CS 591: Formal Methods in Security and Privacy

Formal Proofs for Cryptography

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Cryptographic Security

- Cryptographic schemes (e.g., encryption) and protocols (e.g., key-exchange) can be specified at a high-level using our Probabilistic While (pWhile) language.
    - They generally make use of randomness, which can be modeled by random assignments from (sub-)distributions.
        - When these high-level specifications are implemented, this randomness must be realized using pseudorandom number generators, whose seeds make use of randomness from the underlying operating system.
    - They also often make use of primitives like pseudorandom functions (PRFs).
        - These primitives must also be implemented; e.g., PRFs can be implemented using hash functions like SHA-?.

Cryptographic Security

- Our focus in this course will be at the specification level.
- But there is research that addresses how to specify and prove the security of implementations of cryptographic schemes and protocols.

pWhile in EasyCrypt

- E.g., here is a pWhile procedure that exclusive-ors two booleans chosen from the uniform distribution on booleans (each of true and false will be chosen with probability 1/2):

```
module M = {
  proc f() : bool = {
    var x, y : bool;
    x <$ {0,1}; y <$ {0,1};
    return x ^^ y;
  }
}.
```

- And here is how we can state the lemma that `M.f()` returns true with probability 1/2 no matter what memory it's run in:

```
lemma M_f_true &m :
  Pr[M.f() @ &m : res] = 1%r / 2%r.
```

Building Encryption from PRF + Randomness

- Our running example will be a symmetric encryption scheme built out of a pseudorandom function plus randomness.
  - Symmetric encryption means the same key is used for both encryption and decryption.
- We'll first define when a symmetric encryption scheme is secure under indistinguishability under chosen plaintext attack (IND-CPA).
- Next we'll define our instance of this scheme, and informally analyze adversaries' strategies for breaking security.
- We'll return later in the course (in lecture and/or lab) to look at the proof in EasyCrypt of the IND-CPA security of our scheme.

Symmetric Encryption Schemes

- Our treatment of symmetric encryption schemes is parameterized by three types:

```
type key.  (* encryption keys, key_len bits *)
type text.  (* plaintexts, text_len bits *)
type cipher.  (* ciphertexts - scheme specific *)
```

- An encryption scheme is a *stateless* implementation of this module interface:

```
module type ENC = {
  proc key_gen() : key  (* key generation *)
  proc enc(k : key, x : text) : cipher  (* encryption *)
  proc dec(k : key, c : cipher) : text  (* decryption *)
}.
```

Scheme Correctness

- An encryption scheme is *correct* if and only if the following procedure returns true with probability 1 for all arguments:

```
module Cor (Enc : ENC) = {
  proc main(x : text) : bool = {
    var k : key; var c : cipher; var y : text;
    k <@ Enc.key_gen();
    c <@ Enc.enc(k, x);
    y <@ Enc.dec(k, c);
    return x = y;
  }
}.
```

- The module `Cor` is parameterized (may be applied to) an arbitrary encryption scheme, `Enc`.

Encryption Oracles

- To define IND-CPA security of encryption schemes, we need the notion of an *encryption oracle*, which both the adversary and IND-CPA game will interact with:

```
module type EO = {
  (* initialization - generates key *)
  proc * init() : unit
  (* encryption by adversary before game's encryption *)
  proc enc_pre(x : text) : cipher
  (* one-time encryption by game *)
  proc genc(x : text) : cipher
  (* encryption by adversary after game's encryption *)
  proc enc_post(x : text) : cipher
}.
```

Standard Encryption Oracle

- Here is the standard encryption oracle, parameterized by an encryption scheme, `Enc`:

```
module EncO (Enc : ENC) : EO = {
  var key : key
  var ctr_pre : int
  var ctr_post : int

  proc init() : unit = {
    key <@ Enc.key_gen();
    ctr_pre <- 0; ctr_post <- 0;
  }
```

Standard Encryption Oracle

```
proc enc_pre(x : text) : cipher = {
  var c : cipher;
  if (ctr_pre < limit_pre) {
    ctr_pre <- ctr_pre + 1;
    c <@ Enc.enc(key, x);
  }
  else {
    c <- ciph_def;  (* default result *)
  }
  return c;
}
```

Standard Encryption Oracle

```
proc genc(x : text) : cipher = {
  var c : cipher;
  c <@ Enc.enc(key, x);
  return c;
}
```

Standard Encryption Oracle

```
  proc enc_post(x : text) : cipher = {
    var c : cipher;
    if (ctr_post < limit_post) {
      ctr_post <- ctr_post + 1;
      c <@ Enc.enc(key, x);
    }
    else {
      c <- ciph_def;  (* default result *)
    }
    return c;
  }
}.
```

Encryption Adversary

- An *encryption adversary* is parameterized by an encryption oracle:

```
module type ADV (EO : EO) = {
  (* choose a pair of plaintexts, x1/x2 *)
  proc * choose() : text * text {EO.enc_pre}

  (* given ciphertext c based on a random boolean b
     (the encryption using EO.genc of x1 if b = true,
      the encryption of x2 if b = false), try to guess b
  *)
  proc guess(c : cipher) : bool {EO.enc_post}
}.
```

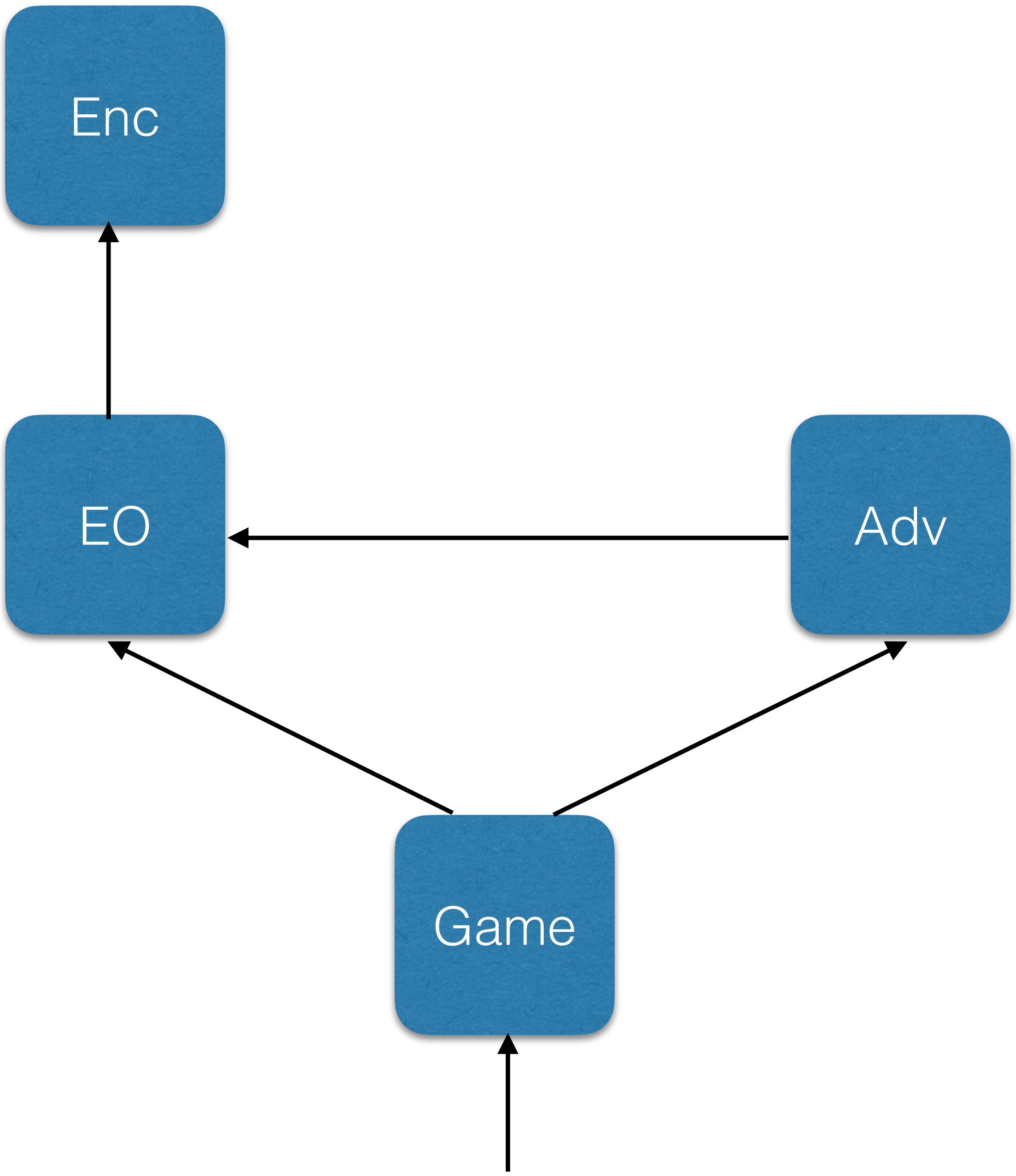
- Adversaries may be probabilistic.

IND-CPA Game

- The IND-CPA Game is parameterized by an encryption scheme and an encryption adversary:

```
module INDCPA (Enc : ENC, Adv : ADV) = {
  module EO = EncO(Enc)           (* make EO from Enc *)
  module A = Adv(EO)              (* connect Adv to EO *)
  proc main() : bool = {
    var b, b' : bool; var x1, x2 : text; var c : cipher;
    EO.init();                    (* initialize EO *)
    (x1, x2) <@ A.choose();       (* let A choose x1/x2 *)
    b <$ {0,1};                   (* choose boolean b *)
    c <@ EO.genc(b ? x1 : x2);    (* encrypt x1 or x2 *)
    b' <@ A.guess(c);             (* let A guess b from c *)
    return b = b';                (* see if A won *)
  }
}.
```

IND-CPA Game



IND-CPA Game

- If the value `b'` that `Adv` returns is independent of the random boolean `b`, then the probability that `Adv` wins the game will be exactly 1/2.
  - E.g., if `Adv` always returns true, it'll win half the time.
- The question is how much better it can do—and we want to prove that it can't do much better than win half the time.
  - But this will depend upon the quality of the encryption scheme.
- An adversary that *wins* with probability greater than 1/2 can be converted into one that *loses* with that probability, and vice versa. When formalizing security, it's convenient to upper-bound the *distance* between the probability of the adversary winning and 1/2.

IND-CPA Security

- In our security theorem for a given encryption scheme `Enc` and adversary Adv, we prove an upper bound on the absolute value of the difference between the probability that `Adv` wins the game and 1/2:

```
`|Pr[INDCPA(Enc, Adv).main() @ &m : res] - 1%r / 2%r|
  <= … Adv …
```

- Ideally, we'd like the upper bound to be 0, so that the probability that `Enc` wins is exactly 1/2, but this won't be possible.

- The upper bound may also be a function of the number of bits `text_len` in `text` and the encryption oracle limits `limit_pre` and `limit_post`.

IND-CPA Security

- Q: Because the adversary can call the encryption oracle with the plaintexts x_1/x_2 it goes on to choose, why isn't it impossible to define a secure scheme?
  - A: Because encryption can (must!) involve randomness.
- Q: What is the rationale for letting the adversary call `enc_pre` and `enc_post` at all?
  - A: It models the possibility that the adversary may be able to influence which plaintexts are encrypted.
- Q: What is the rationale for limiting the number of times `enc_pre` and `enc_post` may be called?
  - A: There will probably be some limit on the adversary's influence on what is encrypted.

Next class: Defining an encryption scheme from a pseudorandom function and randomness, and informally analyzing adversaries' strategies for breaking security