CS 511: Formal Methods in Security and Privacy

Differential Privacy

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(ε,δ)-Differential Privacy

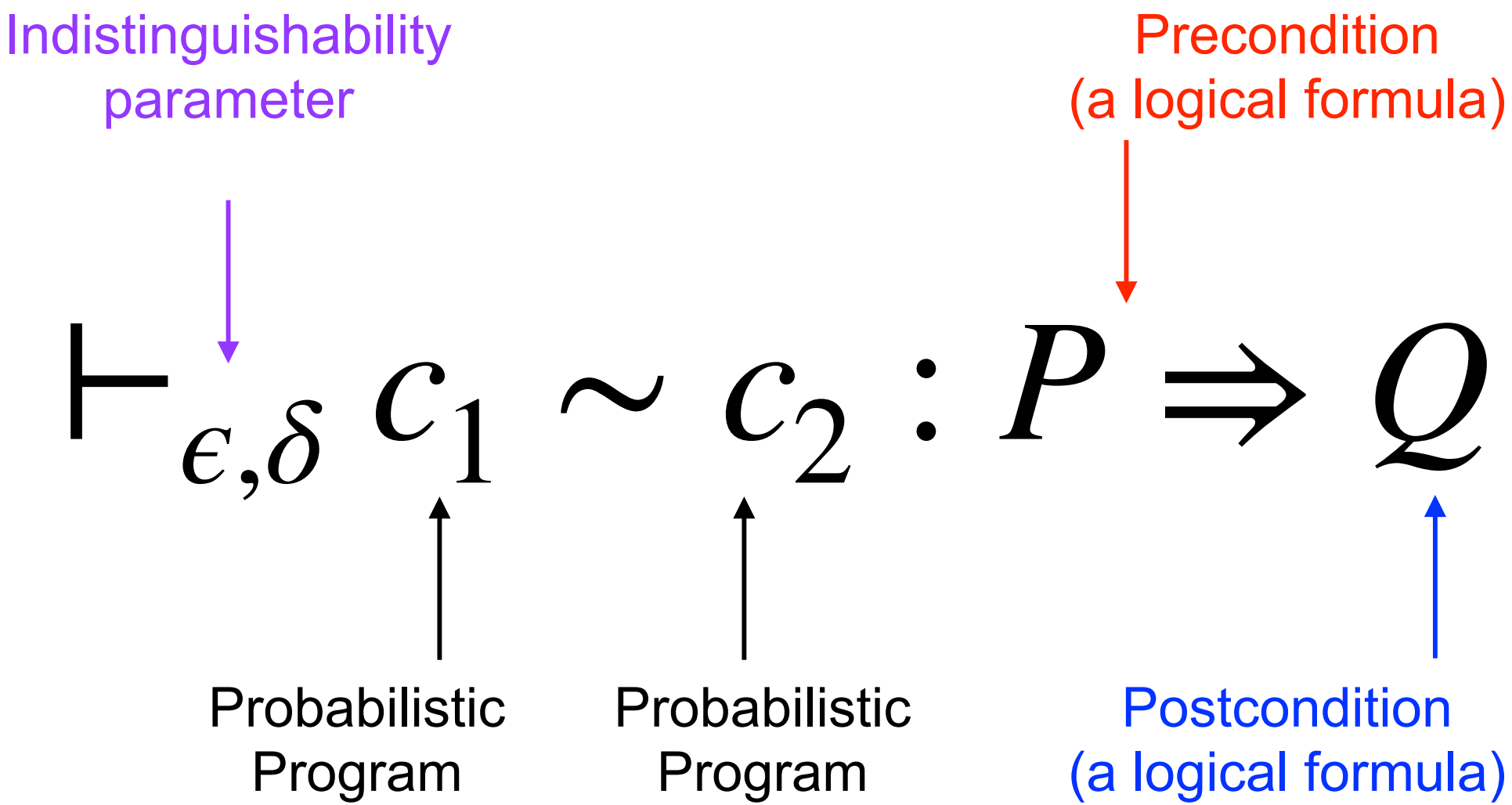
Definition

Given $\varepsilon, \delta \geq 0$, a probabilistic query $Q: X^n \rightarrow R$ is (ε,δ)-differentially private iff

for all adjacent database b_1, b_2 and for every $S \subseteq R$:

$$\Pr[Q(b_1) \in S] \leq \exp(\varepsilon)\Pr[Q(b_2) \in S] + \delta$$

apRHL


Indistinguishability parameter
Precondition (a logical formula)
$\vdash_{\epsilon,\delta} c_1 \sim c_2 : P \Rightarrow Q$
Probabilistic Program
Probabilistic Program
Postcondition (a logical formula)

apRHL: More general Lap rule (still restricted)

$$\vdash_{k*\varepsilon,0} \frac{}{x_1 :=\$\ \mathrm{Lap}(1/\varepsilon, y_1) \sim x_2 :=\$\ \mathrm{Lap}(1/\varepsilon, y_2) : |y_1 - y_2| \leq k \Rightarrow =}$$

Composition

Let M_1:DB $\rightarrow R_1$ be a (ε_1,δ_1)-differentially private program and M_2:DB $\rightarrow R_2$ be a (ε_2,δ_1)-differentially private program. Then, their composition $M_{1,2}$:DB$\rightarrow R_1 x R_2$ defined as

$$M_{1,2}(D)=(M_1(D),M_2(D))$$

is $(\varepsilon_1+\varepsilon_2,\delta_1+\delta_2)$-differentially private.

Parallel Composition

Let M_1:DB $\rightarrow$R be a (ε_1,δ_1)-differentially private program and M_2:DB $\rightarrow$R be a (ε_2,δ_2)-differentially private program. Suppose that we partition D in a data-independent way into two datasets D_1 and D_2. Then, the composition $M_{1,2}$:DB$\rightarrow$R defined as

$$MP_{1,2}(D)=(M_1(D_1),M_2(D_2))$$

is $(\max(\varepsilon_1,\varepsilon_2),\max(\delta_1,\delta_2))$-differentially private.

Probabilistic Relational Hoare Logic

Composition

$$\frac{\vdash_{\varepsilon_1,\delta_1} c_1 \sim c_2 : P \Rightarrow R \qquad \vdash_{\varepsilon_2,\delta_2} c_1' \sim c_2' : R \Rightarrow S}{\vdash_{\varepsilon_1+\varepsilon_2,\delta_1+\delta_2} c_1; c_1' \sim c_2; c_2' : P \Rightarrow S}$$

Releasing partial sums

```
DummySum(d : {0,1} list) : real list
   i:= 0;
   s:= 0;
   r:= [];
   while (i<size d)
      s:= s + d[i]
      z:=$ Lap(1/eps,s)
      r:= r ++ [z];
      i:= i+1;
   return r
```

Releasing partial sums

```
DummySum(d : {0,1} list) : real list
   i:=0;
   s:=0;
   r:=[];
   while (i<size d)
      z:=$ Lap(1/eps,d[i])
      s:= s + z
      r:= r ++ [s];
      i:= i+1;
   return r
```

More examples of differentially private programs

Histogram



(a)



(b)



(c)



(d)

Suppose that each one of us can vote for one star, and we want to say who is the star that receives most votes.

Histogram



(a)



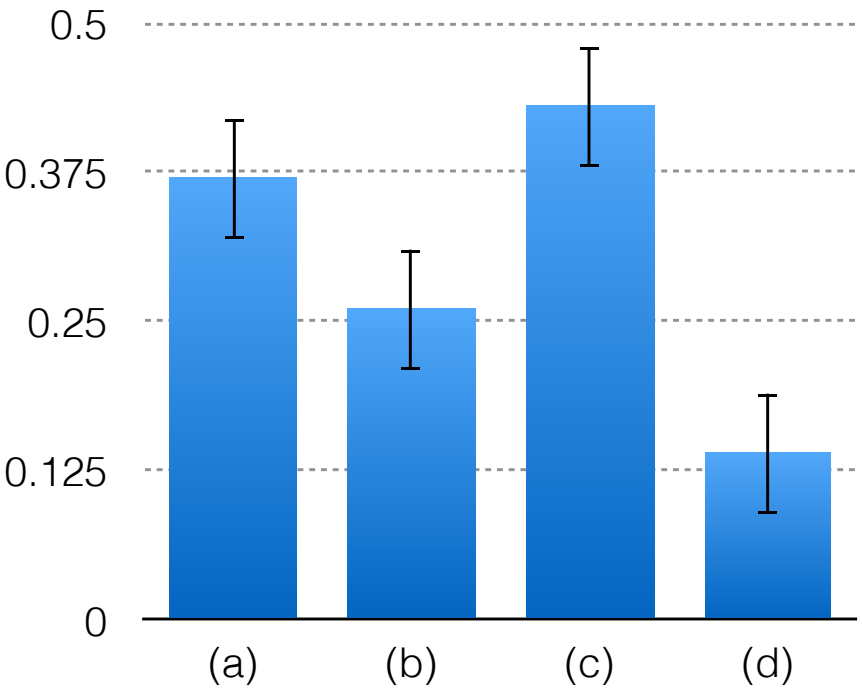
(b)



(c)

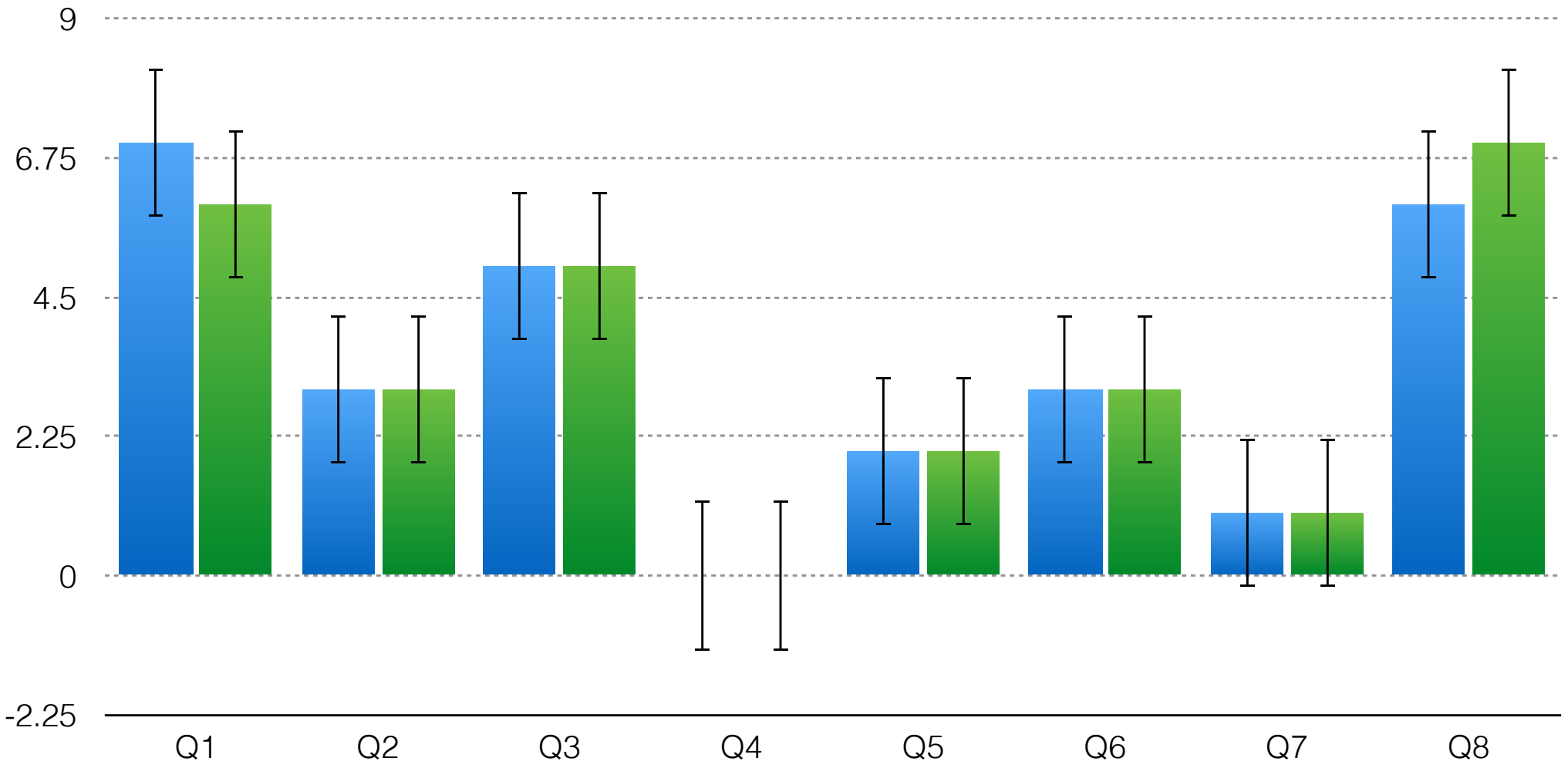


(d)



Intuition:
We can compute the histogram add Laplace noise to each score and then select the maximal noised score.

Histogram



Report Noisy Max

Given a set of queries with sensitivity l, return the index of the noised query with the max value

$q_1(D)$+noise

$q_2(D)$+noise

$q_3(D)$+noise

.....

$q_k(D)$+noise



Report Noisy Max

Given a set of queries with sensitivity l, return the index of the noised query with the max value

$q_1(D)$+noise

$q_2(D)$+noise

$q_3(D)$+noise

.....

$q_k(D)$+noise



A naive analysis gives $k\varepsilon$-differentially private

Report Noisy Max - intuition

We can prove this algorithm
ε-differentially private

Report Noisy Max - intuition

We can prove this algorithm
ε-differentially private



Databases differing
in one individual

Report Noisy Max - intuition

$q_1(D)$+noise $q_1(D')$+noise

$q_2(D)$+noise $q_2(D')$+noise

$q_3(D)$+noise $q_3(D')$+noise

....

$q_k(D)$+noise $q_k(D')$+noise



Databases differing in one individual

We can prove this algorithm ε-differentially private

Report Noisy Max - intuition

l sensitive queries

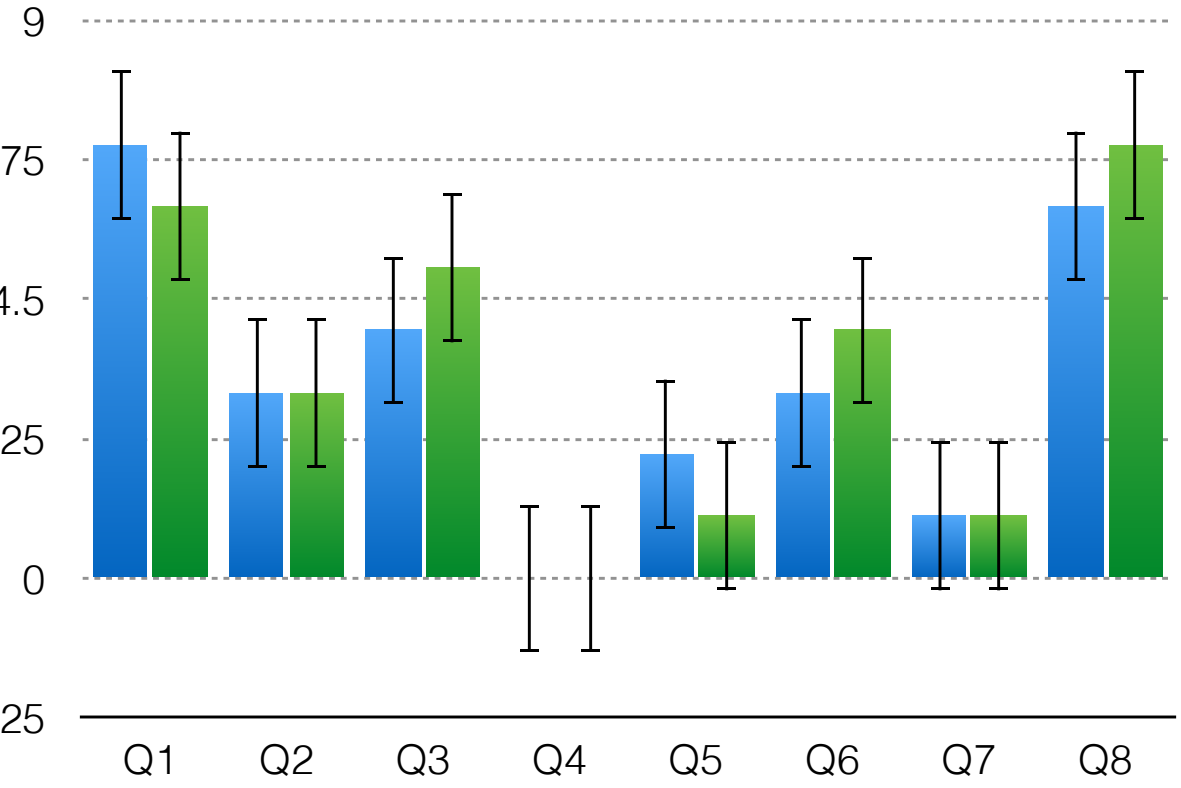
$q_1(D)$+noise	$q_1(D')$+noise
$q_2(D)$+noise	$q_2(D')$+noise
$q_3(D)$+noise	$q_3(D')$+noise
.....	.....
$q_k(D)$+noise	$q_k(D')$+noise



Databases differing in one individual

We can prove this algorithm ε-differentially private

Report Noisy Max - intuition



I sensitive queries

$q_1(D)$+noise $q_1(D')$+noise

$q_2(D)$+noise $q_2(D')$+noise

$q_3(D)$+noise $q_3(D')$+noise

....

$q_k(D)$+noise $q_k(D')$+noise

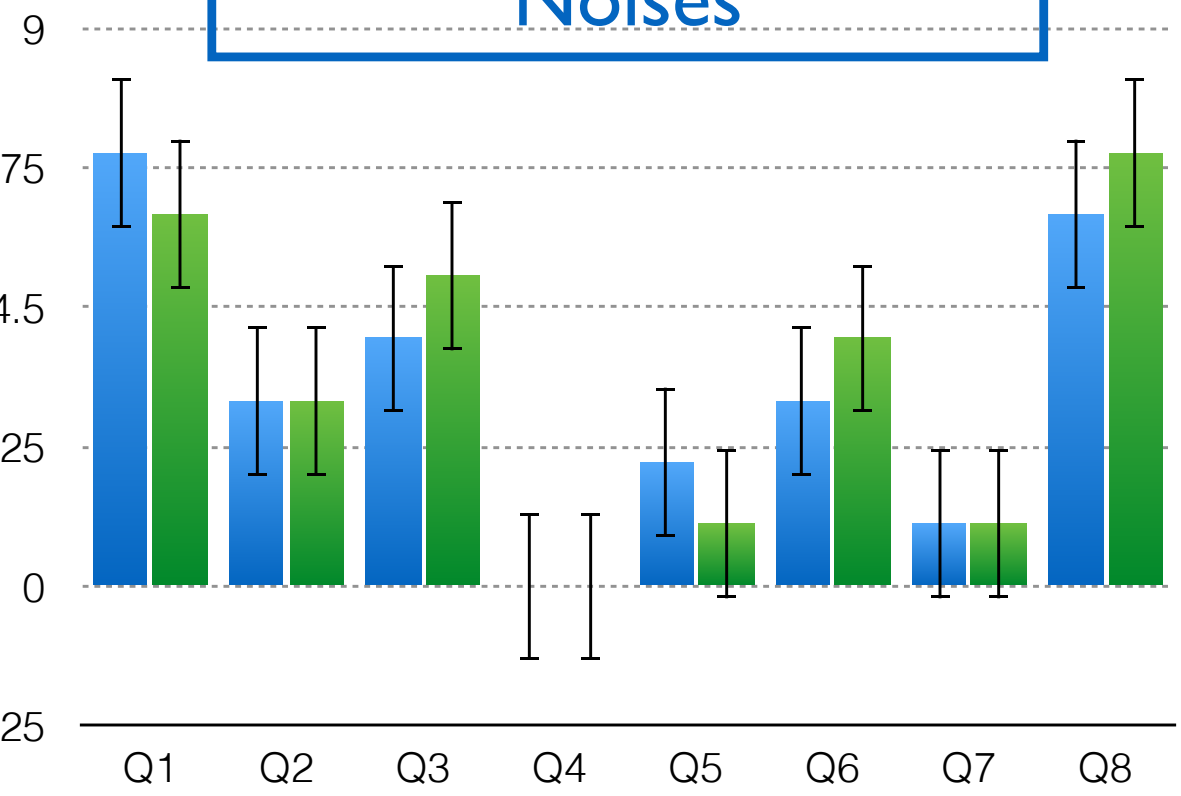


Databases differing in one individual

We can prove this algorithm ε-differentially private

Report Noisy Max - intuition

We need to coordinate Noises



We can prove this algorithm ε-differentially private

I sensitive queries

$q_1(D)$+noise $q_1(D')$+noise

$q_2(D)$+noise $q_2(D')$+noise

$q_3(D)$+noise $q_3(D')$+noise

....

$q_k(D)$+noise $q_k(D')$+noise

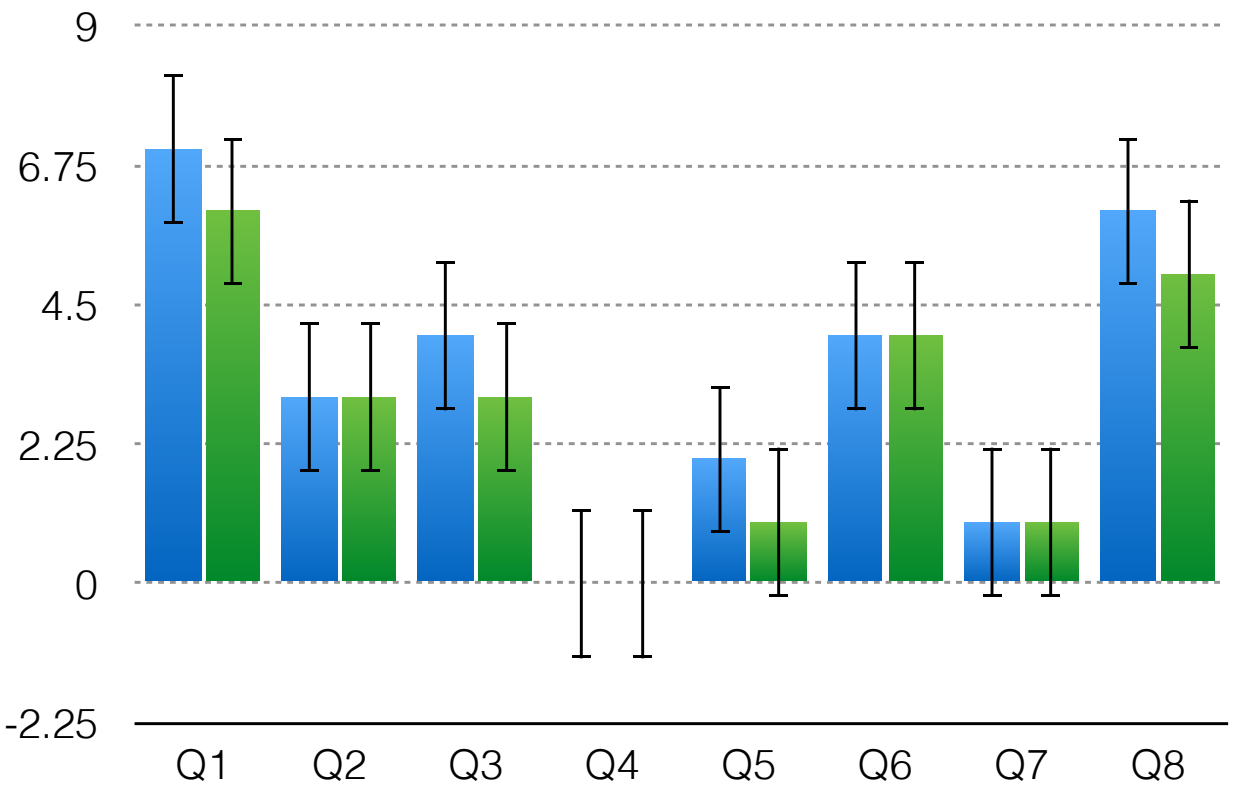


Databases differing in one individual

Report Noisy Max

```
RNM (q_1,...,q_N : (data → R) list,
     b : list data, ε: R) : nat
  i = 0;
  max = 0;
  while (i < N){
     cur = q_i(b) + Lap(1/ε)
     if (cur > max)
          max =  cur ;
          output = i;
  return output;
```

Report Noisy Max

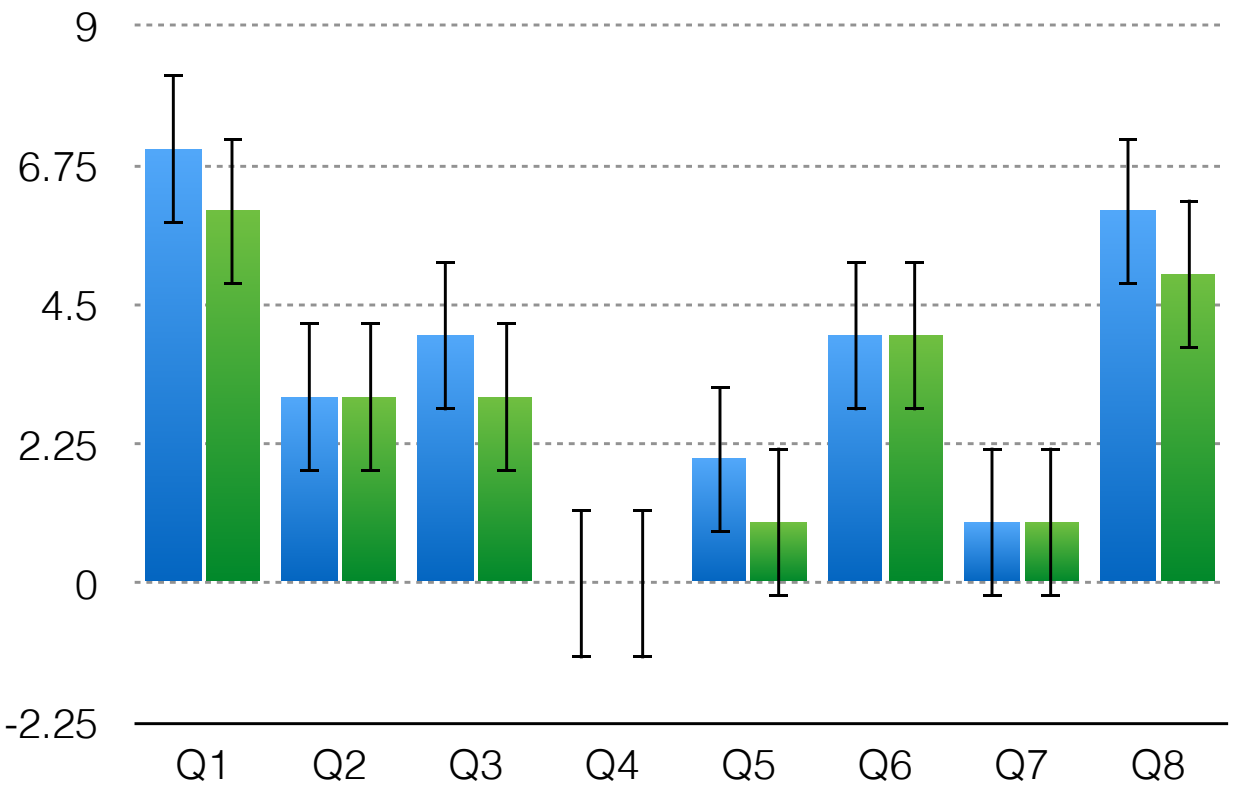


Simplifying assumptions

$c_k \geq c_k'$

$c_k' + 1 \geq c_k$

Report Noisy Max



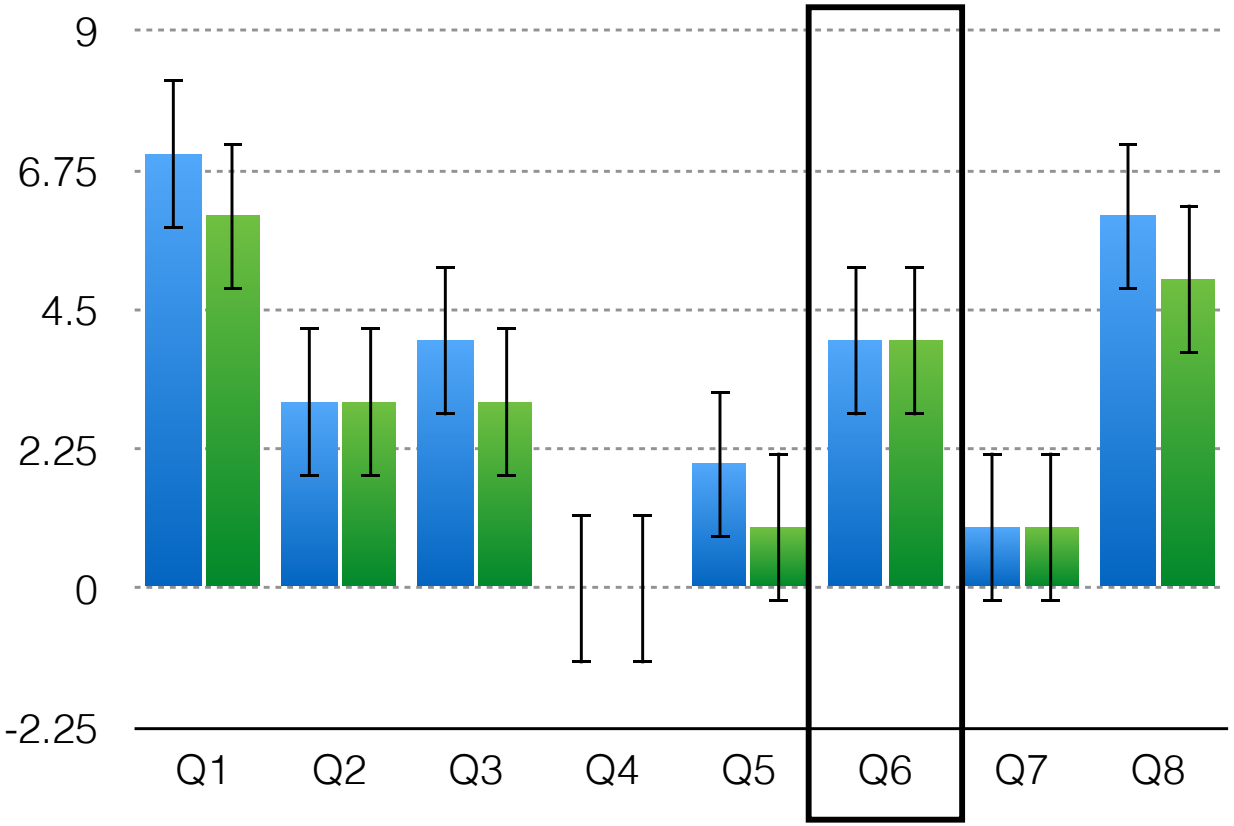
Simplifying assumptions

$c_k \geq c_k'$

$c_k' + 1 \geq c_k$

Notation r_k, r_k' noise added at round k.

Report Noisy Max



Simplifying assumptions

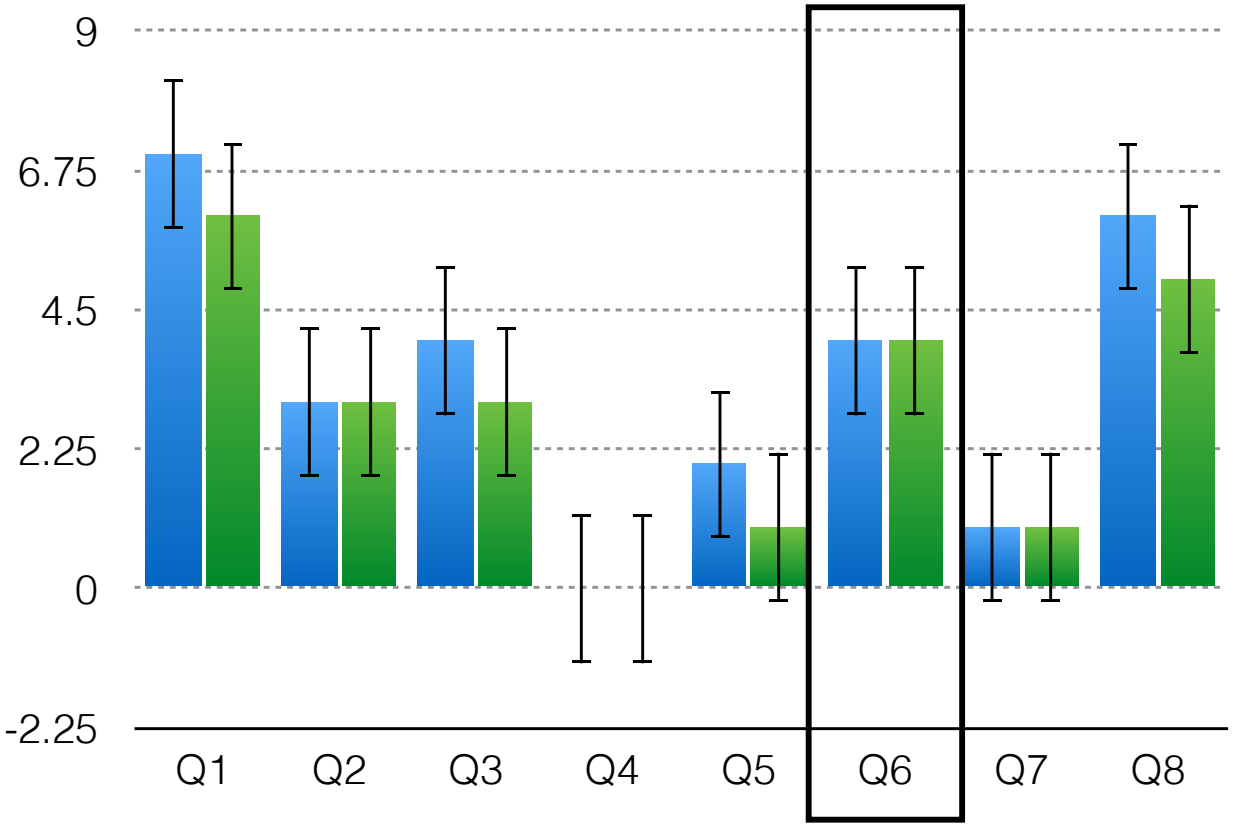
$c_k \geq c_k'$

$c_k' + 1 \geq c_k$

Notation r_k, r_k' noise added at round k.

Let's focus on the iteration i and let's fix the noises r_j for all $j \neq i$

Report Noisy Max



Simplifying assumptions

$c_k \geq c_k'$

$c_k' + 1 \geq c_k$

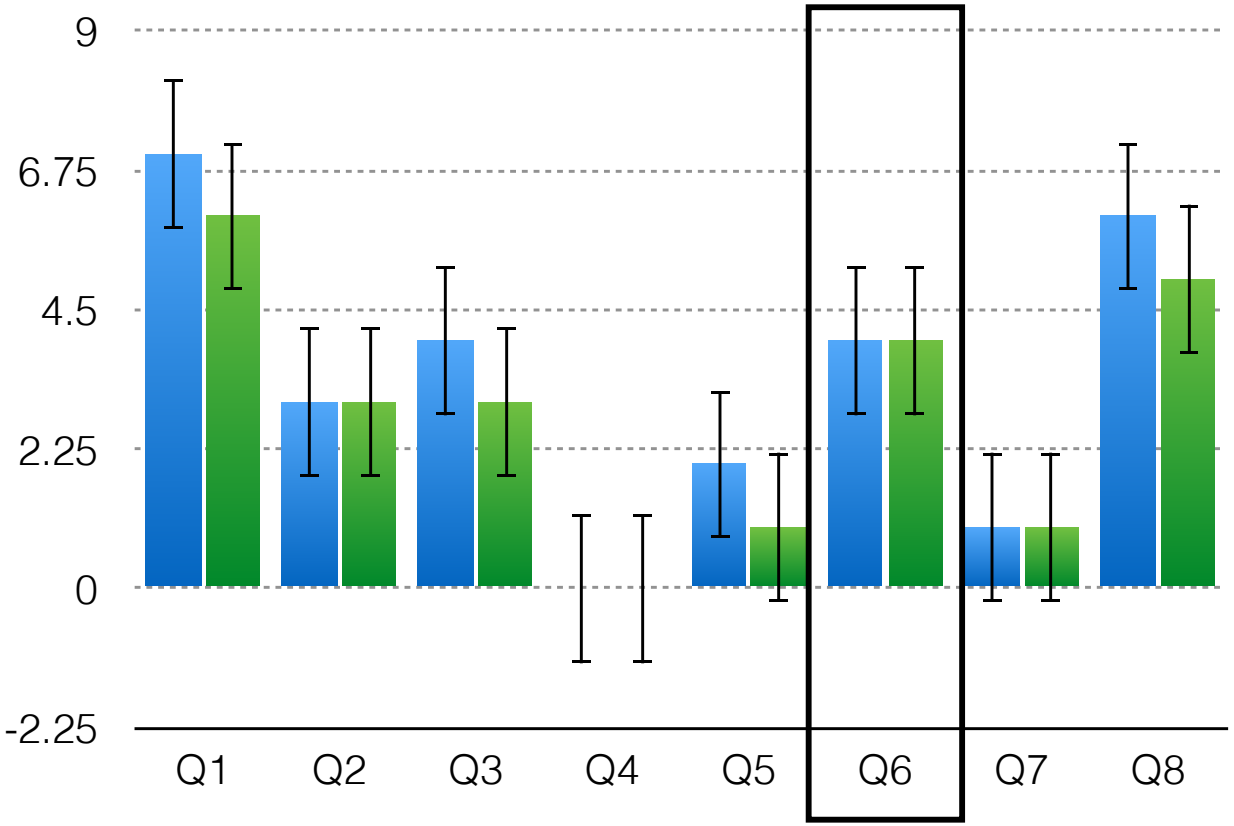
Notation r_k, r_k' noise added at round k.

We want to show:

$$\Pr_{x \sim RNM(D)}[x = i \mid r_{-i}] \leq e^{\epsilon} \Pr_{x \sim RNM(D')}[x = i \mid r_{-i}]$$

Let's focus on the iteration i and let's fix the noises r_j for all $j \neq i$

Report Noisy Max



By fixing the noises r_j for all $j \neq i$ we can compute the following

$r^* = \min_r \; c_i + r \geq c_j + r_j$ for all j

Simplifying assumptions

$c_k \geq c_k'$

$c_k' + 1 \geq c_k$

Notation r_k, r_k' noise added at round k.

Let's focus on the iteration i and let's fix the noises r_j for all $j \neq i$

Report Noisy Max

$r^* = \min_r \; c_i + r \geq c_j + r_j$ for all j

Notice that

$$\Pr_{x \sim RNM(D)}[x = i \mid r_{-i}] = \Pr_{r \sim Lap}[r \geq r^*]$$

Simplifying assumptions

$c_k \geq c_k'$

$c_k' + 1 \geq c_k$

Notation r_k, r_k' noise added at round k.

Let's focus on the iteration i and let's fix the noises r_j for all $j \neq i$

Report Noisy Max

$r^* = \min_r\ c_i + r \geq c_j + r_j$ for all j

Notice that we also have

$$c_i' + 1 + r^* \geq c_j' + r_j$$

Since

$$c_i + r^* \geq c_j + r_j$$

and this says

$$\Pr_{x \sim RNM(D')}[x = i \mid r_{-i}] \geq \Pr_{r \sim Lap}[r \geq 1 + r*]$$

Simplifying assumptions

$c_k \geq c_k'$

$c_k' + 1 \geq c_k$

Notation r_k, r_k' noise added at round k.

Let's focus on the iteration i and let's fix the noises r_j for all $j \neq i$

Report Noisy Max

$r^* = \min_r\ c_i + r \geq c_j + r_j$ for all j

Summarizing we have:

$$\Pr_{x \sim RNM(D)}[x = i \mid r_{-i}] = \Pr_{r \sim Lap}[r \geq r^*]$$

And

$$\Pr_{x \sim RNM(D')}[x = i \mid r_{-i}] \geq \Pr_{r \sim Lap}[r \geq 1 + r^*]$$

Simplifying assumptions

$c_k \geq c_k'$

$c_k' + 1 \geq c_k$

Notation r_k, r_k' noise added at round k.

Let's focus on the iteration i and let's fix the noises r_j for all $j \neq i$

Report Noisy Max

$r^* = \min_r \; c_i + r \geq c_j + r_j$ for all j

Summarizing we have:

$$\Pr_{x \sim RNM(D)}[x = i \,|\, r_{-i}] = \Pr_{r \sim Lap}[r \geq r^*]$$

And

$$\Pr_{x \sim RNM(D')}[x = i \,|\, r_{-i}] \geq \Pr_{r \sim Lap}[r \geq 1 + r^*]$$

How can we connect them?

Simplifying assumptions

$c_k \geq c_k'$

$c_k' + 1 \geq c_k$

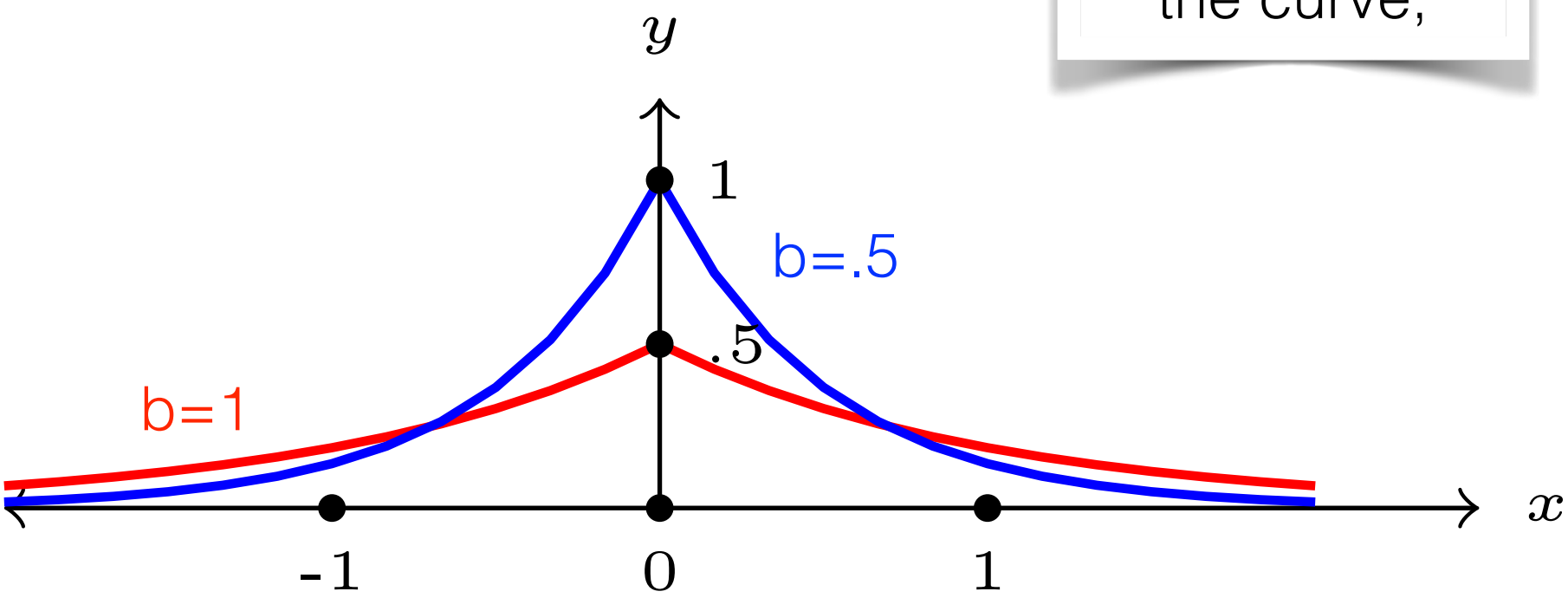
Notation r_k, r_k' noise added at round k.

Let's focus on the iteration i and let's fix the noises r_j for all $j \neq i$

Laplace Distribution

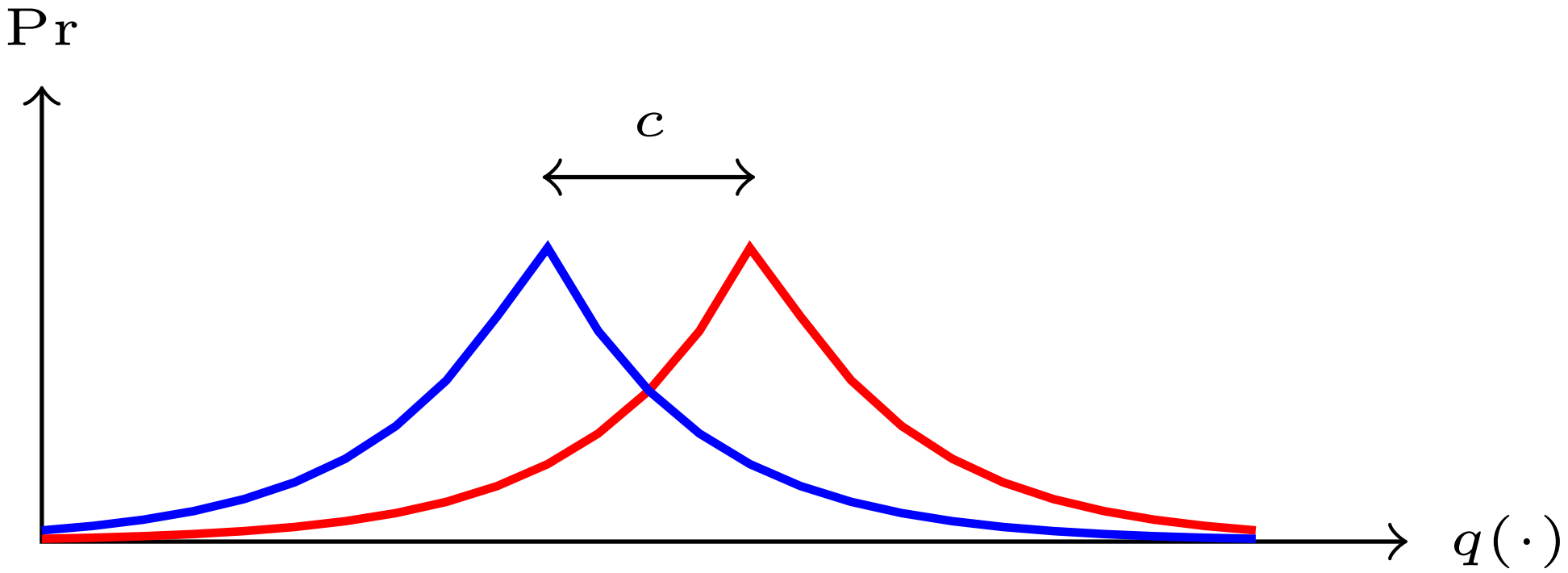
$$\text{Lap}(b, \mu)(X) = \frac{1}{2b} \exp\left(- \frac{|\mu - X|}{b} \right)$$

b regulates the skewness of the curve,



Sliding property of the Laplace Distribution

$$\Pr_{x \sim Lap(\frac{1}{\epsilon}, \mu)}[k \leq x] \leq e^{c\epsilon} \Pr_{x \sim Lap(\frac{1}{\epsilon}, \mu)}[k + c \leq x]$$



Report Noisy Max

Summarizing we have:

$$\Pr_{x \sim RNM(D)}[x = i \mid r_{\_i}]$$

$$= \Pr_{r \sim Lap}[r \geq r*] \leq e^{\epsilon} \Pr_{r \sim Lap}[r \geq 1 + r*]$$

$$\leq e^{\epsilon} \Pr_{x \sim RNM(D')}[x = i \mid r_{\_i}]$$

Report Noisy Max

In a similar way we can prove:

$$\Pr_{x \sim RNM(D')}[x = i \,|\, r\_{i}] \leq e^{\epsilon} \Pr_{x \sim RNM(D)}[x = i \,|\, r\_{i}]$$

Report Noisy Max

```
RNM (q_1,...,q_N : (data → R) list,
     b : list data, ε: R) : nat
  i = 0;
  max = 0;
  while (i < N){
     cur = q_i(b) + Lap(1/ε)
     if (cur > max)
          max =  cur ;
          output = i;
  return output;
```

Report Noisy Max

$\vdash(\varepsilon,0)$

$[\mathrm{adj}\ b_1\ b_2, GS(q_i)\leq 1, \ldots]$

```
RNM (q_1,...,q_N : (data → R) list,
     b : list data, ε: R) : nat
  i = 0;
  max = 0;
  while (i < N){
     cur = q_i(b) + Lap(1/ε)
     if (cur > max)
          max =  cur ;
          output = i;
   return output;
```

$[\mathrm{output}_1=\mathrm{output}_2]$

Point-wise reformulation of differential privacy

Given $\varepsilon,\delta \geq 0$, a mechanism M: db $\rightarrow$O where O is discrete, is (ε,δ)-differentially private iff $\forall b_1 \sim 1\ b_2$ and $\forall s \in O$:

$$\Pr[M(b_1) = s] \leq \exp(\varepsilon) \cdot \Pr[M(b_2) = s] + \delta_s$$

with $\sum \delta_s \leq \delta$.

Can we turn this definition into a rule?

apRHL: pointwise DP rule

$$\frac{\text{forall } r \in R \quad \vdash_{\varepsilon, \delta r} c_1 \sim c_2 : P \Longrightarrow x\langle 1\rangle = r \Rightarrow x\langle 2\rangle = r \qquad \sum \delta r \leq \delta}{\vdash_{\varepsilon, \delta} c_1 \sim c_2 : P \Longrightarrow x\langle 1\rangle = x\langle 2\rangle}$$

Report Noisy Max

forall s, |-(ε,0)
[adj b_1 b_2,GS(q_i)≤1,…]

```
RNM (q1,…,qN : (data → R) list,
     b : list data, ε: R) : nat
  i = 0;
  max = 0;
  while (i < N){
     cur = qi(b) + Lap(1/ε)
     if (cur > max)
          max =  cur ;
          output = i;
  return output;
```

[$output_1$=s => $output_2$=s]

By applying the pointwise rule we get a different post

Report Noisy Max

```
forall s, |-(ε,0)
[adj b1 b2,GS(qi)≤1,…]
RNM (q1,…,qN : (data → R) list,
     b : list data, ε: R) : nat
  i = 0;
  max = 0;
  while (i < N){
     cur = qi(b) + Lap(1/ε)
     if (cur > max)
          max =  cur ;
          output = i;
  return output;
[output1=s => output2=s]
```

By applying the pointwise rule we get a different post

Notice that we focus on a single general s.

Report Noisy Max

```
forall s, |-(ε,0)
RNM (q1,…,qN : (data → R) list,
     b : list data, ε: R) : nat
  i = 0;
  max = 0;
[adj b1 b2,GS(qi)≤1,…]
  while (i < N){
      cur = qi(b) + Lap(1/ε)
      if (cur > max)
           max =  cur ;
           output = i;
   return output;
[output1=s => output2=s]
```

We can apply standard RHL

Report Noisy Max

```
forall s, |-(ε,0)
RNM (q1,…,qN : (data → R) list,
     b : list data, ε: R) : nat
  i = 0;
  max = 0;
  while (i < N){
[adj b1 b2,GS(qi)≤1,…, invariant]
     cur = qi(b) + Lap(1/ε)
     if (cur > max)
          max =  cur ;
          output = i;
   return output;
[output1=s => output2=s]
```

Invariant

$$\ldots\ (max_1 < cur_1 \Rightarrow output_1 = i_1)$$
$$\wedge\ (max_2 < cur_2 \Rightarrow output_2 = i_2)$$
$$\wedge\ i_1 = i_2 \wedge output_1 = output_2$$

Report Noisy Max

forall s, |-(ε,0)

RNM ($q_1,\ldots,q_N$: (data → R) list,
b : list data, ε: R) : nat

```
  i = 0;
  max = 0;
  while (i < N){
```

[adj b_1 b_2,GS(q_i)≤1,…, inv]<fun k => if k=s then ε else 0>

```
    cur = q_i(b) + Lap(1/ε)
    if (cur > max)
        max =  cur ;
        output = i;
  return output;
```

[$output_1$=s => $output_2$=s]

Report Noisy Max

forall s, |-(ε,0)

RNM (q_1,…,q_N : (data → R) list,
b : list data, ε: R) : nat

```
  i = 0;
  max = 0;
  while (i < N){
```

[adj b_1 b_2,GS(q_i)≤1,…, inv,(i_1=s \/ i_1<>s)] <fun k … >

```
      cur = q_i(b) + Lap(1/ε)
      if (cur > max)
            max =  cur ;
            output = i;
   return output;
```

[$output_1$=s => $output_2$=s]

We can now proceed by cases

Report Noisy Max

forall s, |-(ε,0)

RNM ($q_1,\ldots,q_N$: (data → R) list,
b : list data, ε: R) : nat

```
  i = 0;
  max = 0;
  while (i < N){
```

[adj b_1 b_2,GS(q_i)≤1,…, inv, i_1=s] <fun k => if k=s then ε else 0>

```
      cur = q_i(b) + Lap(1/ε)
      if (cur > max)
            max =  cur ;
            output = i;
   return output;
```

[$output_1$=s => $output_2$=s]

Case 1

Report Noisy Max

forall s, |-(ε,0)

RNM (q_1,…,q_N : (data → R) list,
b : list data, ε: R) : nat

```
  i = 0;
  max = 0;
  while (i < N){
```

[adj b_1 b_2,GS(q_i)≤1,…, inv, i_1=s] <ε>

```
    cur = q_i(b) + Lap(1/ε)
    if (cur > max)
        max =  cur ;
        output = i;
  return output;
```

[$output_1$=s => $output_2$=s]

We can simplify

Report Noisy Max

forall s, $\vdash_{(\varepsilon,0)}$

RNM ($q_1,\ldots,q_N$: (data → R) list,
b : list data, ε: R) : nat

```
i = 0;
max = 0;
while (i < N){
```

[adj b_1 b_2,GS(q_i)≤1,…, inv, i_1=s] <ε>

```
    cur = q_i(b) + Lap(1/ε)
    if (cur > max)
        max = cur ;
        output = i;
return output;
```

[$output_1$=s => $output_2$=s]

What rule shall we apply now?

apRHL: More general Lap rule (still restricted)

$$\vdash_{k*\varepsilon,0} \frac{}{x_1 :=\$\ \mathrm{Lap}(1/\varepsilon, y_1) \sim x_2 :=\$\ \mathrm{Lap}(1/\varepsilon, y_2) : |y_1 - y_2| \leq k \Rightarrow =}$$

Invariant

$$\ldots\ (max_1 < cur_1 \Rightarrow output_1 = i_1)$$
$$\wedge\ (max_2 < cur_2 \Rightarrow output_2 = i_2)$$
$$\wedge\ i_1 = i_2 \wedge output_1 = output_2$$

Report Noisy Max

forall s, |-(ε,0)

RNM ($q_1,\ldots,q_N$: (data → R) list,
b : list data, ε: R) : nat

```
  i = 0;
  max = 0;
  while (i < N){
    cur = q_i(b) + Lap(1/ε)
```

[adj b_1 b_2,GS(q_i)≤1,…, inv, i_1=s, cur_1=cur_2] <0>

```
    if (cur > max)
        max =  cur ;
        output = i;
  return output;
```

[$output_1$=s => $output_2$=s]

Report Noisy Max

forall s, |-(ε,0)

RNM ($q_1,\ldots,q_N$: (data → R) list,
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  while (i < N){
    cur = q_i(b) + Lap(1/ε)
```

[adj b_1 b_2,GS(q_i)≤1,…, inv, i_1=s, cur_1=cur_2] <0>

```
    if (cur > max)
        max =  cur ;
        output = i;
  return output;
```

[$output_1$=s => $output_2$=s]

We can conclude this case by the asynchronous if rule

Report Noisy Max

forall s, |-(ε,0)

RNM ($q_1,\ldots,q_N$: (data → R) list,
b : list data, ε: R) : nat

```
  i = 0;
  max = 0;
  while (i < N){
```

[adj b_1 b_2,GS(q_i)≤1,…, inv, i_1<>s] <fun k => if k=s then ε else 0>

```
      cur = q_i(b) + Lap(1/ε)
      if (cur > max)
            max =  cur ;
            output = i;
   return output;
```

[$output_1$=s => $output_2$=s]

Case 2

Report Noisy Max

forall s, |-(ε,0)

RNM ($q_1,\ldots,q_N$: (data → R) list,
b : list data, ε: R) : nat

```
  i = 0;
  max = 0;
  while (i < N){
```

[adj b_1 b_2,GS(q_i)≤1,…, inv, i_1<>s] <0>

```
    cur = q_i(b) + Lap(1/ε)
    if (cur > max)
        max =  cur ;
        output = i;
  return output;
```

[$output_1$=s => $output_2$=s]

We can simplify

Report Noisy Max

forall s, |-(ε,0)

RNM ($q_1,\ldots,q_N$: (data → R) list,
b : list data, ε: R) : nat

```
  i = 0;
  max = 0;
  while (i < N){
```

[adj b_1 b_2,GS(q_i)≤1,…, inv, i_1<>s] <0>

```
    cur = q_i(b) + Lap(1/ε)
    if (cur > max)
        max =  cur ;
        output = i;
  return output;
```

[$output_1$=s => $output_2$=s]

What rule shall we apply now?

apRHL: More general Lap rule (still restricted)

$$\frac{}{\vdash_{k*\varepsilon,0} x_1 :=\$\ \mathrm{Lap}(1/\varepsilon, y_1) \sim x_2 :=\$\ \mathrm{Lap}(1/\varepsilon, y_2) : |y_1 - y_2| \leq k \Rightarrow =}$$

Can we use this rule here?

apRHL
Generalized Laplace

$x_1 :=\$\ \mathrm{Lap}(\varepsilon, e_1)$

$\vdash_{k2*\varepsilon,0}\ \sim$

$x_2 :=\$\ \mathrm{Lap}(\varepsilon, e_2)$

$: \ |k_1 + e_1\langle 1\rangle - e_2\langle 2\rangle| \leq k_2$

$\Longrightarrow\ x1\langle 1\rangle + k_1 = x\langle 2\rangle$

Report Noisy Max

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[$output_1$=s => $output_2$=s]

We can apply this rule with

$k_1 = q_{i\langle 2\rangle} - q_{i\langle 1\rangle}$

Morally

$\vdash_{(0,0)}$ Pre: true

output = input + Lap(ε)

Post: [output1−output2=input1−input2]

Report Noisy Max

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```
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```

[adj b_1 b_2,GS(q_i)≤1,…, inv, i_1<>s, cur_1+qi<2>-qi<1>=cur_2] <0>

```
      if (cur > max)
          max =  cur ;
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[$output_1$=s => $output_2$=s]

We can apply this rule with

$k_1 = q_{i\langle 2\rangle} - q_{i\langle 1\rangle}$

Report Noisy Max

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```
      if (cur > max)
          max =  cur ;
          output = i;
  return output;
```

[$output_1$=s => $output_2$=s]

We can conclude this case by the asynchronous if rule and conclude