CS 511: Formal Methods in Security and Privacy
An imperative programming language and Hoare Triples

Marco Gaboardi
gaboardi@bu.edu

Alley Stoughton
stough@bu.edu

From the previous class

Does the program comply with the specification?

Precondition: $x \geq 0$ and $y \geq 0$

```
Function Add(x: int, y: int) : int
{
  r = 0;
  n = y;
  while n != 0
  {
    r = r + 1;
    n = n - 1;
  }
  return r
}
```

Postcondition: $r == x + y$

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Fail to meet the specification

How about this one?

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Precondition: x ≥ 0 and y ≥ 0
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  r = x;
  n = y;
  while n != 0
  {
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    n = n - 1;
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Postcondition: r == x + y
```

How about this one?

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Precondition: x ≥ 0 and y ≥ 0
Function Add(x: int, y: int) : int
{
  r = x;
  n = y;
  while n != 0
  {
    r = r + 1;
    n = n - 1;
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Postcondition: r == x + y
```

It meets the specification

How can we make this reasoning mathematically precise?

Formal Semantics

We need to assign a formal meaning to the different components:

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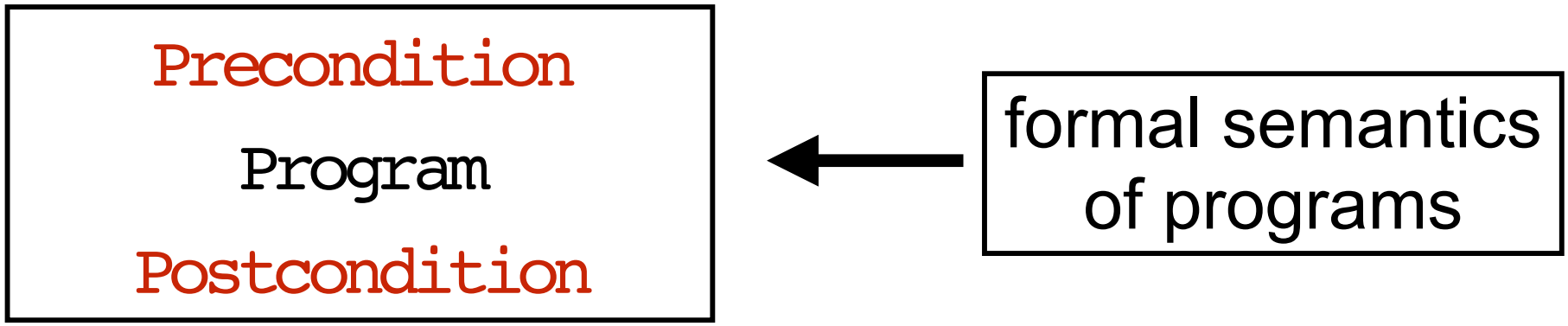

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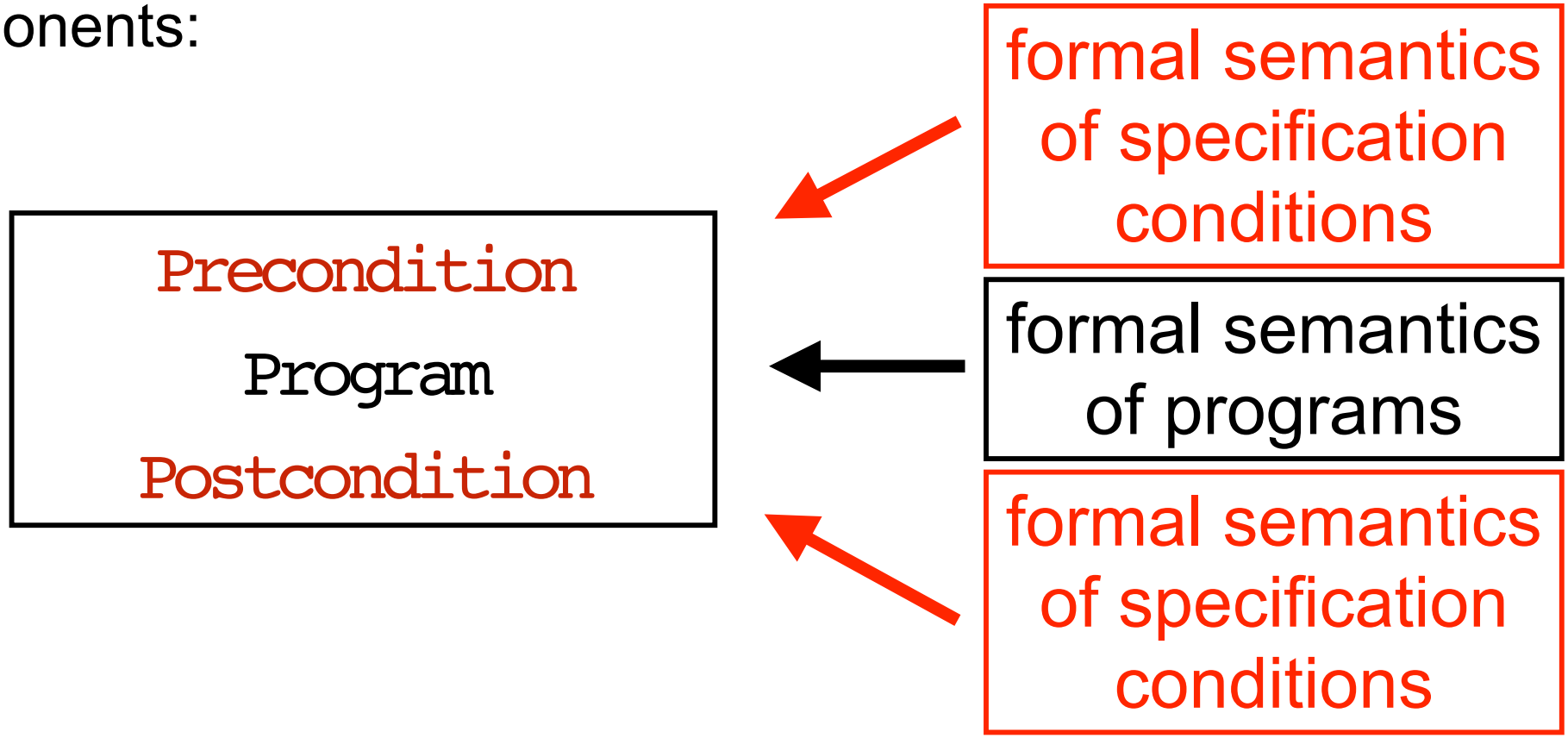

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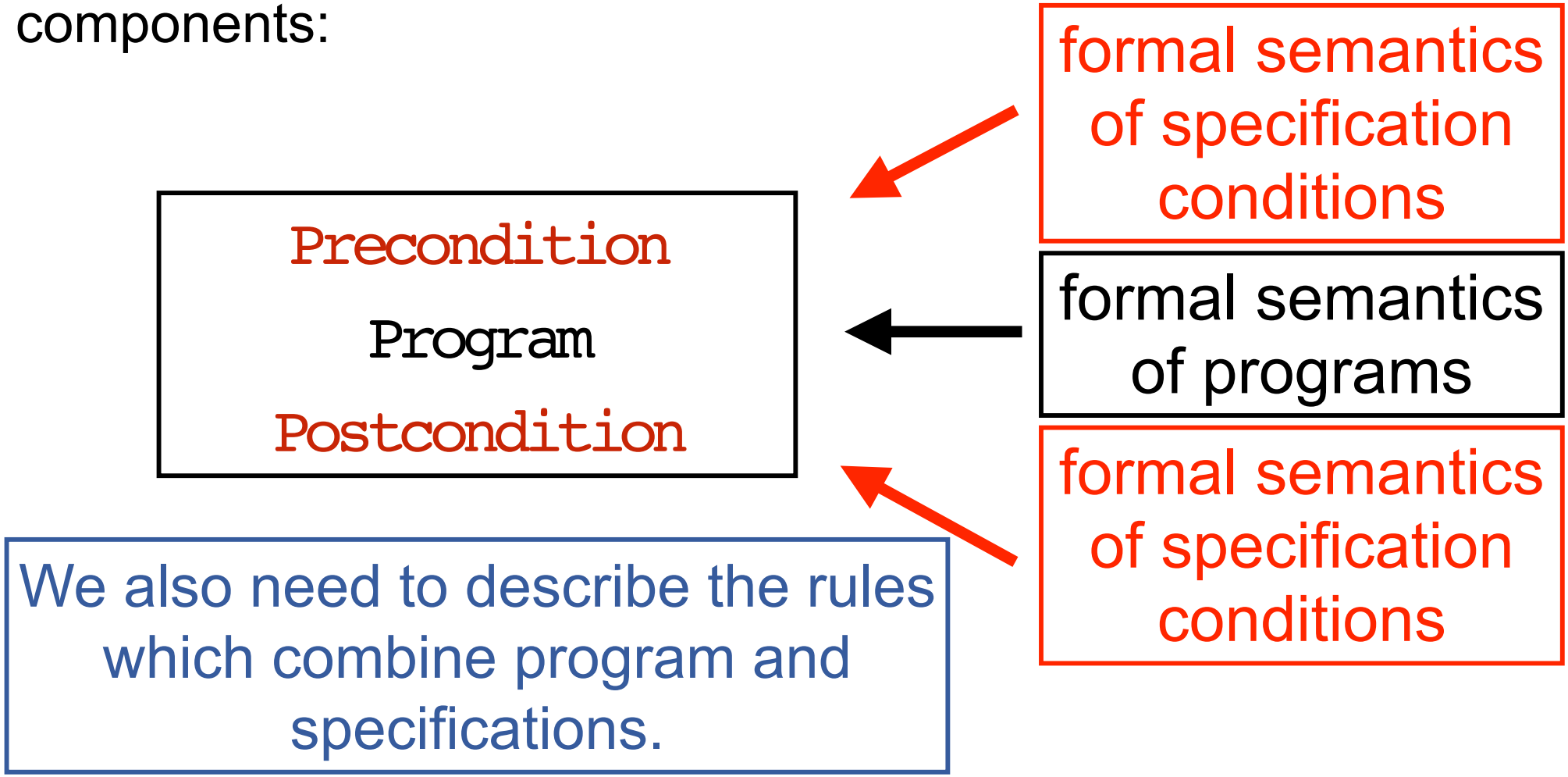

Goal for today

- Formalize the semantics of a simple imperative programming language.

A first example

```
FastExponentiation(n, k : Nat) : Nat
  n' := n; k' := k; r := 1;
  if k' > 0 then
    while k' > 1 do
       if even(k') then
          n' := n' * n';
          k' := k'/2;
       else
          r := n' * r;
          n' := n' * n';
          k' := (k' - 1)/2;
    r := n' * r;
  (* result is r *)
```

Programming Language

```
c::= abort
   | skip
   | x:=e
   | c;c
   | if e then c else c
   | while e do c
```

$x, y, z, \ldots$ program variables

$e_1, e_2, \ldots$ expressions

$c_1, c_2, \ldots$ commands

Expressions

We want to be able to write complex programs with our language.

$$
\begin{array}{l}
e ::= x \\
\quad\quad | \;\; f(e_1, \ldots, e_n)
\end{array}
$$

Where `f` can be any arbitrary operator.

Some expression examples

```
x+5        x mod k        x[i]        (x[i+1] mod 4)+5
```

Types

In expressions we want to be able to use “arbitrary” data types.

$$
\begin{array}{l}
t ::= b \\
\quad\quad \mid T(t_1, \ldots, t_n)
\end{array}
$$

Types

In expressions we want to be able to use “arbitrary” data types.

$$t ::= b \mid T(t_1, \ldots, t_n)$$

We assume a collection of base types `b` including

`Bool` `Int` `Nat` `String`

We also assume a set of type constructors `T` that we can use to build more complex types, such as:

`Bool list` `Int*Bool` `Int*String -> Bool`

Types

We also use types to guarantee that commands are well-formed.

For example, in the commands

`while e do c` `if e then` c_1 `else` c_2

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We omit the details of the type system here but you can find them in the notes by Gilles Barthe

Values

Values are atomic expressions whose semantics is self-evident and which do not need a further analysis.

For example, we have the following values

```
true    5    [1,2,3,4]    "Hello"
```

The following are not values:

```
not true    x+5    [x,x+1]    x[1]
```

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not true    x+5    [x,x+1]    x[1]
```

We could define a grammar for values, but we prefer to leave this at the intuitive level for now.

How can we give semantics to expressions and commands?

Memories

We can formalize a memory as a map m from variables to values.

$$m = [x_1 \longmapsto v_1, \ldots, x_n \longmapsto v_n]$$

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We consider only maps that respect types.

We want to read the value associated to a particular variable:

$$m(x)$$

We want to update the value associated to a particular variable:

$$m[x \leftarrow v]$$

This is defined as

$$m[x \leftarrow v](y) = \begin{cases} v & \text{If } x = y \\ m(y) & \text{Otherwise} \end{cases}$$

Semantics of Expressions

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Exp * Mem -> Val
```

We will denote this relation as:

$$\{e\}_m = v$$

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We will denote this relation as:

$\{e\}_m = v$

This is commonly typeset as:

$$[\![e]\!]_m = v$$

Semantics of Expressions

This is defined on the structure of expressions:

$$\{x\}_m = m(x)$$

$$\{f(e_1,\ldots,e_n)\}_m = \{f\}(\{e_1\}_m,\ldots,\{e_n\}_m)$$

where $\{f\}$ is the semantics associated with the basic operation we are considering.

Semantics of Expressions

Suppose we have a memory

$$\texttt{m=[i}\longmapsto\texttt{1,x}\longmapsto\texttt{[1,2,3],y}\longmapsto\texttt{2]}$$

That `{mod}` is the modulo operation and `{+}` is addition, we can derive the meaning of the following expression:

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$$= (\{x[i+1]\}_m \{mod\} \{y\}_m)\{+\}\{5\}_m$$

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&\quad = (\{x[i+1]\}_m \ \{mod\} \ \{y\}_m)\{+\}\{5\}_m \\
&\quad = (\{x\}_m[\{i\}_m\{+\}\{1\}_m] \ \{mod\} \ \{y\}_m)\{+\}\{5\}_m
\end{aligned}$$

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&\quad = (\{x\}_m[2] \ \{mod\} \ 2)\{+\}5 \\
&\quad = (2 \ \{mod\} \ 2)\{+\}5 = 0 \ \{+\} \ 5 = 5
\end{aligned}
$$

Operational vs Denotational Semantics

The style of semantics we are using is denotational, in the sense that we describe the meaning of an expression by means of the value it denotes.

A different approach, more operational in nature, would be to describe the meaning of an expression by means of the value that the expression evaluates to in an abstract machine.

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k:=2; z:=x mod k; if z=0 then r:=1 else r:=2
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We can give the semantics as a map associating a pair of an command, and a memory with a memory or an error.

$$Com * Mem \rightarrow (Mem \cup \{\bot\})$$

We will denote this relation as:

$\{c\}_m = m'$ Or $\{c\}_m = \bot$

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$$\{\texttt{c;c'}\}_m = \{\texttt{c'}\}_{m'} \quad \text{If} \quad \{\texttt{c}\}_m = m'$$

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$\{\texttt{c;c'}\}_m = \bot$ If $\{\texttt{c}\}_m = \bot$

$\{\texttt{if e then } c_t \texttt{ else } c_f\}_m = \{c_t\}_m$ If $\{e\}_m = \texttt{true}$

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$\{\texttt{if e then } c_t \texttt{ else } c_f\}_m = \{c_f\}_m$ If $\{e\}_m = \texttt{false}$

Semantics of While

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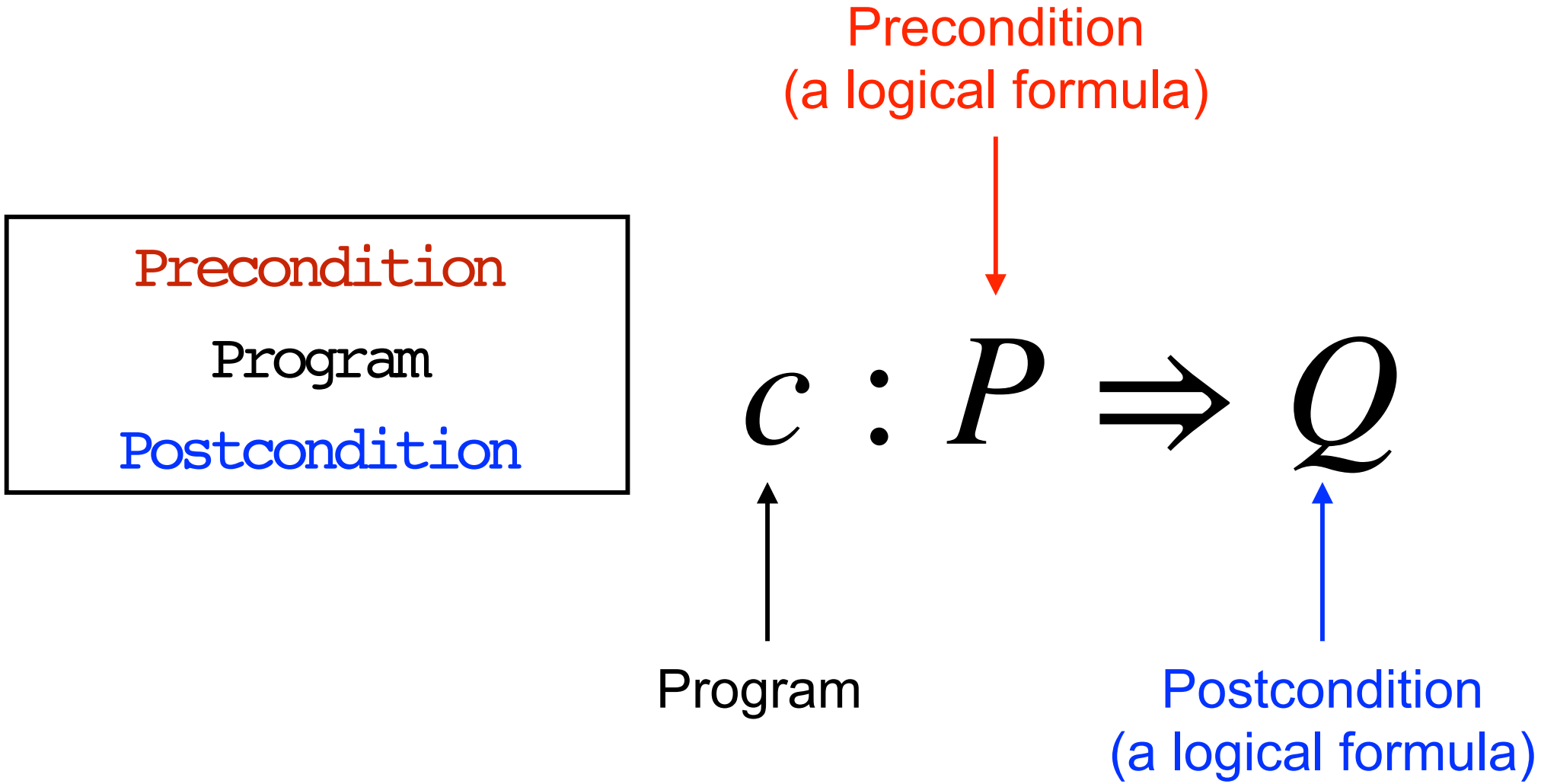
What about while

$$\{\texttt{while e do c}\}_m = ???$$

We omit the semantics of while, you can find it in the notes by Gilles Barthe.
Alternatively, you can look at these notes:
https://groups.seas.harvard.edu/courses/cs152/2016sp/lectures/lec06-denotational.pdf
https://web.cecs.pdx.edu/~apt/cs578_2022/imp.pdf

Hoare Triples

Hoare triple



Some examples

Precondition

$$x := z + 1 : \{z = n\} \Rightarrow \{x = n + 1\}$$

Postcondition

Is it a valid specification?

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Specification can also be imprecise.

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$m_{in} = [z = -1, x = 2]$ $m_{out} = [z = -1, x = 0]$

Some examples

```
i:=0;
r:=1;
while(i≤k)do
 r:=r * n;
 i:=i + 1
```

Precondition

$$: \{0 \leq k\} \Rightarrow \{r = n^k\}$$

Postcondition

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Some examples

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Postcondition

Is it a valid specification?

$$m_{in} = [k = 0, n = 2, i = 0, r = 0]$$

$$m_{out} = [k = 0, n = 2, i = 1, r = 2]$$

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Is it a valid specification?

$$m_{in} = [k = 1, n = 2, i = 0, r = 0]$$

$$m_{out} = [k = 1, n = 2, i = 2, r = 4]$$

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i:=0;
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Precondition

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This is good because there is no memory that satisfies the precondition.