CCC has been introduced by Lawvere [12].

Definition 16 (Fix-point operator)**.** *Let $\mathbb{C}$ be a cartesian closed category. A* fix-point operator *is a family of morphisms $\{Y_A : A \Rightarrow A \to A | A \in Obj(\mathbb{C})\}$ such that for all $A \in Obj(\mathbb{C})$, the following diagram commutes*

$$\begin{array}{ccc} A \Rightarrow A & \xrightarrow{\Delta} & (A \Rightarrow A) \times (A \Rightarrow A) \\ \downarrow{\scriptstyle Y_A} & & \downarrow{\scriptstyle id_{A\Rightarrow A} \times Y_A} \\ A & \xleftarrow[\text{eval}]{} & (A \Rightarrow A) \times A \end{array}$$

3.4 Linear Category

In the next section we will introduce the notion of $\mathcal{S}\ell\lambda$-model extending the one of Linear Categories as defined by Benton et al. [1]. These has been introduced to give a categorical model of intuitionistic linear logic.

In order to present linear categories we need to recall the *symmetric monoidal comonad* notion.

Definition 17 (Monoidal Functor)**.** *Given monoidal categories $\mathbb{C}$ and $\mathbb{D}$ a* monoidal functor *is a triple $(F, \mathrm{m}_{A,B}, \mathrm{m}_{\mathbf{1}})$ where $F : \mathbb{C} \to \mathbb{D}$ is a functor, $\mathrm{m}_{A,B} : FA \otimes_{\mathbb{D}} FB \to F(A \otimes_{\mathbb{C}} B)$ is a natural transformation, and $\mathrm{m}_{\mathbf{1}} : \mathbf{1}_{\mathbb{D}} \to F(\mathbf{1}_{\mathbb{C}})$ is a map making the following diagrams commute:*

$$\begin{array}{ccc} F\mathbf{1}_{\mathbb{C}} \otimes_{\mathbb{D}} FA & \xrightarrow{\mathrm{m}_{\mathbf{1},A}} & F(\mathbf{1}_{\mathbb{C}} \otimes_{\mathbb{C}} A) \\ \uparrow{\scriptstyle \mathrm{m}_{\mathbf{1}} \otimes id_{FA}} & & \downarrow{\scriptstyle F(\lambda_A)} \\ \mathbf{1}_{\mathbb{D}} \otimes_{\mathbb{D}} FA & \xrightarrow{\lambda_{FA}} & FA \end{array} \qquad \begin{array}{ccc} FA \otimes_{\mathbb{D}} F\mathbf{1}_{\mathbb{C}} & \xrightarrow{\mathrm{m}_{A,\mathbf{1}}} & F(A \otimes_{\mathbb{C}} \mathbf{1}) \\ \uparrow{\scriptstyle id_{FA} \otimes \mathrm{m}_{\mathbf{1}}} & & \downarrow{\scriptstyle F(\varrho_A)} \\ FA \otimes_{\mathbb{D}} \mathbf{1}_{\mathbb{D}} & \xrightarrow{\varrho_{FA}} & FA \end{array}$$

$$\begin{array}{ccc} (FA \otimes_{\mathbb{D}} FB) \otimes_{\mathbb{D}} FC & \xrightarrow{\alpha} & FA \otimes_{\mathbb{D}} (FB \otimes_{\mathbb{D}} FC) \\ \downarrow{\scriptstyle \mathrm{m}_{A,B} \otimes id_{FC}} & & \downarrow{\scriptstyle id_{FA} \otimes \mathrm{m}_{B,C}} \\ F(A \otimes_{\mathbb{C}} B) \otimes_{\mathbb{D}} FC & & FA \otimes_{\mathbb{D}} F(B \otimes_{\mathbb{C}} C) \\ \downarrow{\scriptstyle \mathrm{m}_{A\otimes B,C}} & & \downarrow{\scriptstyle \mathrm{m}_{A,B\otimes C}} \\ F((A \otimes_{\mathbb{C}} B) \otimes_{\mathbb{C}} C) & \xrightarrow{F\alpha} & F(A \otimes_{\mathbb{C}} (B \otimes_{\mathbb{C}} C)) \end{array}$$

We simply denote a monoidal functor between $\mathbb{C}$ and $\mathbb{D}$ as $(F, \mathrm{m}) : \mathbb{C} \to \mathbb{D}$.

Definition 18 (Symmetric Monoidal Functor)**.** *Given symmetric monoidal categories $\mathbb{C}$ and $\mathbb{D}$, a*

monoidal functor $(F, \mathrm{m}) : \mathbb{C} \to \mathbb{D}$ *is* symmetric *if it satisfies the following coherence condition:*

$$\begin{array}{ccc} FA \otimes_{\mathbb{D}} FB & \xrightarrow{\mathrm{m}_{FA,FB}} & F(A \otimes_{\mathbb{C}} B) \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle F(\gamma)} \\ FB \otimes_{\mathbb{D}} FA & \xrightarrow{\mathrm{m}_{FB,FA}} & F(B \otimes_{\mathbb{C}} A) \end{array}$$

Definition 19 (Monoidal Natural Transformation)**.** *Given monoidal categories* $\mathbb{C}$ *and* $\mathbb{D}$ *and monoidal functors* $(F, \mathrm{m}) : \mathbb{C} \to \mathbb{D}$ *and* $(G, \mathrm{n}) : \mathbb{C} \to \mathbb{D}$*, a* monoidal natural transformation *from* $(F, \mathrm{m}) : \mathbb{C} \to \mathbb{D}$ *to* $(G, \mathrm{n}) : \mathbb{C} \to \mathbb{D}$*, is a natural transformation* θ *from* F *to* G *making the following diagrams commute:*

$$\begin{array}{ccc} FA \otimes_{\mathbb{D}} FB & \xrightarrow{\mathrm{m}_{A,B}} & F(A \otimes_{\mathbb{C}} B) \\ {\scriptstyle\theta_A \otimes \theta_B}\downarrow & & \downarrow{\scriptstyle\theta_{A\otimes B}} \\ GA \otimes_{\mathbb{D}} GB & \xrightarrow{\mathrm{n}_{A,B}} & G(A \otimes_{\mathbb{C}} B) \end{array} \qquad \begin{array}{ccc} F\mathbf{1}_{\mathbb{C}} & \xrightarrow{\theta_{\mathbf{1}}} & G\mathbf{1}_{\mathbb{C}} \\ & {\scriptstyle\mathrm{m}_{\mathbf{1}}}\nwarrow \quad \nearrow{\scriptstyle\mathrm{n}_{\mathbf{1}}} & \\ & \mathbf{1}_{\mathbb{D}} & \end{array}$$

Definition 20 (Comonad)**.** *A* comonad *over a category* $\mathbb{C}$ *is a triple* (F, δ, ε) *where* $F : \mathbb{C} \to \mathbb{C}$ *is a functor, and* $\delta : F \to F \circ F$ *and* $\varepsilon : F \to id_{\mathbb{C}}$ *are natural transformations, such that the following diagrams commute:*

$$\begin{array}{ccc} F & \xrightarrow{\delta} & F^2 \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta F} \\ F^2 & \xrightarrow{F\delta} & F^3 \end{array} \qquad \begin{array}{ccccc} F & \xleftarrow{\varepsilon F} & F^2 & \xrightarrow{F\varepsilon} & F \\ & {\scriptstyle id}\nwarrow & {\scriptstyle\delta}\uparrow & \nearrow{\scriptstyle id} & \\ & & F & & \end{array}$$

Definition 21 (Symmetric Monoidal Comonad)**.** *A* symmetric monoidal comonad $(F, \delta, \varepsilon, \mathrm{m})$ *is a comonad* (F, δ, ε) *such that* (F, m) *is a symmetric monoidal functor, with* δ *and* ε *being monoidal natural transformation.*

The definition of symmetric monoidal comonad bring us to the notion of Linear Category as defined by Benton et al. [1].

Definition 22 (Linear Category [1])**.** *A* Linear Category, $\mathbb{L}$ *is a symmetric monoidal closed category* $(\mathbb{L}, \otimes, \mathbf{1}, \alpha, \lambda, \varrho, \gamma)$ *equipped with a symmetric monoidal comonad* $(!, \delta, \varepsilon, q)$ *such that:*

- *For every free !-coalgebra* $(!A, \delta_A)$ *there are two distinguished monoidal natural transformations with components* $e_A : !A \to \mathbf{1}$ *and* $d_A : !A \to !A \otimes !A$ *forming a commutative comonoid and are coalgebra morphisms.*
- *If* $f : (!A, \delta_A) \to (!B, \delta_B)$ *is a coalgebra morphism between coalgebras, then it is also a comonoid morphism.*

3.5 Intuitionistic Linear Term Calculus

We conclude this section by recalling the *Intuitionistic Linear Term Calculus* introduced by Benton et al. [2] and the relations between this calculus and linear categories.

Definition 23. *The terms of the* Intuitionistic Linear Term Calculus *are defined by the following grammar.*

$$\begin{array}{rcl} \mathtt{M} & ::= & \mathtt{x} \mid \top(\mathtt{M},\ldots,\mathtt{M}) \mid \lambda\mathtt{x}.\mathtt{M} \mid \mathtt{MM} \mid * \mid \mathtt{let\ M\ be\ *\ in\ M} \mid \langle \mathtt{M},\mathtt{M} \rangle \mid \mathtt{fst(M)} \mid \mathtt{snd(M)} \mid \mathtt{M} \otimes \mathtt{M} \mid \\ & & \mathtt{let\ M\ be\ x_1 \otimes x_2\ in\ M} \mid \mathtt{promote\ M,\ldots,M\ for\ x_1,\ldots,x_n\ in\ M} \mid \mathtt{derelict(M)} \mid \\ & & \mathtt{discard(M)} \mid \mathtt{copy\ M\ as\ x_1,x_2\ in\ M} \end{array}$$

$$\frac{}{\mathtt{x}^{\sigma} \vdash \mathtt{x} : \sigma}\ (\mathrm{v}) \qquad \frac{\Gamma, \mathtt{x}_2^{\sigma_2}, \mathtt{x}_1^{\sigma_1}, \Delta \vdash \mathtt{M} : \tau}{\Gamma, \mathtt{x}_1^{\sigma_1}, \mathtt{x}_2^{\sigma_2}, \Delta \vdash \mathtt{M} : \tau}\ (\mathrm{ex}) \qquad \frac{}{\vdash * : \mathbf{1}}\ (\star) \qquad \frac{\Gamma \cap \Delta = \emptyset \quad \Gamma \vdash \mathtt{M} : \mathbf{1} \quad \Delta \vdash \mathtt{N} : \sigma}{\Gamma, \Delta \vdash \mathtt{let}\ \mathtt{M}\ \mathtt{be}\ *\ \mathtt{in}\ \mathtt{N} : \sigma}\ (\mathrm{let}\mathbf{1})$$

$$\frac{\Gamma \cap \Delta = \emptyset \quad \Gamma \vdash \mathtt{M} : \sigma_1 \quad \Delta \vdash \mathtt{N} : \sigma_2}{\Gamma, \Delta \vdash \mathtt{M} \otimes \mathtt{N} : \sigma_1 \otimes \sigma_2}\ (\otimes) \qquad \frac{\Gamma \cap \Delta = \emptyset \quad \Gamma \vdash \mathtt{M} : \sigma_1 \otimes \sigma_2 \quad \Delta, \mathtt{x}_1^{\sigma_1}, \mathtt{x}_2^{\sigma_2} \vdash \mathtt{N} : \sigma}{\Gamma, \Delta \vdash \mathtt{let}\ \mathtt{M}\ \mathtt{be}\ \mathtt{x}_1 \otimes \mathtt{x}_2\ \mathtt{in}\ \mathtt{N} : \sigma}\ (\mathrm{let})$$

$$\frac{\Gamma, \mathtt{x}^{\sigma} \vdash \mathtt{M} : \tau}{\Gamma \vdash \lambda \mathtt{x}^{\sigma}.\mathtt{M} : \sigma \multimap \tau}\ (\lambda) \qquad \frac{\Gamma \cap \Delta = \emptyset \quad \Gamma \vdash \mathtt{M} : \sigma \multimap \tau \quad \Delta \vdash \mathtt{N} : \sigma}{\Gamma, \Delta \vdash \mathtt{MN} : \tau}\ (\mathrm{ap})$$

$$\frac{\Gamma \cap \Delta = \emptyset \quad \Gamma_1 \vdash \mathtt{M}_1 : !\tau_1 \ \cdots\ \Gamma_n \vdash \mathtt{M}_n : !\tau_n \qquad \mathtt{x}_1^{!\tau_1}, \ldots, \mathtt{x}_n^{!\tau_n} \vdash \mathtt{N} : \sigma}{\Gamma_1, \ldots, \Gamma_n \vdash \mathtt{promote}\ \mathtt{M}_1, \ldots, \mathtt{M}_n\ \mathtt{for}\ \mathtt{x}_1, \ldots, \mathtt{x}_n\ \mathtt{in}\ \mathtt{N} : !\sigma}\ (\mathrm{pr})$$

$$\frac{\Gamma \vdash \mathtt{M} : !\sigma}{\Gamma \vdash \mathtt{derelict}(\mathtt{M}) : \sigma}\ (\mathrm{dr}) \qquad \frac{\Gamma \cap \Delta = \emptyset \quad \Gamma \vdash \mathtt{M} : !\tau \quad \Delta \vdash \mathtt{N} : \sigma}{\Gamma, \Delta \vdash \mathtt{discard}\ \mathtt{M}\ \mathtt{in}\ \mathtt{N} : \sigma}\ (\mathrm{ds})$$

$$\frac{\Gamma \cap \Delta = \emptyset \quad \Gamma \vdash \mathtt{M} : !\tau \quad \Delta, \mathtt{x}_1^{!\tau}, \mathtt{x}_2^{!\tau} \vdash \mathtt{N} : \sigma}{\Gamma, \Delta \vdash \mathtt{copy}\ \mathtt{M}\ \mathtt{as}\ \mathtt{x}_1, \mathtt{x}_2\ \mathtt{in}\ \mathtt{N} : \sigma}\ (\mathrm{cp}) \qquad \frac{\Gamma_1 \vdash \mathtt{M}_1 : \sigma_1 \quad \cdots \quad \Gamma_n \vdash \mathtt{M}_n : \sigma_n}{\Gamma_1, \ldots, \Gamma_n \vdash \top(\mathtt{M}_1, \ldots, \mathtt{M}_n) : \top}\ (\top)$$

$$\frac{\Gamma \vdash \mathtt{M} : \sigma_1 \quad \Gamma \vdash \mathtt{N} : \sigma_2}{\Gamma \vdash \langle \mathtt{M}, \mathtt{N} \rangle : \sigma_1 \& \sigma_2}\ (\&_I) \qquad \frac{\Gamma \vdash \mathtt{M} : \sigma_1 \& \sigma_2}{\Gamma \vdash \mathtt{fst}(\mathtt{M}) : \sigma_1}\ (\&_1) \qquad \frac{\Gamma \vdash \mathtt{M} : \sigma_1 \& \sigma_2}{\Gamma \vdash \mathtt{snd}(\mathtt{M}) : \sigma_2}\ (\&_2)$$

Table 3: Type Assignment System for the Intuitionistic Linear Term Calculus

Types in the grammar:

$$\sigma \quad ::= \quad \mathbf{1} \mid \top \mid !\sigma \mid \sigma \multimap \sigma \mid \sigma \& \sigma \mid \sigma \otimes \sigma$$

can be assigned to terms by means of the type system presented in Table 3.

The intuitionistic term calculus is equipped with a *reduction relation* $\rightarrow$ defined as the contextual closure of several beta-reduction and commutative rules. We remind in Table 4 the beta-reduction rules where the notation $\overrightarrow{\mathtt{M}}$ is a short for $\mathtt{M}_1, \ldots, \mathtt{M}_n$. For what concerns the plethora of commutative rules we refer an interested to [2, 4].

Linear categories have been introduced by Benton et al. in [1] in order to obtain the following result.

Theorem 2. *A Linear Category* $\mathbb{L}$ *is a categorical model of the Intuitionistic Linear Term Calculus.*

4 $\mathcal{S}\ell\lambda$-categories

In this section we extend Linear Categories presented in the previous section in order to give a categorical model of the $\mathcal{S}\ell\lambda$-calculus.

An $\mathcal{S}\ell\lambda$-category is a Linear Category admitting a monoidal object of numerals which behaves well with respect to the comonad !, together with a "conditional-like" morphism and a fix-point morphism for every object B in the Kleisli category over the comonad !, which is cartesian closed. This leads to the following definition.

Definition 24 ($\mathcal{S}\ell\lambda$-category)**.** *An $\mathcal{S}\ell\lambda$-category is a linear category $\mathcal{L} = \langle \mathbb{L}, !, \delta, \varepsilon, q, d, e \rangle$ such that:*

Numerals. $\mathbb{L}$ *admits a !-coalgebra $\langle N, p \rangle$ such that*

1. *N is a monoidal object of numerals*

$(\lambda \mathtt{x}.\mathtt{M})\mathtt{N}$	$\to$	$\mathtt{M}[\mathtt{N}/\mathtt{x}]$
$\mathtt{let} * \mathtt{be} * \mathtt{in}\, \mathtt{N}$	$\to$	$\mathtt{N}$
$\mathtt{let}\, \mathtt{M}_1 \otimes \mathtt{M}_2\, \mathtt{be}\, \mathtt{x}_1 \otimes \mathtt{x}_2\, \mathtt{in}\, \mathtt{N}$	$\to$	$\mathtt{N}[\mathtt{M}_1/\mathtt{x}_1, \mathtt{M}_2/\mathtt{x}_2]$
$\mathtt{fst}(\langle \mathtt{M}_1, \mathtt{M}_2 \rangle)$	$\to$	$\mathtt{M}_1$
$\mathtt{snd}(\langle \mathtt{M}_1, \mathtt{M}_2 \rangle)$	$\to$	$\mathtt{M}_2$
$\mathtt{derelict}(\mathtt{promote}\, \overrightarrow{\mathtt{M}}\, \mathtt{for}\, \overrightarrow{\mathtt{x}}\, \mathtt{in}\, \mathtt{N})$	$\to$	$\mathtt{N}[\overrightarrow{\mathtt{M}}/\overrightarrow{\mathtt{x}}]$
$\mathtt{discard}\, (\mathtt{promote}\, \overrightarrow{\mathtt{M}}\, \mathtt{for}\, \overrightarrow{\mathtt{x}}\, \mathtt{in}\, \mathtt{N})\, \mathtt{in}\, \mathtt{N}$	$\to$	$\mathtt{discard}\, \overrightarrow{\mathtt{M}}\, \mathtt{in}\, \mathtt{N}$
$\mathtt{copy}\, (\mathtt{promote}\, \overrightarrow{\mathtt{M}}\, \mathtt{for}\, \overrightarrow{\mathtt{x}}\, \mathtt{in}\, \mathtt{N})\, \mathtt{as}\, \mathtt{y},\mathtt{z}\, \mathtt{in}\, \mathtt{Q}$	$\to$	$\mathtt{copy}\, \overrightarrow{\mathtt{M}}\, \mathtt{as}\, \overrightarrow{\mathtt{y}}, \overrightarrow{\mathtt{z}}\, \mathtt{in}\, \mathtt{Q}[\mathtt{promote}\, \overrightarrow{\mathtt{y}}\, \mathtt{for}\, \overrightarrow{\mathtt{x}}\, \mathtt{in}\, \mathtt{N}/\mathtt{y}, \mathtt{promote}\, \overrightarrow{\mathtt{z}}\, \mathtt{for}\, \overrightarrow{\mathtt{x}}\, \mathtt{in}\, \mathtt{N}/\mathtt{z}]$

Table 4: Beta-reduction rules for the Intuitionistic Linear Term Calculus

2. *$\langle N, c_N, w_N \rangle$ is a commutative comonoid where $p : N \to !N$ is a comonoid morphism and all numerals $n : \mathbf{1} \to N$ are !-coalgebra morphisms.*

Conditional Operator. *$\mathbb{L}$ is cartesian and admits a conditional operator.*

Fix-Point Operator. *The Kleisli category $\mathbb{L}_!$ (that is Cartesian Closed) admits a fix-point operator $fix_B : !(!B \multimap B) \to B$ for any object B. We remind that, by the Kleisli construction, we have that the following diagram commutes*

$$\begin{array}{ccc} !(!B \multimap B) & \xrightarrow{d_{!B \multimap B}} & !(!B \multimap B) \otimes !(!B \multimap B) \\ \downarrow{\scriptstyle fix_B} & & \downarrow{\scriptstyle \varepsilon_{!B \multimap B} \otimes (! fix_B \circ \delta_{!B \multimap B})} \\ B & \xleftarrow{\text{eval}} & (!B \multimap B) \otimes !B \end{array}$$

For an $\mathcal{S}\ell\lambda$-category we have the following.

Lemma 2 (Contraction on numerals is definable)**.** $c_N = (\varepsilon_N \otimes \varepsilon_N) \circ d_N \circ p$

Proof. We have:

$$c_N = id_N \otimes id_N \circ c_N = (\varepsilon_N \otimes \varepsilon_N) \circ (p \otimes p) \circ c_N = (\varepsilon_N \otimes \varepsilon_N) \circ d_N \circ p$$

where the first equivalence follows by bifunctorialty, the second one follows by !-coalgebraicity and the third one follows because p is a comonoid morphism. □

Proposition 3. *All numerals $n : \mathbf{1} \to N$ are comonoid morphisms.*

Proof. To prove the comonoidality of $n : \mathbf{1} \to N$ we need to show both $(n \otimes n) \circ \varrho_{\mathbf{1}} = c_N \circ n$ and $id_{\mathbf{1}} = w_N \circ n$. For the first equality we have that

$$\begin{aligned} c_N \circ n &= (\varepsilon_N \otimes \varepsilon_N) \circ d_N \circ p \circ n = (\varepsilon_N \otimes \varepsilon_N) \circ d_N \circ !n \circ q = (\varepsilon_N \otimes \varepsilon_N) \circ (!n \otimes !n) \circ d_{\mathbf{1}} \circ q \\ &= (\varepsilon_N \otimes \varepsilon_N) \circ (!n \otimes !n) \circ (q \otimes q) \circ \varrho_{\mathbf{1}} = (n \otimes n) \circ (\varepsilon_{\mathbf{1}} \circ \varepsilon_{\mathbf{1}}) \circ (q \circ q) \circ \varrho_{\mathbf{1}} = (n \otimes n) \circ \varrho_{\mathbf{1}} \end{aligned}$$

where the first equivalence follows by Lemma 2, the second one follows by coalgebraicity of n, the third one follows by comonoidality of $!n$, the fourth one follows because d is a monoidal natural transformation, the fifth one follows by naturality of ε and bifunctoriality and the sixth one follows by coalgebraicity of q and bifunctoriality. For the second equality, we have that

$$w_N \circ n = e_N \circ p \circ n = e_N \circ !n \circ q = e_{\mathbf{1}} \circ q = id_{\mathbf{1}}$$

where the first equivalence follows by comonoidality of p, the second one follows by coalgebraicity of 0, the third one follows by naturality of ε and the fourth one follows because e is a monoidal natural transformation. □

The following theorem establishes some sufficient conditions for a Linear Category $\mathcal{L}$ to be an $\mathcal{S}\ell\lambda$-category. Observe that these conditions are not necessary.

Theorem 3. *Let $\mathcal{L} = \langle \mathbb{L}, !, \delta, \varepsilon, q, d, e \rangle$ be a Linear Category such that*

1. *$\mathbb{L}$ is N-pointed and cartesian;*
2. *$\mathbb{L}_!$ admits a fix-point operator.*

Then $\mathcal{L}$ is an $\mathcal{S}\ell\lambda$-category.

Proof. Let $p : N \to !N$ be the unique morphism making the following diagrams commute.

$$\begin{array}{ccc} \mathbf{1} & \xrightarrow{0} & N \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ !\mathbf{1} & \xrightarrow{!0} & !N \end{array} \qquad \begin{array}{ccc} N & \xrightarrow{succ} & N \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ !N & \xrightarrow{!succ} & !N \end{array}$$

To prove that p is a !-coalgebra, we need to prove both $\varepsilon_N \circ p = id_N$ and $!p \circ p = \delta_N \circ p$. Since $\mathbb{L}$ is N-pointed, it suffice to prove that for all numerals $n : \mathbf{1} \to N$ we have both $\varepsilon_N \circ p \circ n = n$ and $!p \circ p \circ n = \delta_N \circ p \circ n$. We will show this by induction on n.

- Case $n = 0$. Then, for the first equality we have

  $$\varepsilon_N \circ p \circ 0 = \varepsilon_N \circ !0 \circ q = 0 \circ \varepsilon_{\mathbf{1}} \circ q = 0$$

  where the first equivalence follows by definition of p, the second one follows by naturality of ε and the third one follows because q is a !-coalgebra. For the second equality, we have

  $$\begin{array}{rcl} !p \circ p \circ 0 & = & !p \circ !0 \circ q = !(p \circ 0) \circ q = !(!0 \circ q) \circ q = !!0 \circ !q \circ q \\ & = & !!0 \circ \delta_{\mathbf{1}} \circ q = \delta_N \circ !0 \circ q = \delta_N \circ p \circ 0 \end{array}$$

  where the first equivalence follows by definition of p, the second one follows by functoriality of !, the third one follows again by definition of p, the fourth one follows again by functoriality of !, the fifth one follows because q is a !-coalgebra, the sixth one follows by naturality of δ and the seventh one follows again by definition of p.

- Case $n = succ \circ m$. Then, for the first equality we have

  $$\varepsilon_N \circ p \circ succ \circ m = \varepsilon_N \circ !succ \circ p \circ m = succ \circ \varepsilon_N \circ p \circ m = succ \circ m$$

where the first equivalence follows by definition of p, the second one follows by naturality of ε and the third one follows by inductive hypothesis. For the second equality we have

$$\begin{array}{rcl} !p \circ p \circ succ \circ m & = & !p \circ !succ \circ p \circ m = !(p \circ succ) \circ p \circ m = !(!succ \circ p) \circ p \circ m = !!succ \circ !p \circ p \circ m \\ & = & !!succ \circ \delta_N \circ p \circ m = \delta_N \circ !succ \circ p \circ m = \delta_N \circ p \circ succ \circ m \end{array}$$

where the first equivalence follows by definition of p, the second one follows by functoriality of !, the third one follows again by definition of p, the fourth one follow again by functoriality of !, the fifth one follows by inductive hypothesis, the sixth one follows by naturality of δ and the seventh one follows again by definition of p.

After having proved that p is a !-coalgebra, it is straightforward to see that every numeral morphism is a coalgebra morphism, because $0 : \mathbf{1} \to N$ and $succ : N \to N$ are coalgebra morphisms. It remains to prove that p is a comonoid morphism; we use also here the fact that $\mathbb{L}$ is N-pointed, proving that for all numerals n we have both $d_N \circ p \circ n = (p \otimes p) \circ c_N \circ n$ and $w_N \circ n = e_N \circ p \circ n$. The proof is by induction on n.

- Case $n = 0$. For the first equality, we have that

$$\begin{array}{rcl} d_N \circ p \circ 0 & = & d_N \circ !0 \circ q = (!0 \otimes !0) \circ d_{\mathbf{1}} \circ q = (!0 \otimes !0) \circ (q \otimes q) \circ \varrho_{\mathbf{1}} \\ & = & (!0 \circ q) \otimes (!0 \circ q) \circ \varrho_{\mathbf{1}} = (p \circ 0) \otimes (p \circ 0) \circ \varrho_{\mathbf{1}} = (p \otimes p) \circ (0 \otimes 0) \circ \varrho_{\mathbf{1}} = (p \otimes p) \circ c_N \circ 0 \end{array}$$

where the first equivalence follows by definition of p, the second one follows by comonoidality of !0, the third one follows because d is a monoidal natural transformation, the fourth one follows by bifunctoriality, the fifth one follows by definition of p, the sixth one follows again by bifunctoriality and the seventh follows by definition of c_N, when N is natural number object. For the second equality, we have that

$$e_N \circ p \circ 0 = e_N \circ !0 \circ q = e_{\mathbf{1}} \circ q = id_{\mathbf{1}} = w_N \circ 0$$

where the first equivalence follows by definition of p, the second one follows by comonoidality of !0, the third one follows because e is a monoidal natural transformation and the fourth one follows by definition of w_N, when N is a natural number object.

- Case $n = m + 1$. For the first equality, we have that

$$\begin{array}{rcl} d_N \circ p \circ succ \circ m & = & d_N \circ !succ \circ p \circ m = (!succ \otimes !succ) \circ d_N \circ p \circ m = (!succ \otimes !succ) \circ (p \otimes p) \circ c_N \circ m \\ & = & ((!succ \circ p) \otimes (!succ \circ p)) \circ c_N \circ m = ((p \circ succ) \otimes (p \circ succ)) \circ c_N \circ m \\ & = & (p \otimes p) \circ (succ \otimes succ) \circ c_N \circ m = (p \otimes p) \circ c_N \circ succ \circ m \end{array}$$

where the first equivalence follows by definition of p, the second one follows by comonoidaility of $!succ$, the third one follows by inductive hypothesis, the fourth one follows by bifunctoriality, the fifth one follows again by definition of p, the sixth one follows again by bifunctoriality and the seventh one follows by definition of c_N, when N is a natural number object. For the second equality, we have that

$$e_N \circ p \circ succ \circ m = e_N \circ !succ \circ p \circ m = e_N \circ p \circ m = w_N \circ m = w_N \circ succ \circ m$$

where the first equivalence follows by definition of p, the second one follows by comonoidality of $!succ$, the third follows by inductive hypothesis and the fourth one follows by definition of w_N, when N is a natural number object.

It is now immediate to see that $\mathcal{L}$ is a $\mathcal{S}\ell\lambda$-category, as required. □

5 A linear Interpretation

In this section we give a first interpretation of the $\mathcal{S}\ell\lambda$-terms and types in the $\mathcal{S}\ell\lambda$-category $\mathcal{L}$. Such an interpretation has been suggested in [9] and it is a generalization of the interpretation given in [18]. We prove the interpretation to be sound with respect to the equivalence $=_{\mathcal{S}\ell}$. However, we also show that this is not complete. This motivates the introduction of a further interpretation in the next section.

5.1 Categorical $\mathcal{S}\ell\lambda$-model

We introduce a first notion of categorical model for the $\mathcal{S}\ell\lambda$-calculus.

Definition 25 (Categorical $\mathcal{S}\ell\lambda$-model)**.** *A categorical $\mathcal{S}\ell\lambda$-model consists of*

- *A $\mathcal{S}\ell\lambda$ Category $\langle \mathcal{L}, N, p, c_N, w_N, \ell if, fix\rangle$, where $\mathcal{L} = \langle \mathbb{L}, !, \delta, \varepsilon, q, e, d\rangle$.*
- *A mapping associating to every $\mathcal{S}\ell\lambda$-type σ, an object $[\![\sigma]\!]$ of $\mathbb{L}$ such that $[\![\iota]\!] = N$ and $[\![\sigma \multimap \tau]\!] = [\![\sigma]\!] \multimap [\![\tau]\!]$.*
- *Given a basis Γ we define $[\![\Gamma]\!]$ by induction as $[\![\emptyset]\!] = \mathbf{1}$, $[\![\mathtt{x}^\sigma, \Delta]\!] = [\![\sigma]\!] \otimes [\![\Delta]\!]$ and $[\![F^\sigma, \Delta]\!] = ![\![\sigma]\!] \otimes [\![\Delta]\!]$. Moreover, given a basis Γ such that $\Gamma^\iota = \mathtt{x}_1^\iota, \ldots, \mathtt{x}_n^\iota$ (resp. $\Gamma^* = F_1^{\sigma_1}, \ldots, F_n^{\sigma_n}$) we denote with $p_\Gamma = p \otimes \cdots \otimes p$ n-times (resp. $\delta_\Gamma = \delta_{[\![\sigma_1]\!]} \otimes \cdots \otimes \delta_{[\![\sigma_n]\!]}$).*
  Given a term $\mathtt{M}$ such that $\Gamma \vdash \mathtt{M} : \sigma$ we associate it a morphism $[\![\Gamma \vdash \mathtt{M} : \sigma]\!] : [\![\Gamma]\!] \to [\![\sigma]\!]$, such that
  - $[\![\vdash \underline{\mathtt{0}} : \iota]\!] = 0$, $[\![\vdash \mathtt{succ} : \iota \multimap \iota]\!] = \mathrm{curry}(succ)$, $[\![\vdash \mathtt{pred} : \iota \multimap \iota]\!] = \mathrm{curry}(pred)$
  - $[\![\mathtt{x}^\iota \vdash \mathtt{x} : \iota]\!] = id_N$
  - $[\![\mathtt{f}^{\sigma\multimap\tau} \vdash \mathtt{f} : \sigma \multimap \tau]\!] = id_{[\![\sigma\multimap\tau]\!]}$
  - $[\![F^\sigma \vdash F : \sigma]\!] = \varepsilon_{[\![\sigma]\!]}$
  - $[\![\Gamma^\iota, \Delta^* \vdash \mu F.\mathtt{M} : \sigma]\!] = fix_{[\![\sigma]\!]} \circ !\mathrm{curry}([\![\Gamma^\iota, \Delta^*, F^\sigma \vdash \mathtt{M} : \sigma]\!]) \circ q \circ (p_\Gamma \otimes \delta_\Delta)$
  - $[\![\Gamma \vdash \lambda \mathtt{x}^\sigma.\mathtt{M} : \sigma \multimap \tau]\!] = \mathrm{curry}([\![\Gamma, \mathtt{x}^\sigma \vdash \mathtt{M} : \tau]\!])$
  - $[\![\Gamma, \varkappa_1^{\sigma_1}, \varkappa_2^{\sigma_2}, \Delta \vdash \mathtt{M} : \tau]\!] = [\![\Gamma, \varkappa_2^{\sigma_2}, \varkappa_1^{\sigma_1}, \Delta \vdash \mathtt{M} : \tau]\!] \circ (id_{[\![\Gamma]\!]^{\mathbb{L}}} \otimes \gamma_{[\![\sigma_1]\!],[\![\sigma_2]\!]} \otimes id_{[\![\Delta]\!]})$
  - $[\![\Gamma, \Delta \vdash \mathtt{MN} : \tau]\!] = \mathrm{eval} \circ ([\![\Gamma \vdash \mathtt{M} : \sigma \multimap \tau]\!] \otimes [\![\Delta \vdash \mathtt{N} : \sigma]\!])$.
  - $[\![\Gamma, \Delta \vdash \ell\mathtt{if}\,\mathtt{M}\,\mathtt{L}\,\mathtt{R} : \iota]\!] = \ell if \circ ([\![\Gamma \vdash \mathtt{M} : \iota]\!] \otimes \langle [\![\Delta \vdash \mathtt{L} : \iota]\!], [\![\Delta \vdash \mathtt{R} : \iota]\!]\rangle)$.
  - $[\![\Gamma, \mathtt{x}^\iota \vdash \mathtt{M}[\mathtt{x}/\mathtt{x}_1, \mathtt{x}_2] : \tau]\!] = [\![\Gamma, \mathtt{x}_1{}^\iota, \mathtt{x}_2{}^\iota \vdash \mathtt{M} : \tau]\!] \circ id_{[\![\Gamma]\!]} \otimes c_N$
  - $[\![\Gamma, F^\sigma \vdash \mathtt{M}[F/F_1, F_2] : \tau]\!] = [\![\Gamma, F_1{}^\sigma, F_2{}^\sigma \vdash \mathtt{M} : \tau]\!] \circ id_{[\![\Gamma]\!]} \otimes d_{[\![\sigma]\!]}$
  - $[\![\Gamma, \mathtt{x}^\iota \vdash \mathtt{M} : \tau]\!] = [\![\Gamma \vdash \mathtt{M} : \tau]\!] \circ id_{[\![\Gamma]\!]} \otimes w_N$
  - $[\![\Gamma, F^\sigma \vdash \mathtt{M} : \tau]\!] = [\![\Gamma \vdash \mathtt{M} : \tau]\!] \circ id_{[\![\Gamma]\!]} \otimes e_{[\![\sigma]\!]}$

5.2 Soundness

In order to prove the soundness of the interpretation, we first need to prove some standard semantic *substitution lemmas*. The key point to prove these lemmas is to show that the transformation induced by the typing rules is natural on the unchanged components of the sequent. Let us observe that the substitution of a ground or linear variable respectively with a numeral or a term is modelled directly with the composition in $\mathbb{L}$ (see Lemma 3 and Lemma 4), while the substitution of a stable variable with a term is modelled with the composition in the category of !-coalgebras (see Lemma 5).

For the sake of simplicity, in the proofs we will relax a bit the definition of types and basis. First of all, in the syntax of types, we allow types prefixed with a !. Thus, given a basis $\Gamma = \varkappa_1^{\sigma_1}, \ldots, \varkappa_n^{\sigma_n}$, we denote with $!\Gamma = \varkappa_1^{!\sigma_1}, \ldots \varkappa_n^{!\sigma_n}$. Observe that, for us ! is just a syntactical annotation which will be interpreted with the corresponding exponential comonad; for this reason, we will adapt in the canonical way the interpretation function on the so obtained types and basis. For example, if $\langle \mathbb{L}, !, \delta, \varepsilon, q, e, d, N, p, c_N, w_N, \ell if, fix \rangle$ is an $\mathcal{S}\ell\lambda$ Category and given a basis Γ^* (resp. Γ^ι), we have $\delta_\Gamma : [\![\Gamma^*]\!] \to [\![!\Gamma]\!]$ (resp. $p_\Gamma : [\![\Gamma^\iota]\!] \to [\![!\Gamma]\!]$).

Lemma 3. *Let* $\mathtt{M}$ *be such that* $\Gamma, \mathtt{x}^\iota, \Delta \vdash \mathtt{M} : \sigma$. *Then:*

$$[\![\Gamma, \Delta \vdash \mathtt{M}[\underline{\mathtt{n}}/\mathtt{x}] : \sigma]\!] = [\![\Gamma, \mathtt{x}^\iota, \Delta \vdash \mathtt{M} : \sigma]\!] \circ (id_{[\![\Gamma]\!]} \otimes n \otimes id_{[\![\Delta]\!]})$$

Proof. The proof is by induction on the derivation proving $\Gamma, \mathtt{x}^\iota \vdash \mathtt{M} : \sigma$ and by cases on the last applied rule.

- case **(ex)**. Straightforward.
- case **(gv)**. Obvious, since $[\![\mathtt{x}[\underline{\mathtt{n}}/\mathtt{x}]]\!] = id_N \circ n = n$ as expected.
- case **(λ)**. Then $\Gamma, \mathtt{x}^\iota \vdash \lambda \mathtt{f}.\mathtt{M} : \sigma \multimap \tau$ is direct consequence of $\Gamma, \mathtt{f}^\sigma \vdash \mathtt{M} : \tau$. Thus, we have

$$\begin{array}{rcl} [\![\Gamma, \mathtt{x}^\iota \vdash \lambda \mathtt{f}.\mathtt{M}[\underline{\mathtt{n}}/\mathtt{x}] : \sigma \multimap \tau]\!] & = & \mathrm{curry}([\![\Gamma, \mathtt{f}^\sigma \vdash \mathtt{M}[\underline{\mathtt{n}}/\mathtt{x}] : \tau]\!]) \\ & = & \mathrm{curry}([\![\Gamma, \mathtt{x}^\iota, \mathtt{f}^\sigma \vdash \mathtt{M} : \tau]\!] \circ (id_{[\![\Gamma]\!]} \otimes n \otimes id_\sigma)) \\ & = & \mathrm{curry}([\![\Gamma, \mathtt{x}^\iota, \mathtt{f}^\sigma \vdash \mathtt{M} : \tau]\!]) \circ (id_{[\![\Gamma]\!]} \otimes n) \\ & = & [\![\Gamma, \mathtt{x}^\iota \vdash \lambda \mathtt{f}.\mathtt{M} : \sigma \multimap \tau]\!] \circ (id_{[\![\Gamma]\!]} \otimes n) \end{array}$$

  where the first row follows by interpretation, the second row follows by induction, the third row follows by naturality of curry(−) and the fourth row follows again by interpretation.
- case **(ap)**. This case follows by induction and by functoriality of $\otimes$.
- case **(gc)**. The only interesting case is $\Gamma, \mathtt{x}^\iota \vdash \mathtt{M}[\mathtt{x}/\mathtt{x}_1, \mathtt{x}_2] : \sigma$ consequence of $\Gamma, \mathtt{x}_1{}^\iota, \mathtt{x}_2{}^\iota \vdash \mathtt{M} : \sigma$. Thus we have

$$\begin{array}{rcl} [\![\Gamma \vdash \mathtt{M}[\mathtt{x}/\mathtt{x}_1, \mathtt{x}_2][\underline{\mathtt{n}}/\mathtt{x}] : \sigma]\!] & = & [\![\Gamma \vdash \mathtt{M}[\underline{\mathtt{n}}/\mathtt{x}_1, \underline{\mathtt{n}}/\mathtt{x}_2] : \sigma]\!] \\ & = & [\![\Gamma, \mathtt{x}_1{}^\iota, \mathtt{x}_2{}^\iota \vdash \mathtt{M} : \sigma]\!] \circ id_{[\![\Gamma]\!]} \otimes n \otimes n \\ & = & [\![\Gamma, \mathtt{x}_1{}^\iota, \mathtt{x}_2{}^\iota \vdash \mathtt{M} : \sigma]\!] \circ (id_{[\![\Gamma]\!]} \otimes c_N) \circ (id_{[\![\Gamma]\!]} \otimes n) \\ & = & [\![\Gamma, \mathtt{x}^\iota \vdash \mathtt{M} : \sigma]\!] \circ id_{[\![\Gamma]\!]} \otimes n \end{array}$$

  where the first row follows by definition of substitution, the second row follows by induction, the third row follows by Proposition 3 and the fourth row follows by interpretation.
- case **(gw)**. The only interesting case is $\Gamma, \mathtt{x}^\iota \vdash \mathtt{M} : \sigma$ consequence of $\Gamma \vdash \mathtt{M} : \sigma$. Thus we have

$$\begin{array}{rcl} [\![\Gamma \vdash \mathtt{M}[\underline{\mathtt{n}}/\mathtt{x}] : \sigma]\!] & = & [\![\Gamma \vdash \mathtt{M} : \sigma]\!] \\ & = & [\![\Gamma \vdash \mathtt{M} : \sigma]\!] \circ \varrho \circ id_{[\![\Gamma]\!]} \otimes id_{\mathbf{1}} \circ \varrho^{-1} \\ & = & [\![\Gamma \vdash \mathtt{M} : \sigma]\!] \circ \varrho \circ (id_{[\![\Gamma]\!]} \otimes w_N) \circ (id_{[\![\Gamma]\!]} \otimes n) \circ \varrho^{-1} \\ & = & [\![\Gamma, \mathtt{x}^\iota \vdash \mathtt{M} : \sigma]\!] \circ id_{[\![\Gamma]\!]} \otimes n \circ \varrho^{-1} \end{array}$$

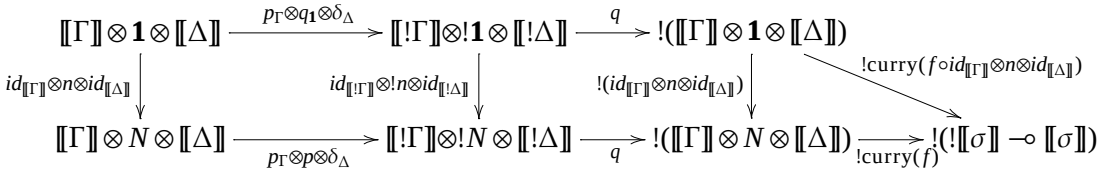
where the first row follows by definition of substitution, the second row follows since ϱ is a natural isomorphism, the third row follows by Proposition 3 and by observing that $\langle \mathbf{1}, \varrho^{-1}, id_{\mathbf{1}} \rangle$ is trivially a commutative comonoid and the fourth row follows by interpretation.

- case **(ℓif)**. The only interesting case is $\Gamma, \Delta, \mathtt{x}^{\iota} \vdash \ell\mathtt{if}\ \mathtt{M}\ \mathtt{L}\ \mathtt{R}$ direct consequence of $\Gamma \vdash \mathtt{M} : \iota$ and $\Delta, \mathtt{x}^{\iota} \vdash \mathtt{L} : \iota$ and $\Delta, \mathtt{x}^{\iota} \vdash \mathtt{R} : \iota$. Thus we have

$$\begin{aligned}
&[\![\Gamma, \Delta \vdash (\ell\mathtt{if}\ \mathtt{M}\ \mathtt{L}\ \mathtt{R})[\underline{\mathtt{n}}/\mathtt{x}] : \iota]\!] = [\![\Gamma, \Delta \vdash \ell\mathtt{if}\ \mathtt{M}\ \mathtt{L}[\underline{\mathtt{n}}/\mathtt{x}]\ \mathtt{R}[\underline{\mathtt{n}}/\mathtt{x}] : \iota]\!] \\
&= \ell if \circ [\![\Gamma \vdash \mathtt{M} : \iota]\!] \otimes \langle [\![\Delta, \mathtt{x}^{\iota} \vdash \mathtt{L} : \iota]\!] \circ id_{[\![\Delta]\!]} \otimes n, [\![\Delta, \mathtt{x}^{\iota} \vdash \mathtt{R} : \iota]\!] \circ id_{[\![\Delta]\!]} \otimes n \rangle \\
&= \ell if \circ [\![\Gamma \vdash \mathtt{M} : \iota]\!] \otimes \langle [\![\Delta, \mathtt{x}^{\iota} \vdash \mathtt{L} : \iota]\!], [\![\Delta, \mathtt{x}^{\iota} \vdash \mathtt{R} : \iota]\!] \rangle \circ id_{[\![\Delta]\!]} \otimes n \\
&= [\![\Gamma, \Delta, \mathtt{x}^{\iota} \vdash \ell\mathtt{if}\ \mathtt{M}\ \mathtt{L}\ \mathtt{R} : \iota]\!] \circ id_{[\![\Gamma]\!]} \otimes id_{[\![\Delta]\!]} \otimes n
\end{aligned}$$

where the first row follows by definition of substitution, the second row follows by induction, the third row follows by naturality of pairing and the fourth row follows by interpretation.

- case **(sc)** and **(sw)** are straightforward.
- case **(μ)**. Then $\Gamma^{\iota}, \mathtt{x}^{\iota}, !\Delta \vdash \mu F.\mathtt{M} : \sigma$ is direct consequence of $\Gamma^{\iota}, \mathtt{x}^{\iota}, !\Delta, F^{\sigma} \vdash \mathtt{M} : \sigma$. Let $f = [\![\Gamma^{\iota}, \mathtt{x}^{\iota}, !\Delta, F^{\sigma} \vdash \mathtt{M} : \sigma]\!]$. Then this case follows from the commutativity of the following diagram.



where the square on the left commutes by definition of $\mathcal{S}\ell\lambda$ Linear Category, the central square commutes since $\langle !, q \rangle$ is a monoidal functor and the triangle on the right commutes since the category is monoidal closed and by functoriality. Thus, we have

$$\begin{aligned}
[\![\Gamma^{\iota}, \Delta^{*} \vdash \mu F.\mathtt{M}[\underline{\mathtt{n}}/\mathtt{x}] : \sigma]\!] &= fix_{[\![\sigma]\!]} \circ !\mathrm{curry}([\![\Gamma^{\iota}, \Delta^{*}, F^{\sigma} \vdash \mathtt{M}[\underline{\mathtt{n}}/\mathtt{x}] : \sigma]\!]) \circ q \circ (p_\Gamma \otimes \delta_\Delta) \\
&= fix_{[\![\sigma]\!]} \circ !\mathrm{curry}(f \circ id_{[\![\Gamma]\!]} \otimes n \otimes id_{[\![\Delta]\!]}) \circ q \circ (p_\Gamma \otimes \delta_\Delta) \\
&= fix_{[\![\sigma]\!]} \circ !\mathrm{curry}(f) \circ q \circ (p_\Gamma \otimes p \otimes \delta_\Delta) \circ (id_{[\![\Gamma]\!]} \otimes n \otimes id_{[\![\Delta]\!]}) \\
&= [\![\Gamma^{\iota}, \mathtt{x}^{\iota}, \Delta^{*} \vdash \mu F.\mathtt{M} : \sigma]\!] \circ (id_{[\![\Gamma]\!]} \otimes n \otimes id_{[\![\Delta]\!]})
\end{aligned}$$

where the first row follows by interpretation, the second row follows by induction, the third row follows by commutativity of the above diagram and the fourth row follows again by interpretation. □

Lemma 4. *Let* $\mathtt{M}, \mathtt{N}$ *be such that* $\Gamma, \mathtt{f}^{\sigma} \vdash \mathtt{M} : \tau$ *and* $\Delta \vdash \mathtt{N} : \sigma$, *with* $\Gamma \cap \Delta = \emptyset$. *Then:*

$$[\![\Gamma, \Delta \vdash \mathtt{M}[\mathtt{N}/\mathtt{f}] : \tau]\!] = [\![\Gamma, \mathtt{f}^{\sigma} \vdash \mathtt{M} : \tau]\!] \circ id_{[\![\Gamma]\!]} \otimes [\![\Delta \vdash \mathtt{N} : \sigma]\!]$$

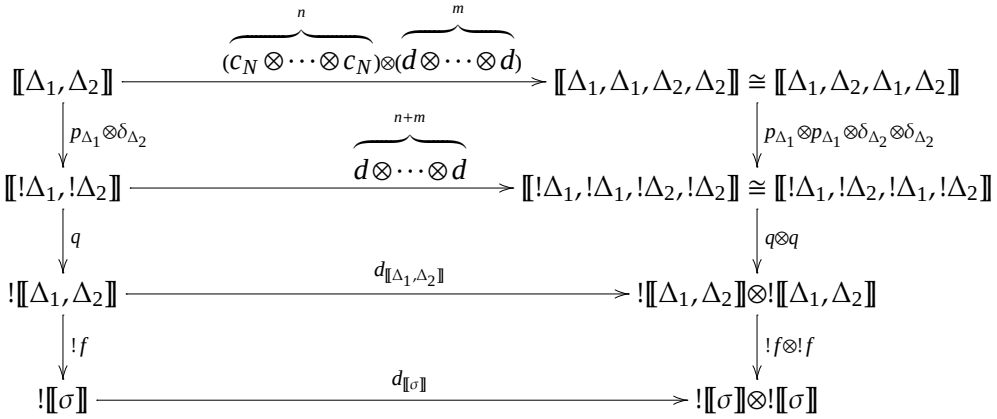
Proof. The proof is by induction on the derivation proving $\Gamma, \mathtt{f}^{\sigma} \vdash \mathtt{M} : \tau$. All the cases are straightforward. It suffices to observe that the operations on morphisms induced by the interpretation are natural in the interpretation of the unchanged components of the sequent. □

Lemma 5. *Let* $\mathtt{M},\mathtt{N}$ *be such that* $\Gamma,F^\sigma \vdash \mathtt{M}:\tau$ *and* $\Delta_1^\iota,\Delta_2^* \vdash \mathtt{N}:\sigma$, *with* $\Gamma\cap\Delta_1\cap\Delta_2=\emptyset$. *Then:*

$$[\![\Gamma,\Delta_1^\iota,\Delta_2^* \vdash \mathtt{M}[\mathtt{N}/F]:\tau]\!] = [\![\Gamma,F^\sigma\vdash \mathtt{M}:\tau]\!]\circ(id_{[\![\Gamma]\!]}\otimes(![\![\Delta_1^\iota,\Delta_2^*\vdash \mathtt{N}:\sigma]\!]\circ q\circ(p_{\Delta_1}\otimes\delta_{\Delta_2})))$$

Proof. The proof is by induction on the derivation proving $\Gamma,F^\sigma\vdash\mathtt{M}:\tau$. All the cases are straightforward except the following.

- case **(sc)**. The only interesting case is $\Gamma,F^\sigma\vdash\mathtt{M}[F/F_1,F_2]:\tau$ direct consequence of $\Gamma,F_1^\sigma,F_2^\sigma\vdash\mathtt{M}:\tau$. If we let $f=[\![\Delta_1^\iota,\Delta_2^*\vdash\mathtt{N}:\sigma]\!]$, then this case follows by the commutativity of the following diagram (to light the notation, we omit some subscripts).

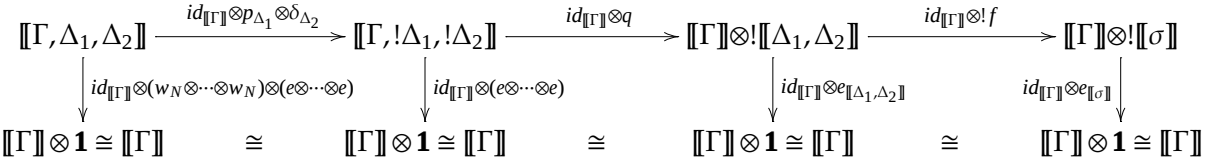


  where the first square on the top commutes since p and δ are comonoid morphisms and since the involved comonoid are commutative, the central square commutes since d is a monoidal natural transformation and the square on the bottom commutes since being $!f$ a coalgebra morphism between free coalgebra, it is also a comonoid morphism. Thus we have

$$\begin{aligned}
&[\![\Gamma,\Delta_1,\Delta_2^*\vdash\mathtt{M}[F/F_1,F_2][\mathtt{N}/F]:\tau]\!]\\
&\quad=[\![\Gamma,\Delta_1,\Delta_2,\Delta_1,\Delta_2\vdash\mathtt{M}[\mathtt{N}/F_1][\mathtt{N}/F_2]:\tau]\!]\circ\gamma\circ(id_{[\![\Gamma]\!]}\otimes(c_N\otimes\cdots\otimes c_N)\otimes(d\otimes\cdots\otimes d))\\
&\quad=[\![\Gamma,F_1^\sigma,F_2^\sigma\vdash\mathtt{M}:\tau]\!]\circ(id_{[\![\Gamma]\!]}\otimes((!f\otimes!f)\circ(q\otimes q)\circ(p_{\Delta_1}\otimes\delta_{\Delta_2}\otimes p_{\Delta_1}\otimes\delta_{\Delta_2})\circ\gamma\circ\\
&\qquad\qquad((c_N\otimes\cdots\otimes c_N)\otimes(d\otimes\cdots\otimes d))\\
&\quad=[\![\Gamma,F_1^\sigma,F_2^\sigma\vdash\mathtt{M}:\tau]\!]\circ id_{[\![\Gamma]\!]}\otimes(d\circ!f\circ q\circ p_{\Delta_1}\otimes\delta_{\Delta_2})\\
&\quad=[\![\Gamma,F^\sigma\vdash\mathtt{M}[F/F_1,F_2]]\!]\circ id_{[\![\Gamma]\!]}\otimes(!f\circ q\circ p_{\Delta_1}\otimes\delta_{\Delta_2})
\end{aligned}$$

  where the first row follows by interpretation, the second row follows by induction, the third row follows by commutativity of the above square and the fourth row follows by interpretation.

- case **(sw)**. The only interesting case is $\Gamma,F^\sigma\vdash\mathtt{M}:\tau$ direct consequence of $\Gamma\vdash\mathtt{M}:\sigma$. If we let $f=[\![\Delta_1^\iota,\Delta_2^*\vdash\mathtt{N}:\sigma]\!]$ then this case follows by commutativity of the following diagram.



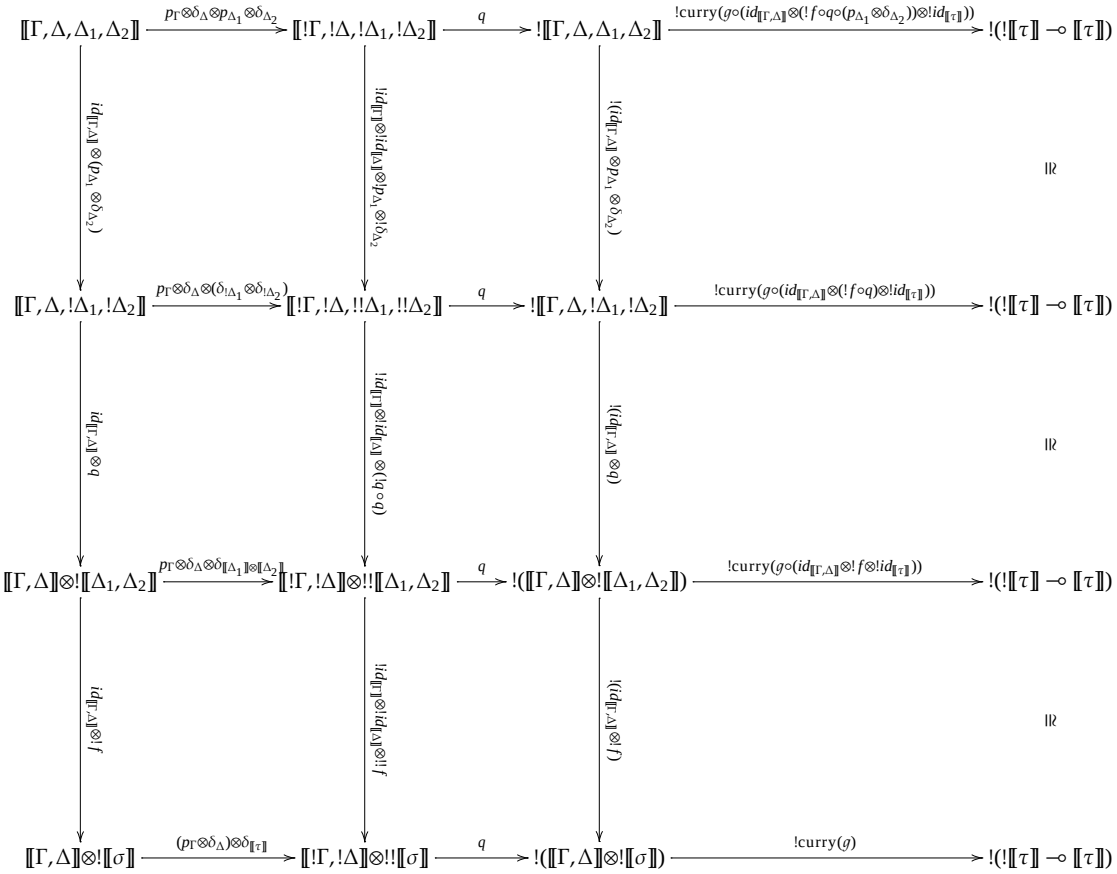


Figure 1: Diagram for the proof of Lemma 5

where the first square on the left commutes since p and δ are comonoid morphisms, the central square commutes since e is a monoidal natural transformation and the last square on the right commutes since, being $!f$ a coalgebra morphism between free coalgebras, it is also a comonoid morphism. Thus we have

$$\begin{aligned}
[\![\Gamma,\Delta_1^\iota,\Delta_2^*\vdash \mathtt{M}[\mathtt{N}/F]:\tau]\!] &= [\![\Gamma,\Delta_1^\iota,\Delta_2^*\vdash \mathtt{M}:\tau]\!]\\
&= [\![\Gamma\vdash \mathtt{M}:\tau]\!]\circ\circ\varrho\circ((w_N\otimes\cdots\otimes w_N)\otimes(e\otimes\cdots\otimes e)\otimes id_{[\![\Gamma]\!]})\\
&= [\![\Gamma\vdash \mathtt{M}:\tau]\!]\circ\varrho\circ(id_{[\![\Gamma]\!]}\otimes(e_{[\![\sigma]\!]}\circ!f\circ q\circ(p_{\Delta_1}\otimes\delta_{\Delta_2}))\\
&= [\![\Gamma,F^\sigma\vdash \mathtt{M}:\tau]\!]\circ(id_{[\![\Gamma]\!]}\otimes(!f\circ q\circ(p_{\Delta_1}\otimes\delta_{\Delta_2}))
\end{aligned}$$

where the first row follows by definition of substitution, the second row follows by interpretation, the third row follows by the commutativity of the above square and the fourth row follows by interpretation.

- case **(μ)**. Then $\Gamma^\iota,\Delta^*,F^\sigma\vdash \mu F_1.\mathtt{M}:\tau$ is direct consequence of $\Gamma^\iota,\Delta^*,F^\sigma,F_1^\tau\vdash \mathtt{M}:\tau$. If we let $f=[\![\Delta_1^\iota,\Delta_2^*\vdash \mathtt{N}:\sigma]\!]$ and $g=[\![\Gamma^\iota,\Delta^*,F^\sigma,F_1^\tau\vdash \mathtt{M}:\tau]\!]$, then this case follows by commutativity of the diagram in Figure 1. For its commutativity, we use the fact that p and δ are coalgebras, δ and q are monoidal natural transformation and $!f$ is a morphism between free coalgebras.

Hence, we can conclude the following

$$\begin{aligned}
[\![\Gamma^{\iota}, \Delta^{*}, \Delta_1^{\iota}, \Delta_2^{*} \vdash \mu F_1.\mathtt{M}[\mathtt{N}/F] : \tau]\!] &= fix_{[\![\tau]\!]} \circ !\mathrm{curry}([\![\Gamma, \Delta, \Delta_1, \Delta_2, F_1^{\tau} \vdash \mathtt{M}[\mathtt{N}/F] : \tau]\!]) \circ q \circ \\
&\qquad (p_\Gamma \otimes \delta_\Delta \otimes p_{\Delta_1} \otimes \delta_{\Delta_2}) \\
&= fix_{[\![\tau]\!]} \circ !\mathrm{curry}(g \circ (id_{[\![\Gamma,\Delta]\!]} \otimes (!f \circ q \circ (p_{\Delta_1} \otimes \delta_{\Delta_2})) \otimes !id_{[\![\tau]\!]})) \circ q \circ (p_\Gamma \otimes \delta_\Delta \otimes p_{\Delta_1} \otimes \delta_{\Delta_2}) \\
&= fix_{[\![\tau]\!]} \circ !\mathrm{curry}(g) \circ q \circ (p_\Gamma \otimes \delta_\Delta \otimes \delta_{[\![\tau]\!]}) \circ (id_{[\![\Gamma,\Delta]\!]} \otimes (!f \circ q \circ (p_{\Delta_1} \otimes \delta_{\Delta_2}))) \\
&= [\![\Gamma, \Delta, F^{\sigma} \vdash \mu F_1.\mathtt{M} : \tau]\!] \circ (id_{[\![\Gamma,\Delta]\!]} \otimes (!f \circ q \circ (p_{\Delta_1} \otimes \delta_{\Delta_2})))
\end{aligned}$$

where the first row follows by interpretation, the second row follows by induction, the third row follows by the commutativity of the diagram in Figure 1 and the fourth row follows again by interpretation. □

Given the above substitution lemmas we can now prove the soundness of the interpretation

Theorem 4 (Soundness)**.** *Let* $\mathtt{M},\mathtt{N}$ *such that* $\Gamma \vdash \mathtt{M} : \sigma$ *and* $\Gamma \vdash \mathtt{N} : \sigma$.

$$\textit{If } \mathtt{M} =_{\mathscr{L}} \mathtt{N} \textit{ then } [\![\Gamma \vdash \mathtt{M} : \sigma]\!] = [\![\Gamma \vdash \mathtt{N} : \sigma]\!]$$

Proof. The proof is by induction on the derivation of $\mathtt{M} =_{\mathscr{L}} \mathtt{N}$. We develop a few cases.

- case $\mathtt{M} = (\lambda \mathtt{f}^{\tau}.\mathtt{M}_1)\mathtt{M}_2$ and $\mathtt{N} = \mathtt{M}_1[\mathtt{M}_2/\mathtt{f}]$. Then we have

$$\begin{aligned}
[\![\Gamma, \Delta \vdash \mathtt{M} : \sigma]\!] &= \mathrm{eval} \circ (\mathrm{curry}([\![\Gamma, \mathtt{f}^{\tau} \vdash \mathtt{M}_1 : \sigma]\!]) \otimes [\![\Delta \vdash \mathtt{M}_2 : \tau]\!]) \\
&= [\![\Gamma, \mathtt{f}^{\tau} \vdash \mathtt{M}_1 : \sigma]\!] \circ (id_{[\![\Gamma]\!]} \otimes [\![\Delta \vdash \mathtt{M}_2 : \tau]\!]) \\
&= [\![\Gamma, \Delta \vdash \mathtt{M}_1[\mathtt{M}_2/\mathtt{f}] : \sigma]\!] \\
&= [\![\Gamma, \Delta \vdash \mathtt{N} : \sigma]\!]
\end{aligned}$$

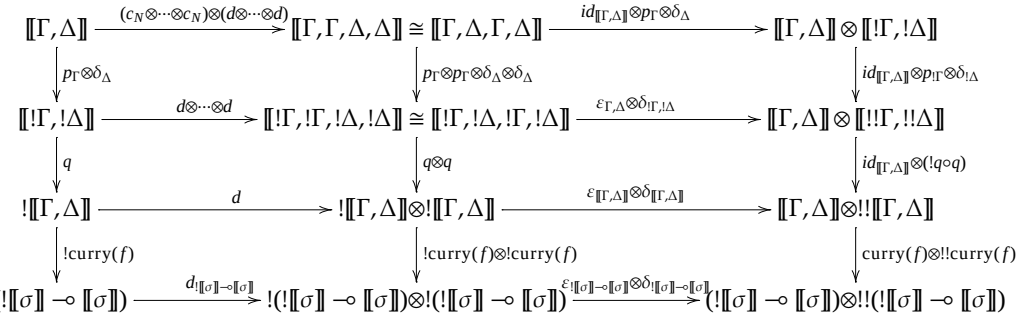
  where in the third line we use Lemma 4

- case $\mathtt{M} = (\lambda \mathtt{x}^{\iota}.\mathtt{M}_1)\underline{\mathtt{n}}$ and $\mathtt{N} = \mathtt{M}_1[\underline{\mathtt{n}}/\mathtt{x}^{\iota}]$. Then we have

$$\begin{aligned}
[\![\Gamma \vdash \mathtt{M} : \sigma]\!] &= \mathrm{eval} \circ (\mathrm{curry}([\![\Gamma, \mathtt{x}^{\iota} \vdash \mathtt{M}_1 : \sigma]\!]) \otimes n) \\
&= [\![\Gamma, \mathtt{x}^{\iota} \vdash \mathtt{M}_1 : \sigma]\!] \circ (id_{[\![\Gamma]\!]} \otimes n) \\
&= [\![\Gamma \vdash \mathtt{M}_1[\underline{\mathtt{n}}/\mathtt{x}] : \sigma]\!] \\
&= [\![\Gamma \vdash \mathtt{N} : \sigma]\!]
\end{aligned}$$

  where in the third line we use Lemma 3.

- case $\mathtt{M} = \mu F.\mathtt{M}_1$ and $\mathtt{N} = \mathtt{M}_1[\mu F.\mathtt{M}_1/F]$. First of all, if we let $f = [\![\Gamma^{\iota}, \Delta^{*}, F^{\sigma} \vdash \mathtt{M}_1 : \sigma]\!]$, let us observe that the following diagram commutes



  where the left square on the top commutes since p and δ are comonoid morphisms, the

right square on the top commutes since p and δ are coalgebras (observe that we used both commutative diagrams of the definition of coalgebra) and by bifunctoriality of $\otimes$, the left square on the middle commutes since d is a monoidal natural transformation, the right square on the middle commutes since δ and ε are monoidal natural transformations, and finally the two squares on the bottom commutes respectively because being !curry(f) a coalgebra morphism between free coalgebra, it is also a comonoid morphism, by naturality of ε and δ and by bifunctoriality of $\otimes$. Thus, we have,

$$\begin{aligned}
&[\![\Gamma^{\iota}, \Delta^{*} \vdash \mathtt{M} : \sigma]\!] = fix_{[\![\sigma]\!]} \circ !\mathrm{curry}(f) \circ q \circ (p_{\Gamma} \otimes \delta_{\Delta}) \\
&= \mathrm{eval} \circ (\varepsilon_{![\![\sigma]\!] \multimap [\![\sigma]\!]} \otimes (! fix_{[\![\sigma]\!]} \circ \delta_{![\![\sigma]\!] \multimap [\![\sigma]\!]})) \circ d_{![\![\sigma]\!] \multimap [\![\sigma]\!]} \circ !\mathrm{curry}(f) \circ q \circ (p_{\Gamma} \otimes \delta_{\Delta}) \\
&= \mathrm{eval} \circ (\mathrm{curry}(f) \otimes (! fix_{[\![\sigma]\!]} \circ !!\mathrm{curry}(f) \circ (!q \circ q) \circ (p_{!\Gamma} \otimes \delta_{!\Delta})) \circ (p_{\Gamma} \otimes \delta_{\Delta})) \\
&\qquad\qquad \circ ((c_N \otimes \cdots \otimes c_N) \otimes (d \otimes \cdots \otimes d)) \\
&= f \circ id_{[\![\Gamma, \Delta]\!]} \otimes (![\![\Gamma^{\iota}, \Delta^{*} \vdash \mu F.\mathtt{M} : \sigma]\!] \circ q \circ (p_{\Gamma} \otimes \delta_{\Delta})) \circ ((c_N \otimes \cdots \otimes c_N) \otimes (d \otimes \cdots \otimes d)) \\
&= [\![\Gamma^{\iota}, \Delta^{*}, \Gamma^{\iota}, \Delta^{*} \vdash \mathtt{N} : \sigma]\!] \circ ((c_N \otimes \cdots \otimes c_N) \otimes (d \otimes \cdots \otimes d))
\end{aligned}$$

where in the second line we use the fix-point law, in the third line we use the commutativity of the above diagram, in the fourth line we use the definition of interpretation and the naturality of q and since the category is monoidal closed and finally in the fifth line we use Lemma 5. Then we can conclude by interpretation. □

5.3 Lack of Completeness

In the previous section we have proved the soundness of the interpretation with respect to the equivalence $=_{\mathscr{S}\ell}$. Now, a natural question is whether this interpretation is also *complete* with respect to this equivalence or not. That is, does $[\![\Gamma \vdash \mathtt{M} : \sigma]\!] = [\![\Gamma \vdash \mathtt{N} : \sigma]\!]$ imply $\mathtt{M} =_{\mathscr{S}\ell} \mathtt{N}$?
The answer is negative. To understand why, let us consider the judgment

$$\Gamma, \Delta \vdash (\lambda \mathtt{x}^{\iota}.\mathtt{M})\mathtt{N} : \tau$$

where $\Gamma \vdash \lambda \mathtt{x}.\mathtt{M} : \iota \multimap \tau$ and $\Delta \vdash \mathtt{N} : \iota$. The interpretation of this judgment is

$$[\![\Gamma, \Delta \vdash (\lambda \mathtt{x}^{\iota}.\mathtt{M})\mathtt{N} : \tau]\!] = \mathrm{eval} \circ ([\![\Gamma \vdash \lambda \mathtt{x}^{\iota}.\mathtt{M} : \iota \multimap \tau]\!] \otimes [\![\Delta \vdash \mathtt{N} : \iota]\!])$$

The term `eval` above represents the standard evaluation morphism of the symmetric monoidal closed category. So, since $[\![\Gamma \vdash \lambda \mathtt{x}^{\iota}.\mathtt{M} : \iota \multimap \tau]\!] = \mathtt{curry}([\![\Gamma, \mathtt{x}^{\iota} \vdash \mathtt{M} : \tau]\!])$, in particular it is easy to verify that the above interpretation is equal to

$$[\![\Gamma, \Delta \vdash \mathtt{M}[\mathtt{N}/\mathtt{x}^{\iota}] : \tau]\!]$$

So, we clearly have:

$$[\![\Gamma, \Delta \vdash (\lambda \mathtt{x}^{\iota}.\mathtt{M})\mathtt{N} : \tau]\!] = [\![\Gamma, \Delta \vdash \mathtt{M}[\mathtt{N}/\mathtt{x}^{\iota}] : \tau]\!]$$

Unfortunately, in the $\mathscr{S}\ell\lambda$-calculus we have

$$(\lambda \mathtt{x}^{\iota}.\mathtt{M})\mathtt{N} \neq_{\mathscr{S}\ell} \mathtt{M}[\mathtt{N}/\mathtt{x}^{\iota}]$$

unless $\mathtt{N}$ is a numeral. So we have a counterexample to completeness. For this reason, in the next section we introduce a second interpretation exploiting the coalgebraic properties of the promotion in order to recover the completeness.

6 An exponential interpretation

In this section we give the second interpretation for the $\mathcal{S}\ell\lambda$-calculus. This interpretation exploit the use of the coalgebraic properties of the coalgebra morphism p in order to simulate the calculus reduction. We prove the interpretation to be sound with respect to the equivalence $=_{\mathcal{S}\ell}$. Moreover, we also show the interpretation to be complete with respect to the operational equivalence $\sim_\sigma$. We prove this result by first giving an embedding from $\mathcal{S}\ell\lambda$-terms in an extension of the intuitionistic linear term calculus and then by using a computability argument. As a corollary we obtain also the completeness with respect to the smallest equivalence $=_{\mathcal{S}\ell}$ for the fix-point free fragment of the $\mathcal{S}\ell\lambda$-calculus.

6.1 Categorical $\mathcal{S}\ell\lambda_!$-model

We introduce here the second notion of categorical model for the $\mathcal{S}\ell\lambda$-calculus.

Definition 26 (Categorical $\mathcal{S}\ell\lambda_!$-model)**.** *A categorical $\mathcal{S}\ell\lambda_!$-model consists of*

- *A $\mathcal{S}\ell\lambda$ Category $\langle \mathcal{L}, N, p, c_N, w_N, \ell if, fix\rangle$, where $\mathcal{L} = \langle \mathbb{L}, !, \delta, \varepsilon, q, e, d\rangle$.*
- *A mapping associating to every $\mathcal{S}\ell\lambda$-type σ, an object $[\![\sigma]\!]^!$ of $\mathbb{L}$ such that $[\![\iota]\!]^! = !N$ and $[\![\sigma \multimap \tau]\!]^! = [\![\sigma]\!]^! \multimap [\![\tau]\!]^!$.*
- *Given a basis Γ we define $[\![\Gamma]\!]^!$ by induction as $[\![\emptyset]\!]^! = \mathbf{1}$, $[\![\mathtt{x}^\sigma, \Delta]\!]^! = [\![\sigma]\!]^! \otimes [\![\Delta]\!]^!$ and $[\![F^\sigma, \Delta]\!]^! = ![\![\sigma]\!]^! \otimes [\![\Delta]\!]^!$.*
  Given a term $\mathtt{M}$ such that $\Gamma \vdash \mathtt{M} : \sigma$ we associate it a morphism $[\![\Gamma \vdash \mathtt{M} : \sigma]\!]^! : [\![\Gamma]\!]^! \to [\![\sigma]\!]^!$, such that:
  - $[\![\vdash \underline{\mathtt{0}} : \iota]\!]^! = !0 \circ q$, $[\![\vdash \mathtt{succ} : \iota \multimap \iota]\!]^! = \mathrm{curry}(!succ)$, $[\![\vdash \mathtt{pred} : \iota \multimap \iota]\!]^! = \mathrm{curry}(!pred)$
  - $[\![\mathtt{x}^\sigma \vdash \mathtt{x} : \sigma]\!]^! = id_{[\![\sigma]\!]^!}$
  - $[\![F^\sigma \vdash F : \sigma]\!]^! = \varepsilon_{[\![\sigma]\!]^!}$
  - $[\![\Gamma \vdash \lambda \mathtt{x}^\sigma.\mathtt{M} : \tau]\!]^! = \mathrm{curry}([\![\Gamma, \mathtt{x}^\sigma \vdash \mathtt{M} : \tau]\!]^!)$
  - $[\![\Gamma^\iota, \Delta^* \vdash \mu F.\mathtt{M} : \sigma]\!]^! = fix_{[\![\sigma]\!]^!} \circ !\mathrm{curry}([\![\Gamma^\iota, \Delta^*, F^\sigma \vdash \mathtt{M} : \sigma]\!]^!) \circ q \circ \delta_{\Gamma,\Delta}$
  - $[\![\Gamma, \varkappa_1^{\sigma_1}, \varkappa_2^{\sigma_2}, \Delta \vdash \mathtt{M} : \tau]\!]^! = [\![\Gamma, \varkappa_2^{\sigma_2}, \varkappa_1^{\sigma_1}, \Delta \vdash \mathtt{M} : \tau]\!]^! \circ (id_{[\![\Gamma]\!]^!} \otimes \gamma_{[\![\sigma_1]\!]^!, [\![\sigma_2]\!]^!} \otimes id_{[\![\Delta]\!]^!})$
  - $[\![\Gamma, \Delta \vdash \mathtt{M}^{\sigma\multimap\tau}\mathtt{N}^\sigma : \tau]\!]^! = \begin{cases} \mathrm{eval} \circ ([\![\Gamma \vdash \mathtt{M} : \sigma \multimap \tau]\!]^! \otimes (p \circ \varepsilon_N \circ [\![\Delta \vdash \mathtt{N} : \sigma]\!]^!)) & \text{if } \sigma = \iota \\ \mathrm{eval} \circ ([\![\Gamma \vdash \mathtt{M} : \sigma \multimap \tau]\!]^! \otimes [\![\Delta \vdash \mathtt{N} : \sigma]\!]^!) & \text{otherwise} \end{cases}$.
  - $[\![\Gamma, \Delta \vdash \ell\mathtt{if}\,\mathtt{M}\,\mathtt{L}\,\mathtt{R} : \iota]\!]^! = \ell if \circ ([\![\Gamma \vdash \mathtt{M} : \iota]\!]^! \otimes \langle [\![\Delta \vdash \mathtt{L} : \iota]\!]^!, [\![\Delta \vdash \mathtt{R} : \iota]\!]^!\rangle)$.
  - $[\![\Gamma, \mathtt{x}^\iota \vdash \mathtt{M}[\mathtt{x}/\mathtt{x}_1, \mathtt{x}_2] : \tau]\!]^! = [\![\Gamma, \mathtt{x}_1{}^\iota, \mathtt{x}_2{}^\iota \vdash \mathtt{M} : \tau]\!]^! \circ id_{[\![\Gamma]\!]^!} \otimes d_N$
  - $[\![\Gamma, F^\sigma \vdash \mathtt{M}[F/F_1, F_2] : \tau]\!]^! = [\![\Gamma, F_1{}^\sigma, F_2{}^\sigma \vdash \mathtt{M} : \tau]\!]^! \circ id_{[\![\Gamma]\!]^!} \otimes d_{[\![\sigma]\!]^!}$
  - $[\![\Gamma, \mathtt{x}^\iota \vdash \mathtt{M} : \tau]\!]^! = [\![\Gamma \vdash \mathtt{M} : \tau]\!]^! \circ id_{[\![\Gamma]\!]^!} \otimes e_N$
  - $[\![\Gamma, F^\sigma \vdash \mathtt{M} : \tau]\!]^! = [\![\Gamma \vdash \mathtt{M} : \tau]\!]^! \circ id_{[\![\Gamma]\!]^!} \otimes e_{[\![\sigma]\!]^!}$

The interpretation above mainly differs from the one given in Section 5 for the interpretation given to the type ι and for the interpretation of the application in the case the function argument is of ground type. These are the key ingredients that allow us to obtain the completeness results.

6.2 Soundness

The soundness proof follows the structure of the one given in Section 5. Some adaptation are however needed. In particular, we need the following substitution lemma for numerals (analogous of Lemma 3).

Lemma 6. *Let* $\mathtt{M}$ *be such that* $\Gamma, \mathtt{x}^{\iota}, \Delta \vdash \mathtt{M} : \sigma$. *Then:*

$$[\![\Gamma, \Delta \vdash \mathtt{M}[\underline{\mathtt{n}}/\mathtt{x}] : \sigma]\!]^{!} = [\![\Gamma, \mathtt{x}^{\iota}, \Delta \vdash \mathtt{M} : \sigma]\!]^{!} \circ (id_{[\![\Gamma]\!]^{!}} \otimes (!n \circ q) \otimes id_{[\![\Delta]\!]^{!}})$$

Proof. The proof is by induction on the derivation of $\Gamma, \mathtt{x}^{\iota} \vdash \mathtt{M} : \sigma$ and by cases on the last applied rule.

- case **(ex)**. Straightforward.
- case **(gv)**. Obvious, since $[\![\mathtt{x}[\underline{\mathtt{n}}/\mathtt{x}]]\!]^{!} = !id_N \circ !n \circ q = !n \circ q$ as expected.
- case **(λ)**. Then $\Gamma, \mathtt{x}^{\iota} \vdash \lambda \mathtt{f}.\mathtt{M} : \sigma \multimap \tau$ is direct consequence of $\Gamma, \mathtt{f}^{\sigma} \vdash \mathtt{M} : \tau$. Thus, we have

$$\begin{aligned}
[\![\Gamma, \mathtt{x}^{\iota} \vdash \lambda \mathtt{f}.\mathtt{M}[\underline{\mathtt{n}}/\mathtt{x}] : \sigma \multimap \tau]\!]^{!} &= \mathrm{curry}([\![\Gamma, \mathtt{f}^{\sigma} \vdash \mathtt{M}[\underline{\mathtt{n}}/\mathtt{x}] : \tau]\!]^{!}) \\
&= \mathrm{curry}([\![\Gamma, \mathtt{x}^{\iota}, \mathtt{f}^{\sigma} \vdash \mathtt{M} : \tau]\!]^{!} \circ (id_{[\![\Gamma]\!]^{!}} \otimes (!n \circ q) \otimes id_{\sigma})) \\
&= \mathrm{curry}([\![\Gamma, \mathtt{x}^{\iota}, \mathtt{f}^{\sigma} \vdash \mathtt{M} : \tau]\!]^{!}) \circ (id_{[\![\Gamma]\!]^{!}} \otimes (!n \circ q)) \\
&= [\![\Gamma, \mathtt{x}^{\iota} \vdash \lambda \mathtt{f}.\mathtt{M} : \sigma \multimap \tau]\!]^{!} \circ (id_{[\![\Gamma]\!]^{!}} \otimes ((!n \circ q))
\end{aligned}$$

  where the first row follows by interpretation, the second row follows by induction, the third row follows by naturality of curry(−) and the fourth row follows again by interpretation.
- case **(ap)**. This case follows by induction and by functoriality of $\otimes$.
- case **(gc)**. The only interesting case is $\Gamma, \mathtt{x}^{\iota} \vdash \mathtt{M}[\mathtt{x}/\mathtt{x}_1, \mathtt{x}_2] : \sigma$ consequence of $\Gamma, \mathtt{x}_1{}^{\iota}, \mathtt{x}_2{}^{\iota} \vdash \mathtt{M} : \sigma$. Thus we have

$$\begin{aligned}
[\![\Gamma \vdash \mathtt{M}[\mathtt{x}/\mathtt{x}_1, \mathtt{x}_2][\underline{\mathtt{n}}/\mathtt{x}] : \sigma]\!]^{!} &= [\![\Gamma \vdash \mathtt{M}[\underline{\mathtt{n}}/\mathtt{x}_1, \underline{\mathtt{n}}/\mathtt{x}_2] : \sigma]\!]^{!} \\
&= [\![\Gamma, \mathtt{x}_1{}^{\iota}, \mathtt{x}_2{}^{\iota} \vdash \mathtt{M} : \sigma]\!]^{!} \circ id_{[\![\Gamma]\!]^{!}} \otimes (!n \circ q) \otimes (!n \circ q) \circ \varrho^{-1} \\
&= [\![\Gamma, \mathtt{x}_1{}^{\iota}, \mathtt{x}_2{}^{\iota} \vdash \mathtt{M} : \sigma]\!]^{!} \circ (id_{[\![\Gamma]\!]^{!}} \otimes d_N) \circ (id_{[\![\Gamma]\!]^{!}} \otimes (!n \circ q)) \\
&= [\![\Gamma, \mathtt{x}^{\iota} \vdash \mathtt{M} : \sigma]\!]^{!} \circ id_{[\![\Gamma]\!]^{!}} \otimes (!n \circ q)
\end{aligned}$$

  where the first row follows by definition of substitution, the second row follows by induction, the third row follows by definition of Linear Category (namely by comonoidality of $!n$) and the fourth row follows by interpretation.
- case **(gw)**. The only interesting case is $\Gamma, \mathtt{x}^{\iota} \vdash \mathtt{M} : \sigma$ consequence of $\Gamma \vdash \mathtt{M} : \sigma$. Thus we have

$$\begin{aligned}
[\![\Gamma \vdash \mathtt{M}[\underline{\mathtt{n}}/\mathtt{x}] : \sigma]\!]^{!} &= [\![\Gamma \vdash \mathtt{M} : \sigma]\!]^{!} \\
&= [\![\Gamma \vdash \mathtt{M} : \sigma]\!]^{!} \circ \varrho \circ id_{[\![\Gamma]\!]^{!}} \otimes id_{\mathbf{1}} \circ \varrho^{-1} \\
&= [\![\Gamma \vdash \mathtt{M} : \sigma]\!]^{!} \circ \varrho \circ (id_{[\![\Gamma]\!]^{!}} \otimes e_N) \circ (id_{[\![\Gamma]\!]^{!}} \otimes (!n \circ q)) \circ \varrho^{-1} \\
&= [\![\Gamma, \mathtt{x}^{\iota} \vdash \mathtt{M} : \sigma]\!]^{!} \circ id_{[\![\Gamma]\!]^{!}} \otimes (!n \circ q) \circ \varrho^{-1}
\end{aligned}$$

  where the first row follows by definition of substitution, the second row follows since ϱ is a natural isomorphism, the third row follows by definition of Linear Category (namely by comonoidality of $!n$) and by observing that $\langle \mathbf{1}, \varrho^{-1}, id_{\mathbf{1}} \rangle$ is trivially a commutative comonoid and the fourth row follows by interpretation.

- case **(ℓif)**. The only interesting case is $\Gamma,\Delta,\mathtt{x}^\iota \vdash \ell\mathtt{if}\,\mathtt{M}\,\mathtt{L}\,\mathtt{R} : \iota$ direct consequence of $\Gamma \vdash \mathtt{M} : \iota$ and $\Delta,\mathtt{x}^\iota \vdash \mathtt{L} : \iota$ and $\Delta,\mathtt{x}^\iota \vdash \mathtt{R} : \iota$. Thus we have

$$\begin{aligned}
&[\![\Gamma,\Delta \vdash (\ell\mathtt{if}\,\mathtt{M}\,\mathtt{L}\,\mathtt{R})[\underline{\mathtt{n}}/\mathtt{x}] : \iota]\!]^! = [\![\Gamma,\Delta \vdash \ell\mathtt{if}\,\mathtt{M}\,\mathtt{L}[\underline{\mathtt{n}}/\mathtt{x}]\,\mathtt{R}[\underline{\mathtt{n}}/\mathtt{x}] : \iota]\!]^! \\
&= \ell if \circ [\![\Gamma \vdash \mathtt{M} : \iota]\!]^! \otimes \langle [\![\Delta,\mathtt{x}^\iota \vdash \mathtt{L} : \iota]\!]^! \circ id_{[\![\Delta]\!]^!} \otimes (!n \circ q), [\![\Delta,\mathtt{x}^\iota \vdash \mathtt{R} : \iota]\!]^! \circ id_{[\![\Delta]\!]^!} \otimes (!n \circ q)\rangle \\
&= \ell if \circ [\![\Gamma \vdash \mathtt{M} : \iota]\!]^! \otimes \langle [\![\Delta,\mathtt{x}^\iota \vdash \mathtt{L} : \iota]\!]^!, [\![\Delta,\mathtt{x}^\iota \vdash \mathtt{R} : \iota]\!]^! \rangle \circ id_{[\![\Delta]\!]^!} \otimes (!n \circ q) \\
&= [\![\Gamma,\Delta,\mathtt{x}^\iota \vdash \ell\mathtt{if}\,\mathtt{M}\,\mathtt{L}\,\mathtt{R} : \iota]\!]^! \circ id_{[\![\Gamma]\!]^!} \otimes id_{[\![\Delta]\!]^!} \otimes (!n \circ q)
\end{aligned}$$

  where the first row follows by definition of substitution, the second row follows by induction, the third row follows by naturality of pairing and the fourth row follows by interpretation.

- case **(sc)** and **(sw)** are straightforward.

- case **(μ)**. Then $\Gamma^\iota,\mathtt{x}^\iota,!\Delta \vdash \mu F.\mathtt{M} : \sigma$ is direct consequence of $\Gamma^\iota,\mathtt{x}^\iota,!\Delta,F^\sigma \vdash \mathtt{M} : \sigma$. Let $f = [\![\Gamma^\iota,\mathtt{x}^\iota,!\Delta,F^\sigma \vdash \mathtt{M} : \sigma]\!]^!$. Then we have

$$\begin{aligned}
[\![\Gamma^\iota,\mathtt{x}^\iota,!\Delta \vdash \mu F.\mathtt{M}[\underline{\mathtt{n}}/x] : \sigma]\!]^! &= fix_{[\![\sigma]\!]^!} \circ !\mathrm{curry}(f \circ id_\Gamma \otimes (!n \circ q) \otimes id_\Delta) \circ q \circ \delta_{\Gamma,\Delta} \\
&= fix_{[\![\sigma]\!]^!} \circ !\mathrm{curry}(f) \circ !(!id_\Gamma \otimes !(n \circ q) \otimes !id_\Delta) \circ q \circ \delta_{\Gamma,\Delta} \\
&= fix_{[\![\sigma]\!]^!} \circ !\mathrm{curry}(f) \circ q \circ (!!id_\Gamma \otimes !!n \otimes !!id_\Delta) \circ (\delta_\Gamma \otimes q \otimes \delta_\Delta) \\
&= fix_{[\![\sigma]\!]^!} \circ !\mathrm{curry}(f) \circ q \circ \delta_{\Gamma,\mathtt{x},\Delta} \circ (id_\Gamma \otimes (!n \circ q) \otimes id_\Delta) \\
&= [\![\Gamma,x,\Delta \vdash \mu F.\mathtt{M} : \sigma]\!]^! \circ (id_\Gamma \otimes (!n \circ q) \otimes id_\Delta)
\end{aligned}$$

  where the first equality follows by induction, the second equality follows by naturality of curry(), the third equality follows by naturality of q, the fourth equality follows by naturality of δ and the fifth equality follows by definition of interpretation. □

Similarly, we have the following substitution lemmas for linear and stable variables analogous of Lemma 4 and Lemma 5, respectively.

Lemma 7. *Let* $\mathtt{M},\mathtt{N}$ *be such that* $\Gamma,\mathtt{f}^\sigma \vdash \mathtt{M} : \tau$ *and* $\Delta \vdash \mathtt{N} : \sigma$*, with* $\Gamma \cap \Delta = \emptyset$*. Then:*

$$[\![\Gamma,\Delta \vdash \mathtt{M}[\mathtt{N}/\mathtt{f}] : \tau]\!]^! = [\![\Gamma,\mathtt{f}^\sigma \vdash \mathtt{M} : \tau]\!]^! \circ id_{[\![\Gamma]\!]^!} \otimes [\![\Delta \vdash \mathtt{N} : \sigma]\!]^!$$

Proof. The proof is similar to the proof of Lemma 4. □

Lemma 8. *Let* $\mathtt{M},\mathtt{N}$ *be such that* $\Gamma,F^\sigma \vdash \mathtt{M} : \tau$ *and* $\Delta_1^\iota,\Delta_2^* \vdash \mathtt{N} : \sigma$*, with* $\Gamma \cap \Delta_1 \cap \Delta_2 = \emptyset$*. Then:*

$$[\![\Gamma,\Delta_1^\iota,\Delta_2^* \vdash \mathtt{M}[\mathtt{N}/F] : \tau]\!]^! = [\![\Gamma,F^\sigma \vdash \mathtt{M} : \tau]\!]^! \circ (id_{[\![\Gamma]\!]^!} \otimes (![\![\Delta_1^\iota,\Delta_2^* \vdash \mathtt{N} : \sigma]\!]^! \circ q \circ (p_{\Delta_1} \otimes \delta_{\Delta_2})))$$

Proof. The proof is similar to the proof of Lemma 5. □

With the help of the three above substitution lemmas we can now prove the soundness of the $\mathcal{S}\ell\lambda_!$-model interpretation.

Theorem 5 (Soundness)**.** *Let* $\mathtt{M},\mathtt{N}$ *such that* $\Gamma \vdash \mathtt{M} : \sigma$ *and* $\Gamma \vdash \mathtt{N} : \sigma$*.*

$$\textit{If } \mathtt{M} =_{\mathscr{L}} \mathtt{N} \textit{ then } [\![\Gamma \vdash \mathtt{M} : \sigma]\!]^! = [\![\Gamma \vdash \mathtt{N} : \sigma]\!]^!$$

Proof. The proof is similar to the proof of Theorem 4 and proceeds by induction on the derivation of $\mathtt{M} =_{\mathcal{S}\ell} \mathtt{N}$. The only interesting case is when $\mathtt{M} = (\lambda \mathtt{x}^{\iota}.\mathtt{M}_1)\underline{\mathtt{n}}$ and $\mathtt{N} = \mathtt{M}_1[\underline{\mathtt{n}}/\mathtt{x}^{\iota}]$. Then we have

$$\begin{aligned}
[\![\Gamma \vdash \mathtt{M} : \sigma]\!]^{!} &= \mathrm{eval} \circ (\mathrm{curry}([\![\Gamma, \mathtt{x}^{\iota} \vdash \mathtt{M}_1 : \sigma]\!]^{!}) \otimes (p \circ \varepsilon_N \circ !n \circ q)) \\
&= [\![\Gamma, \mathtt{x}^{\iota} \vdash \mathtt{M}_1 : \sigma]\!]^{!} \circ (p \circ \varepsilon_N \circ !n \circ q) \\
&= [\![\Gamma, \mathtt{x}^{\iota} \vdash \mathtt{M}_1 : \sigma]\!]^{!} \circ (p \circ n) \\
&= [\![\Gamma, \mathtt{x}^{\iota} \vdash \mathtt{M}_1 : \sigma]\!]^{!} \circ (!n \circ q) \\
&= [\![\Gamma \vdash \mathtt{N} : \sigma]\!]^{!}
\end{aligned}$$

where the first equality follows by interpretation, the second equality follows by monoidal closeness, the third equality follows by naturality of ε and coalgebraicity, the fourth equality follows by coalgrebraicity of n and the fifth equality follows by Lemma 6 □

6.3 Enriching the Intuitionistic Linear Term Calculus

We here extend the Benton et al. intuitionistic linear term calculus that we have recalled in Section 3.5 by means of additional term constructions making it a syntactic model corresponding to $\mathcal{S}\ell\lambda$-categories. In particular, we add constructions to deal with numerals, conditional and fix-points. In order to deal with numerals, besides the usual $\underline{\mathtt{0}}$, succ and pred constants, we need constructions making the type ι a comonoid and a syntactic version of the morphism p.

Definition 27. *The terms of the* Enriched Intuitionistic Linear Term Calculus *can be obtained by extending the intuitionistic linear term calculus in the following way:*

$$\mathtt{M} \quad ::= \quad \cdots \mid \underline{\mathtt{0}} \mid \mathtt{succ} \mid \mathtt{pred} \mid \mathtt{Y}^{\sigma} \mid \ell\mathtt{if}\ \mathtt{M}\ \mathtt{M} \mid \mathtt{promote}^{\iota}(\mathtt{M}) \mid \mathtt{discard}^{\iota}\ \mathtt{M}\ \mathtt{in}\ \mathtt{M} \mid \mathtt{copy}^{\iota}\ \mathtt{M}\ \mathtt{as}\ \mathtt{x}_1, \mathtt{x}_2\ \mathtt{in}\ \mathtt{M}$$

Types of the intuitionistic linear term calculus extended by the ground type ι can be assigned to terms by means of the type system presented in Table 5.

The enriched intuitionistic linear term calculus is equipped with a *reduction relation* $\rightarrow$ defined by extending the intuitionistic linear term calculus reduction by means of beta-reduction rules for numerals, conditional, and fix-points and by the corresponding commutative rules. We give the beta-reduction rules in Table 6. Note that the rules to deal with numerals have been designed starting from the categorical equalities.

6.4 Completeness

In this section, we prove that the enriched term model induced by the operational equivalence forms a $\mathcal{S}\ell\lambda$-category. We will prove this by defining an encoding $(\!|-|\!)$ from $\mathcal{S}\ell\lambda$-calculus to the Enriched Intuitionistic Linear Term Calculus. We will show that the encoding is injective, taking as equivalence in the Enriched Intuitionistic Linear Term Calculus, the equality induced by the reduction and conversion rules, denoted with $=_{\beta}$. The encoding is defined in the following way.

- $(\!|\vdash \underline{\mathtt{0}} : \iota|\!) = \mathtt{promote}\ -\ \mathtt{as}\ -\ \mathtt{in}\ \underline{\mathtt{0}}$
- $(\!|\vdash \mathtt{succ} : \iota \multimap \iota|\!) = \lambda \mathtt{y}.\mathtt{promote}\ \mathtt{y}\ \mathtt{as}\ \mathtt{x}\ \mathtt{in}\ (\mathtt{succ}\ \mathtt{x})$
- $(\!|\vdash \mathtt{pred} : \iota \multimap \iota|\!) = \lambda \mathtt{y}.\mathtt{promote}\ \mathtt{y}\ \mathtt{as}\ \mathtt{x}\ \mathtt{in}\ (\mathtt{pred}\ \mathtt{x})$
- $(\!|\mathtt{x}^{\sigma} \vdash \mathtt{x}|\!) = \mathtt{x}^{[\![\sigma]\!]^{!}}$

$$\frac{}{\vdash \underline{0} : \iota}\ (z) \qquad \frac{}{\vdash \mathtt{succ} : \iota \multimap \iota}\ (s) \qquad \frac{}{\vdash \mathtt{pred} : \iota \multimap \iota}\ (p)$$

$$\frac{}{\vdash \mathtt{Y}^{\sigma} : !(!\sigma \multimap \sigma) \multimap \sigma}\ (Y) \qquad \frac{\Gamma \cap \Delta = \emptyset \quad \Gamma \vdash \mathtt{M} : \iota \quad \Delta \vdash \mathtt{N} : \sigma \& \sigma}{\Gamma, \Delta \vdash \ell\mathtt{if}\ \mathtt{M}\ \mathtt{N} : \sigma}\ (\ell\mathtt{if})$$

$$\frac{\Gamma \vdash \mathtt{M} : \iota}{\Gamma \vdash \mathtt{promote}^{\iota}(\mathtt{M}) : !\iota}\ (\mathrm{pr}_{\iota})$$

$$\frac{\Gamma \cap \Delta = \emptyset \quad \Gamma \vdash \mathtt{M} : \iota \quad \Delta \vdash \mathtt{N} : \sigma}{\Gamma, \Delta \vdash \mathtt{discard}^{\iota}\ \mathtt{M}\ \mathtt{in}\ \mathtt{N} : \sigma}\ (\mathrm{ds}_{\iota}) \qquad \frac{\Gamma \cap \Delta = \emptyset \quad \Gamma \vdash \mathtt{M} : \iota \quad \Delta, \mathtt{x_1}^{\iota}, \mathtt{x_2}^{\iota} \vdash \mathtt{N} : \sigma}{\Gamma, \Delta \vdash \mathtt{copy}^{\iota}\ \mathtt{M}\ \mathtt{as}\ \mathtt{x_1}, \mathtt{x_2}\ \mathtt{in}\ \mathtt{N} : \sigma}\ (\mathrm{cp}_{\iota})$$

Table 5: Type Assignment rules for the Enriched Intuitionistic Linear Term Calculus

Promotion on numerals is comonoidal

$\mathtt{discard}\ (\mathtt{promote}^{\iota}\ \mathtt{M})\ \mathtt{in}\ \mathtt{N} = \mathtt{discard}^{\iota}\ \mathtt{M}\ \mathtt{in}\ \mathtt{N}$

$\mathtt{copy}\ (\mathtt{promote}^{\iota}\ \mathtt{M})\ \mathtt{as}\ \mathtt{x}^{!\iota}, \mathtt{y}^{!\iota}\ \mathtt{in}\ \mathtt{N} = \mathtt{copy}^{\iota}\ \mathtt{M}\ \mathtt{as}\ \mathtt{x}^{\iota}, \mathtt{y}^{\iota}\ \mathtt{in}\ \mathtt{N}[(\mathtt{promote}^{\iota}\ \mathtt{x})/\mathtt{x}, (\mathtt{promote}^{\iota}\ \mathtt{y})/\mathtt{y}]$

Promotion on numerals is a !*-coalgebra*

$\mathtt{derelict}\ (\mathtt{promote}^{\iota}\ \mathtt{M}) = \mathtt{M}$

$\mathtt{promote}\ (\mathtt{promote}^{\iota}\ \mathtt{M})\ \mathtt{as}\ \mathtt{z}\ \mathtt{in}\ \mathtt{N}[\mathtt{promote}^{\iota}\ (\mathtt{derelict}\ \mathtt{z})/\mathtt{z}] = \mathtt{promote}(\mathtt{promote}^{\iota}\ \mathtt{M})\ \mathtt{as}\ \mathtt{z}\ \mathtt{in}\ \mathtt{N}$

Numerals are coalgebric

$\mathtt{promote}^{\iota}\ \underline{\mathtt{n}} = \mathtt{promote}\ -\ \mathtt{as}\ -\ \mathtt{in}\ \underline{\mathtt{n}}$

The type ι *is a comonoid*

$\mathtt{copy}^{\iota}\ \mathtt{M}\ \mathtt{as}\ \mathtt{x}, \mathtt{y}\ \mathtt{in}\ (\mathtt{discard}^{\iota}\ \mathtt{x}\ \mathtt{in}\ \mathtt{N})) = \mathtt{N}[\mathtt{M}/\mathtt{y}]$

$\mathtt{copy}^{\iota}\ \mathtt{M}\ \mathtt{as}\ \mathtt{x}, \mathtt{y}\ \mathtt{in}\ (\mathtt{discard}^{\iota}\ \mathtt{y}\ \mathtt{in}\ \mathtt{N})) = \mathtt{N}[\mathtt{M}/\mathtt{x}]$

$\mathtt{copy}^{\iota}\ \mathtt{M}\ \mathtt{as}\ \mathtt{x}, \mathtt{y}\ \mathtt{in}\ \mathtt{N} = \mathtt{copy}^{\iota}\ \mathtt{M}\ \mathtt{as}\ \mathtt{y}, \mathtt{x}\ \mathtt{in}\ \mathtt{N}$

$\mathtt{copy}^{\iota}\ \mathtt{M}\ \mathtt{as}\ \mathtt{x}, \mathtt{w}\ \mathtt{in}\ (\mathtt{copy}^{\iota}\ \mathtt{w}\ \mathtt{as}\ \mathtt{y}, \mathtt{z}\ \mathtt{in}\ \mathtt{N}) = \mathtt{copy}^{\iota}\ \mathtt{M}\ \mathtt{as}\ \mathtt{w}, \mathtt{z}\ \mathtt{in}\ (\mathtt{copy}^{\iota}\ \mathtt{w}\ \mathtt{as}\ \mathtt{x}, \mathtt{y}\ \mathtt{in}\ \mathtt{N})$

Commuting Conversion for numerals

$\mathtt{M}[\mathtt{discard}^{\iota}\ \mathtt{z}\ \mathtt{in}\ \mathtt{N}/\mathtt{w}] = \mathtt{discard}^{\iota}\ \mathtt{z}\ \mathtt{in}\ \mathtt{M}[\mathtt{N}/\mathtt{w}]$

$\mathtt{M}[\mathtt{copy}^{\iota}\ \mathtt{z}\ \mathtt{as}\ \mathtt{x}, \mathtt{y}\ \mathtt{in}\ \mathtt{N}/\mathtt{w}] = \mathtt{copy}^{\iota}\ \mathtt{z}\ \mathtt{as}\ \mathtt{x}, \mathtt{y}\ \mathtt{in}\ \mathtt{M}[\mathtt{N}/\mathtt{w}]$

Conditional and Fix point

$\ell\mathtt{if}\ \mathtt{0}\ \mathtt{M} = \pi_1(\mathtt{M})$

$\ell\mathtt{if}\ (\mathtt{succ}\ \mathtt{n})\ \mathtt{M} = \pi_2(\mathtt{M})$

$\mathtt{Y}^{\sigma}\mathtt{M} = \mathtt{copy}\ \mathtt{M}\ \mathtt{as}\ \mathtt{x}, \mathtt{y}\ \mathtt{in}\ (\mathtt{derelict}\ \mathtt{x})(\mathtt{promote}\ \mathtt{y}\ \mathtt{as}\ \mathtt{w}\ \mathtt{in}\ \mathtt{Y}(\mathtt{w}))$

Table 6: Beta-reduction rules for the Enriched Intuitionistic Linear Term Calculus

- $(\!|F^{\sigma} \vdash F|\!) = \mathtt{derelict}(F^{[\![\sigma]\!]^!})$
- $(\!|\Gamma \vdash \lambda \mathtt{x}^{\sigma}.\mathtt{M} : \sigma \multimap \tau|\!) = \lambda \mathtt{x}^{[\![\sigma]\!]^!}.(\!|\Gamma, \mathtt{x}^{\sigma} \vdash \mathtt{M} : \tau|\!)$
- Suppose $\Gamma = \vec{\mathtt{x}}^{\iota}, \vec{F}^{\vec{\tau}}$; then $(\!|\Gamma \vdash \mu F^{\sigma}.\mathtt{M}|\!) = \mathtt{Y}(\mathtt{promote}\ \vec{\mathtt{x}}, \vec{F}\ \mathtt{for}\ \vec{\mathtt{x}}, \vec{F}\ \mathtt{in}\ \lambda F^{[\![\sigma]\!]^!}.(\!|\Gamma, F^{\sigma} \vdash \mathtt{M} : \sigma|\!))$
- $(\!|\Gamma, \Delta \vdash \mathtt{M}^{\iota \multimap \tau}\mathtt{N} : \tau|\!) = (\!|\Gamma \vdash \mathtt{M}^{\iota \multimap \tau} : \iota \multimap \tau|\!)\ \mathtt{promote}^{\iota}(\mathtt{derelict}((\!|\Delta \vdash \mathtt{N} : \iota|\!)))$
- $(\!|\Gamma, \Delta \vdash \mathtt{M}^{\sigma \multimap \tau}\mathtt{N} : \tau|\!) = (\!|\Gamma \vdash \mathtt{M}^{\sigma \multimap \tau} : \sigma \multimap \tau|\!)(\!|\Delta \vdash \mathtt{N} : \sigma|\!)$
- $(\!|\Gamma, \Delta \vdash \ell\,\mathtt{if}\ \mathtt{M}\ \mathtt{L}\ \mathtt{R} : \iota|\!) = \ell\,\mathtt{if}\ (\!|\Gamma \vdash \mathtt{M} : \iota|\!)\ \langle (\!|\Delta \vdash \mathtt{L} : \iota|\!), (\!|\Delta \vdash \mathtt{R} : \iota|\!) \rangle$
- $(\!|\Gamma, \mathtt{x}^{\iota} \vdash \mathtt{M}[\mathtt{x}/\mathtt{x}_1, \mathtt{x}_2]|\!) = \mathtt{copy}\ \mathtt{x}^{[\![\iota]\!]^!}\ \mathtt{as}\ \mathtt{x}_1, \mathtt{x}_2\ \mathtt{in}\ (\!|\Gamma, \mathtt{x}_1{}^{\iota}, \mathtt{x}_2{}^{\iota} \vdash \mathtt{M} : \sigma|\!)$
- $(\!|\Gamma, F^{\sigma} \vdash \mathtt{M}[F/F_1, F_2] : \tau|\!) = \mathtt{copy}\ F^{[\![\sigma]\!]^!}\ \mathtt{as}\ F_1, F_2\ \mathtt{in}\ (\!|\Gamma, F_1^{\sigma}, F_2^{\sigma} \vdash \mathtt{M} : \tau|\!)$
- $(\!|\Gamma, \mathtt{x}^{\iota} \vdash \mathtt{M} : \tau|\!) = \mathtt{discard}\ \mathtt{x}^{[\![\iota]\!]^!}\ \mathtt{in}\ (\!|\Gamma \vdash \mathtt{M} : \tau|\!)$
- $(\!|\Gamma, F^{\sigma} \vdash \mathtt{M} : \tau|\!) = \mathtt{discard}\ F^{[\![\sigma]\!]^!}\ \mathtt{in}\ (\!|\Gamma \vdash \mathtt{M} : \tau|\!)$

The following lemmas are a syntactic version of Lemma 3, Lemma 4 and Lemma 5.

Lemma 9 (Substitution Lemma)**.**

1. $(\!|\Gamma \vdash \mathtt{M}[\underline{\mathtt{n}}/\mathtt{x}^{\iota}] : \sigma|\!) =_{\beta} (\!|\Gamma, \mathtt{x}^{\iota} \vdash \mathtt{M} : \sigma|\!)[(\!|\underline{\mathtt{n}}|\!)/\mathtt{x}^{\iota}]$
2. $(\!|\Gamma, \Delta \vdash \mathtt{M}[\mathtt{N}/\mathtt{f}^{\sigma}] : \tau|\!) =_{\beta} (\!|\Gamma, \mathtt{f}^{\sigma} \vdash \mathtt{M} : \tau|\!)[(\!|\Delta \vdash \mathtt{N} : \sigma|\!)/\mathtt{f}^{\sigma}]$
3. $(\!|\Gamma, \vec{\mathtt{x}}^{\vec{\iota}}, \vec{F}^{\vec{\sigma}} \vdash \mathtt{M}[\mathtt{N}/F^{\tau}] : \eta|\!) =_{\beta} (\!|\Gamma, F^{\sigma} \vdash \mathtt{M} : \eta|\!)[\mathtt{promote}\ \vec{\mathtt{x}}, \vec{F}\ \mathtt{for}\ \vec{\mathtt{x}}, \vec{F}\ \mathtt{in}\ (\!|\vec{\mathtt{x}}^{\vec{\iota}}, \vec{F}^{\vec{\sigma}} \vdash \mathtt{N} : \tau|\!)/F]$

Proof. (1) is proved by induction on the derivation of $\Gamma, \mathtt{x}^{\iota} \vdash \mathtt{M} : \sigma$. (2) is proved by induction on the derivation of $\Gamma, \mathtt{f}^{\sigma} \vdash \mathtt{M} : \tau$. (3) is proved by induction on the derivation of $\Gamma, F^{\tau} \vdash \mathtt{M} : \eta$. □

Proposition 4. $\mathtt{M} =_{\mathcal{S}\ell} \mathtt{N}$ *implies* $(\!|\mathtt{M}|\!) =_{\beta} (\!|\mathtt{N}|\!)$.

Proof. By induction on the derivation $\mathtt{M} =_{\mathcal{S}\ell} \mathtt{N}$. The non-trivial case is the case $\mathtt{M} = \mu F^{\sigma}.\mathtt{M}_1$ and $\mathtt{N} = \mathtt{M}[\mu F^{\sigma}.\mathtt{M}_1/F]$, with $\vec{\mathtt{x}}^{\vec{\iota}}, \vec{F}^{\vec{\tau}} \vdash \mathtt{M} : \alpha$. Here we have

$(\!|\vec{\mathtt{x}}^{\vec{\iota}}, \vec{F}^{\vec{\tau}} \vdash \mathtt{M} : \alpha|\!) = \mathtt{Y}(\mathtt{promote}\ \vec{\mathtt{x}}, \vec{F}\ \mathtt{for}\ \vec{\mathtt{x}}, \vec{F}\ \mathtt{in}\ \lambda F.(\!|\vec{\mathtt{x}}^{\vec{\iota}}, \vec{F}^{\vec{\tau}}, F^{\sigma} \vdash \mathtt{M}_1 : \eta|\!))$
$= \mathtt{copy}(\mathtt{promote}\ \vec{\mathtt{x}}, \vec{F}\ \mathtt{for}\ \vec{\mathtt{x}}, \vec{F}\ \mathtt{in}\ \lambda F.(\!|\vec{\mathtt{x}}^{\vec{\iota}}, \vec{F}^{\vec{\tau}}, F^{\sigma} \vdash \mathtt{M}_1 : \eta|\!))\ \mathtt{as}\ \mathtt{z}, \mathtt{w}\ \mathtt{in}\ (\mathtt{derelict}\ \mathtt{z})(\mathtt{promote}\ \mathtt{w}\ \mathtt{as}\ \mathtt{w}'\ \mathtt{in}\ \mathtt{Y}(\mathtt{w}'))$
$= \mathtt{copy}\ \vec{\mathtt{x}}, \vec{F}\ \mathtt{as}\ \vec{\mathtt{x}'}, \vec{\mathtt{x}''}, \vec{F'}, \vec{F''}\ \mathtt{in}\ (\lambda F.(\!|\vec{\mathtt{x}}^{\vec{\iota}}, \vec{F}^{\vec{\tau}}, F^{\sigma} \vdash \mathtt{M}_1 : \eta|\!)[\vec{\mathtt{x}'}/\vec{\mathtt{x}}, \vec{F'}/\vec{F}])(\mathtt{promote}\ \vec{\mathtt{x}''}, \vec{F''}\ \mathtt{as}\ \vec{\mathtt{x}}, \vec{F}\ \mathtt{in}(\!|\vec{\mathtt{x}}^{\vec{\iota}}, \vec{F}^{\vec{\tau}} \vdash \mu F.\mathtt{M}_1 : \alpha|\!))$
$= \mathtt{copy}\ \vec{\mathtt{x}}, \vec{F}\ \mathtt{as}\ \vec{\mathtt{x}'}, \vec{\mathtt{x}''}, \vec{F'}, \vec{F''}\ \mathtt{in}\ (\!|\vec{\mathtt{x}}^{\vec{\iota}}, \vec{F}^{\vec{\tau}}, F^{\sigma} \vdash \mathtt{M}_1 : \eta|\!)[\vec{\mathtt{x}'}/\vec{\mathtt{x}}, \vec{F'}/\vec{F}, \mathtt{promote}\ \vec{\mathtt{x}''}, \vec{F''}\ \mathtt{as}\ \vec{\mathtt{x}}, \vec{F}\ \mathtt{in}(\!|\vec{\mathtt{x}}^{\vec{\iota}}, \vec{F}^{\vec{\tau}} \vdash \mu F.\mathtt{M}_1 : \alpha|\!)/F]$
$= (\!|\vec{\mathtt{x}}^{\vec{\iota}}, \vec{F}^{\vec{\tau}} \vdash \mathtt{N} : \alpha|\!)$

where the last equality follows by Lemma 9 point (3). □

By using Proposition 1, we can conclude the following statement.

Corollary 2. *If* $\mathtt{M} \in \mathcal{P}$*, then* $\mathtt{M} \Downarrow \underline{\mathtt{n}}$ *implies* $(\!|\mathtt{M}|\!) =_{\beta} (\!|\underline{\mathtt{n}}|\!)$

For simplicity, in the following we omit type annotation when clear from the context or uninteresting.

Definition 28. *The "computability predicate" is defined by the following cases.*

- *Case* $FV(\mathtt{M}^{\sigma}) = \emptyset$.
  - *Subcase* $\sigma = \iota$. $\mathrm{Comp}(\mathtt{M}^{\iota})$ *if and only if* $(\!|\mathtt{M}|\!) =_{\beta} (\!|\underline{\mathtt{n}}|\!)$ *implies* $\mathtt{M} \Downarrow \underline{\mathtt{n}}$.
  - *Subcase* $\sigma = \mu \multimap \tau$. $\mathrm{Comp}(\mathtt{M}^{\mu \multimap \tau})$ *if and only if* $\mathrm{Comp}(\mathtt{M}^{\mu \multimap \tau}\mathtt{N}^{\mu})$ *for each closed* $\mathtt{N}^{\mu}$ *such that* $\mathrm{Comp}(\mathtt{N}^{\mu})$.

- *Case* $FV(\mathtt{M}^\sigma) = \{\varkappa_1^{\tau_1}, \ldots, \varkappa_n^{\tau_n}\}$, *for some* $n \geq 1$.
  $\mathrm{Comp}(\mathtt{M}^\sigma)$ *if and only if* $\mathrm{Comp}(\mathtt{M}[\mathtt{N}_1/\varkappa_1, \ldots, \mathtt{N}_n/\varkappa_n])$ *for each closed* $\mathtt{N}_i^{\tau_i}$ *such that* $\mathrm{Comp}(\mathtt{N}_i^{\tau_i})$.

Lemma 10 states an equivalent formulation of computability predicate.

Lemma 10. *Let* $\mathtt{M}$ *be such that* $\Gamma \vdash \mathtt{M} : \tau_1 \multimap \ldots \multimap \tau_m \multimap \iota$ *and* $FV(\mathtt{M}) = \{\varkappa_1^{\mu_1}, \ldots, \varkappa_n^{\mu_n}\}$ $(n, m \in \mathbb{N})$. $\mathrm{Comp}(\mathtt{M})$ *if and only if* $(\!|\mathtt{M}[\mathtt{N}_1/\varkappa_1, \ldots, \mathtt{N}_n/\varkappa_n]\mathtt{P}_1 \ldots \mathtt{P}_m|\!) = (\!|\underline{\mathtt{n}}|\!)$ *implies* $\mathtt{M}[\mathtt{N}_1/\varkappa_1, \ldots, \mathtt{N}_n/\varkappa_n]\mathtt{P}_1 \ldots \mathtt{P}_m \Downarrow \underline{\mathtt{n}}$ *for each closed* $\mathtt{N}_i^{\mu_i}$ *and* $\mathtt{P}_j^{\tau_j}$ *such that* $\mathrm{Comp}(\mathtt{N}_i)$ *and* $\mathrm{Comp}(\mathtt{P}_j)$ *where* $i \leq n, j \leq m$.

Adequacy follows immediately by next lemmas.

Lemma 11. $\mathrm{Comp}(\mathtt{M}^\sigma)$.

Proof. The proof is by induction on $\Gamma \vdash \mathtt{M} : \sigma$. The only non trivial case is $\Gamma \vdash \lambda \mathtt{x}^\iota.\mathtt{M} : \iota \multimap \sigma$ consequence of $\Gamma, \mathtt{x}^\iota \vdash \mathtt{M} : \sigma$, with $\sigma = \sigma_1 \multimap \ldots \sigma_k \multimap \iota$. Suppose $(\!|(\lambda \mathtt{x}^\iota.\mathtt{M}[\mathtt{P}_1/\varkappa_1, \ldots, \mathtt{P}_n/\varkappa_n])\mathtt{N}\ \mathtt{N}_1 \ldots \mathtt{N}_k|\!) = (\!|\underline{\mathtt{n}}|\!)$. We suppose by simplicity that for all i, $\sigma_i \neq \iota$ (the other case is similar). This means that $(\!|\lambda \mathtt{x}^\iota.\mathtt{M}[\mathtt{P}_1/\varkappa_1, \ldots, \mathtt{P}_n\varkappa_n]|\!)(\mathtt{promote}^\iota\ \mathtt{derelict}((\!|\mathtt{N}|\!)))(\!|\mathtt{N}_1|\!) \ldots (\!|\mathtt{N}_k|\!) = (\!|\underline{\mathtt{n}}|\!)$. This happens if and only if there is $\underline{\mathtt{n}}$ such that $(\!|\mathtt{N}|\!) = \underline{\mathtt{n}}$ and $(\!|\mathtt{M}[\mathtt{P}_1/\varkappa_1, \ldots, \mathtt{P}_n\varkappa_n, \underline{\mathtt{m}}/\mathtt{x}]\mathtt{N}_1 \ldots, \mathtt{N}_k|\!) = (\!|\underline{\mathtt{n}}|\!)$: thus $\mathtt{N} \Downarrow \underline{\mathtt{n}}$ and $\mathtt{M}[\mathtt{P}_1/\varkappa_1, \ldots, \mathtt{P}_n\varkappa_n, \underline{\mathtt{m}}/\mathtt{x}]\mathtt{N}_1 \ldots, \mathtt{N}_k \Downarrow \underline{\mathtt{n}}$ by induction. So we conclude by applying the evaluation rule. The other cases are simpler: we observe that, for the fix-point case, we need to use suitable approximation theorems. □

Corollary 3. *If* $\mathtt{M} \in \mathcal{P}$, *then* $\mathtt{M} \Downarrow \underline{\mathtt{n}}$ *if and only if* $(\!|\mathtt{M}|\!) =_\beta (\!|\underline{\mathtt{n}}|\!)$

Theorem 6. $(\!|\mathtt{M}|\!) =_\beta (\!|\mathtt{N}|\!)$ *implies* $\mathtt{M} \sim \mathtt{N}$

Proof. Suppose $\mathtt{M}, \mathtt{N}$ such that $(\!|\mathtt{M}|\!) =_\beta (\!|\mathtt{N}|\!)$ and $C[\mathtt{M}] \Downarrow \underline{\mathtt{n}}$. Then $(\!|C[\mathtt{M}]|\!) =_\beta (\!|\underline{\mathtt{n}}|\!)$. This implies $(\!|C[\mathtt{N}]|\!) =_\beta (\!|\underline{\mathtt{n}}|\!)$ by hypothesis and by the fact that $=_\beta$ is a congruence. But this means that $C[\mathtt{N}] \Downarrow \underline{\mathtt{n}}$ by Corollary 3. The other direction is similar. □

Saying that the Intuitionistic Linear Term Calculus is already complete for $\mathcal{S}\ell\lambda$-category is equivalent to say that taking any two terms $\mathtt{M}, \mathtt{N}$ of the Intuitionistic Linear Term Calculus, if the interpretation of $\mathtt{M}$ is equal to the interpretation of $\mathtt{N}$ in all $\mathcal{S}\ell\lambda$-categories, then $\mathtt{M} =_\beta \mathtt{N}$. Now suppose that two terms $\mathtt{M}, \mathtt{N}$ of $\mathcal{S}\ell\lambda$-calculus are equated by the categorical interpretation in all $\mathcal{S}\ell\lambda$-category $\mathbb{C}$. This means that also $(\!|\mathtt{M}|\!)$ and $(\!|\mathtt{N}|\!)$ are equated. So we have $(\!|\mathtt{M}|\!) =_\beta (\!|\mathtt{N}|\!)$. Thus we conclude that $\mathtt{M} \sim \mathtt{N}$ by Theorem 6.

Observe that, from the above theorem it is possible to derive a stronger fact for the strongly normalizing fragment of $\mathcal{S}\ell\lambda$-calculus (i.e. the fragment without fix points), thanks to Separability Theorem. For that fragment, the equational theory on terms induced by the operational equivalence is the same as the the equational theory induced by $=_{\mathcal{S}\ell}$.

Corollary 4. *If* $\mathtt{M}$ *and* $\mathtt{N}$ *are term of the fix-point free fragment of* $\mathcal{S}\ell\lambda$*-calculus, then* $(\!|\mathtt{M}|\!) = (\!|\mathtt{N}|\!)$ *implies* $\mathtt{M} =_{\mathcal{S}\ell} \mathtt{N}$

7 Instances of $\mathcal{S}\ell\lambda$-categories

In this section, we show three interesting concrete instances of $\mathcal{S}\ell\lambda$-category, in the setting of Coherence Spaces and Scott Domains. By means of the results proved in the previous section these three instances give models in which the $\mathcal{S}\ell\lambda$-calculus can be soundly interpreted.

Let us first recall some notions about orders and sets. Given a *partial order* $\langle D, \sqsubseteq \rangle$, a subset $X \subseteq D$ is *directed* if $\forall x, x' \in X\ \exists x'' \in X$ such that $x \sqsubseteq x''$ and $x' \sqsubseteq x''$, namely for each pair of

elements of X there is an upper bound in X. If a poset D is such that for every directed $X \subseteq D$ there is $\bigsqcup X \in D$, namely a least upper bound, then it is a *directed complete partial order* (*cpo*).

A cpo is said to be *bounded complete* if for every $X \subseteq D$ having an upper bound, then $\bigsqcup X \in D$. Observe that a bounded complete cpo always admits a bottom element $\bot$. An element $d \in D$ is said to be *compact* when for all directed $X \subseteq D$, if $d \sqsubseteq \bigsqcup X$ then there is $x \in X$ such that $d \sqsubseteq x$. A cpo D is said to be ω*-algebraic* if, for every $x \in D$, the set $X = \{a \sqsubseteq x \mid a \text{ compact}\}$ is directed and $\bigsqcup X = x$.

Definition 29. *A* Scott Domain *D is an* ω-algebraic bounded complete cpo.

Let A, B be Scott Domains. A function $f : A \to B$ is *monotonic* if and only if $\forall x, x' \in A$ if $x \sqsubseteq_A x'$ then $f(x) \sqsubseteq_B f(x')$. A monotonic function $f : A \to B$ is *continuous* when for every directed set $X \subseteq A$ we have $f(\bigsqcup X) = \bigsqcup f(X)$. A continuous function $f : A \to B$ is *strict* when $f(\bot) = \bot$. Given two function $f, g : A \to B$ we write $f \sqsubseteq g$ if for all $x \in A$ we have $f(x) \sqsubseteq g(x)$. We call this order *pointwise order* or *extensional order*.

We can now present the three instances of $\mathcal{S}\ell\lambda$-categories.

7.1 Coherence Spaces

Coherence spaces have been firstly introduced by Girard in order to describe the semantics of System F. They have been also the starting point of his development of Linear Logic.

Definition 30. *A coherence space is a pair $X = \langle |X|, \frown_X \rangle$, consisting of a finite or countable set of tokens $|X|$ called* web *and a binary reflexive symmetric relation on $|X|$ called* coherence relation.

Given a coherence space X, the set of *cliques* of X is defined as

$$Cl(X) = \{x \subseteq |X| \mid a, b \in x \Rightarrow a \frown_X b\}$$

This set ordered by inclusion forms a Scott Domain whose set of finite elements is the set $Cl_{fin}(X)$ of finite cliques. Two cliques $x, y \in Cl(X)$ are *compatible* when $x \cup y \in Cl(X)$. A continuous function $f : Cl(X) \to Cl(Y)$ is *stable* when it preserves intersections of compatible cliques. A stable function $f : Cl(X) \to Cl(Y)$ is *linear* when it is strict and preserves unions of compatibles cliques. Given a linear function $f : Cl(X) \to Cl(Y)$, we denote its trace with

$$\mathrm{tr}(f) = \{(a, b) \mid b \in f(\{a\})\}$$

We say that a linear function f is less or equal than g according to the *stable order* when $\mathrm{tr}(f) \subseteq \mathrm{tr}(g)$.

Definition 31. *The category* **Coh** *is obtained by taking*

- coherence spaces *as* **objects**,
- linear functions *as* **morphisms**.

This category is monoidal closed. To see this it suffices to consider the tensor product $A \otimes B$ to be defined as the coherence space having $|X \otimes Y| = |X| \times |Y|$ as web, while $(a, b) \frown_{X \otimes Y} (a', b')$ if $a \frown_X a'$ and $b \frown_Y b'$. The tensor unit can be defined as $\mathbf{1} = \langle \{*\}, \frown_{\mathbf{1}} \rangle$ with $* \frown_{\mathbf{1}} *$. Finally, the function space $X \multimap Y$ is the coherence space having $|X \multimap Y| = |X| \times |Y|$ as web, while $(a, b) \frown_{X \multimap Y} (a', b')$ is defined as:

$$a =_X a' \text{ implies } b =_Y b' \text{ and } a \smile_X a' \text{ implies } b \smile_Y b'$$

Given a coherence space X, we can define $!X$ to be the coherence space having as web the set $Cl_{fin}(X)$ and as coherence relation, the compatibility relation between cliques. The operator ! is an exponential comonad, thus **Coh** is a Linear Category [16]. Note also that the Kleisli category over the comonad is the usual category of coherence spaces and stable maps. Moreover, we have the following.

Lemma 12. **Coh** *is an* $\mathcal{SL}\lambda$ *Category.*

Proof. It suffices to show that **Coh** satisfies all the extra conditions of $\mathcal{SL}\lambda$-categories.

- As N it suffices to take the usual flat domain of natural numbers. The coalgebra $p : N \to !N$ can be defined in such a way that:

$$\mathrm{tr}(p) = \{(n,\{n\}) \mid n \in \mathbb{N}\} \cup \{(n,\{\emptyset\}) \mid n \in \mathbb{N}\}$$

  The domain N form a commutative comonoid, by taking $w_N : N \to \mathbf{1}$ be such that $w_N(n) = *$ and $c_N : N \to N \otimes N$ be such that $c_N(n) = \langle n, n\rangle$ for all $n \neq \bot$.

- The category **Coh** is cartesian, by taking $A \& B$ to be the usual cartesian product $\&$ of Scott Domains:

$$A \& B = \{\langle a, b\rangle \mid a \in A, b \in B\}$$

  Thus we can define $\ell if : N \otimes (N \& N) \to N$ to be such that $\ell if(c) = m_1$ if $c = \langle 0, \langle m_1, m_2\rangle\rangle$, $\ell if(c) = m_2$ if $c = \langle n, \langle m_1, m_2\rangle\rangle$ with $n \neq 0$ and $\ell if(c) = \bot$ otherwise.

- Finally, it follows easily by Knaster-Tarsky's Fix-Point Theorem that the considered category admits fix-point for every object. □

The category **Coh** have been used to build the model of $\mathcal{SL}$PCF defined in [18].

7.2 Scott Domains and strict continuous functions

Scott Domains have been introduced to give a mathematically sound model of pure lambda calculus.

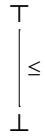
Definition 32. *The category* **StrictBcdom** *(Strict Bounded Complete Domains) is obtained by taking*

- Scott Domains *as* **objects**,
- strict continuous functions *as* **morphisms**.

This category is monoidal closed, by taking the tensor product $A \otimes B$ to be the smash product

$$A \wedge B = \{\langle a, b\rangle \mid a \in A \setminus \{\bot\}, b \in B \setminus \{\bot\}\} \cup \{\bot\}$$

the unit of the tensor product **1** to be the Sierpinsky Domain



and the function space $A \multimap B$ consisting of all strict maps between A and B under the point-wise order. Moreover we can take as exponential comonad !, the lifting constructor $(-)_\bot$ that, given

a Scott Domain A, gives a domain $A_\perp$ obtained from A by adding a new least element below the bottom of A. From this follows that **StrictBcdom** is a linear category. Note moreover that the Kleisli category over the comonad is the usual category of Scott Domains and continuous maps. Moreover, we can prove that the following holds.

Lemma 13. **StrictBcdom** *is an $\mathcal{SL}\lambda$-category.*

Proof. It suffices to show that **StrictBcdom** satisfies all the extra conditions of $\mathcal{SL}\lambda$ Categories.

- As N it suffices to take the usual flat domain of natural numbers. The coalgebra $p : N \to N_\perp$ can be defined as $p(n) = n$ for all $n \neq \perp$. Moreover, the domain N is a commutative comonoid, by taking $w_N : N \to \mathbf{1}$ be such that $w_N(n) = \top$ and $c_N : N \to N \otimes N$ be such that $c_N(n) = \langle n, n\rangle$ for all $n \neq \perp$.
- The category **StrictBcdom** is cartesian, by taking $A \times B$ to be the usual cartesian product $\times$ of Scott Domains:
$$A \times B = \{\langle a, b\rangle \mid a \in A, b \in B\}$$
Thus we can define $\ell if : N \otimes (N \times N) \to N$ to be such that $\ell if(c) = m_1$ if $c = \langle 0, \langle m_1, m_2\rangle\rangle$, $\ell if(c) = m_2$ if $c = \langle n, \langle m_1, m_2\rangle\rangle$ with $n \neq 0$ and $\ell if(c) = \perp$ otherwise.
- Finally, it follows easily by Knaster-Tarsky's Fix-Point Theorem that the considered category admits fix-point for every object. □

The model presented above have been shown to be adequate with respect to the operational semantics of $\mathcal{SL}$PCF in [20].

7.3 Scott Domains and Linear Functions

A similar construction as the one presented above for coherence spaces can be obtained also in the Scott Domain setting. First we need to introduce some further concepts.

A *linear map* is a function which preserves all existing suprema, that is $f : D_1 \to D_2$ is linear if for all bounded $X \subseteq D_1$ we have $f(\bigsqcup X) = \bigsqcup f(X)$, reminding that $\bigsqcup \emptyset = \perp$. If D is a Scott Domain, we write D_0 for its *poset of finite elements*. Note that D is obtained from D_0 by adding all suprema of directed subsets of D_0.

Definition 33. *The category* **LinBcdom** *(Linear Bounded Complete Domains) is obtained by taking*

- Scott Domains *as* **objects**,
- linear maps *as* **morphisms**.

This category is monoidal closed, since the tensor product $D_1 \otimes D_2$ classifies maps $D_1 \times D_2 \to D$ linear in each argument, while the unit of the tensor product $\mathbf{1}$ is again the Sierpinsky Domain; the linear function space $B \multimap C$ consists of all linear maps from B to C ordered pointwise.

Moreover we can describe the exponential comonad ! in terms of finite elements. Given D, we let the set $(!D)_0$ to be the set obtained from D_0 by freely adding suprema of bounded finite subsets of D_0. We complete $(!D)_0$ with all directed limits, to obtain $!D$.

From this follows that **LinBcdom** is a linear category. Note that also in this case, the Kleisli category over the comonad is the usual category of Scott Domains and continuous maps.

Moreover, we can prove that the following holds.

Lemma 14. **LinBcdom** *is an $\mathcal{SL}\lambda$-category.*

Proof. It suffices to show that **LinBcdom** satisfies all the extra conditions of $\mathcal{SL}\lambda$-categories. The only interesting case is the natural number, the other constructions are similar to the case of **StrictBcdom**.

- As N it suffices to take the usual flat domain of natural numbers. The coalgebra $p : N \to !N$ can be defined as $p(\bot) = \bot$ and $p(n) = \sqcup\{\bot, n\}$. Moreover, the domain N is a commutative comonoid, by taking $w_N : N \to \mathbf{1}$ be such that $w_N(\bot) = \bot$ and $w_N(n) = \top$, and $c_N : N \to N \otimes N$ be such that $c_N(\bot) = \bot$ and $c_N(n) = \langle n, n\rangle$. □

8 Conclusion

In the present work we have introduced the notion of $\mathcal{SL}\lambda$-category. Such a notion provide a categorical model for $\mathcal{SL}\lambda$-calculus introduced in [18]. We show that a generalization of the interpretation given in [18] is sound with respect to the $\mathcal{SL}\lambda$-calculus reduction, but not complete. To achieve completeness, we need to define another interpretation that makes explicit use of the (co)-monadic properties of !. In spite of that, a completeness result similar to Corollary 3 has been proved in [18] for a concrete instance of the first interpretation. This suggests that at the categorical level we have to few the information in order to be able to prove a completeness result in full generality also in the case of the first interpretation.

We have shown three concrete model examples in the setting of Scott Domains and Coherence Spaces. The concrete denotational models presented in Section 7 can be useful in the study of linear higher type computability [14, 17]. In this setting one interesting research theme is the study of paradigmatic programming languages fitting models founded on different higher type functionals. Such an approach, already known to Kleene, was recently rediscovered for PCF in [14] and pursued in [17]. This approach is interesting in order to compare different denotationally linear paradigmatic programming languages having different type-respecting computational power.

On this matter, we have already obtained some preliminary results. In [18] the interpretation of $\mathcal{SL}$PCF into the category **Coh** is studied and a partial full abstraction result is presented. In [8], we have extended such a result obtaining a complete full abstraction result with respect to **Coh** for a suitable extension of $\mathcal{SL}$PCF. In future works we plan to systematically extend $\mathcal{SL}$PCF with suitable operators in order to establish definability results with respect to **StrictBcdom** and **LinBcdom**.

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