CS 591: Formal Methods in Security and Privacy

Differential Privacy

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Feedback

Please fill out the (late) mid-semester evaluation.

Recording

This is a reminder that we will record the class and we will post the link on Piazza.

This is also a reminder to myself to start recording!

From the previous classes

A more realistic example

```
StreamCipher(m : private msg[n]) : public msg[n]
   pkey :=$ PRG(Uniform({0,1}^k));
   cipher := msg xor pkey;
   return cipher
```

How can we prove this secure?

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$$m \longrightarrow m \oplus k \qquad\qquad m \longrightarrow m \oplus pk$$

$$\Delta(\, m \oplus k \;,\; m \oplus pk \,) \leq \delta$$

Approximate Probabilistic Relational Hoare Logic

Indistinguishability parameter

Precondition (a logical formula)

$$\vdash_{\delta} c_1 \sim c_2 : P \Rightarrow Q$$

Probabilistic Program

Probabilistic Program

Postcondition (a logical formula)

Validity of approximate Probabilistic Hoare judgments

We say that the quadruple $\vdash_\delta \mathtt{c}_1 \sim \mathtt{c}_2 : P \Rightarrow Q$ is valid if and only if for every pair of memories $\mathtt{m}_1, \mathtt{m}_2$ such that $P(\mathtt{m}_1, \mathtt{m}_2)$ we have:

$\{\mathtt{c}_1\}_{\mathtt{m}1} = \mu_1$ and $\{\mathtt{c}_2\}_{\mathtt{m}2} = \mu_2$ implies $Q_\delta{}^*(\mu_1, \mu_2)$.

$\mathtt{R}$–δ–Coupling

Given two distributions $\mu_1 \in D(A)$, and $\mu_2 \in D(B)$, we have an $\mathtt{R}$-δ-coupling between them, for $\mathtt{R} \subseteq A x B$ and $0 \leq \delta \leq \mathtt{1}$, if there are two joint distributions $\mu_{L,}\mu_R \in D(AxB)$ such that:

1) $\pi_1(\mu_L)=\mu_1$ and $\pi_2(\mu_R)=\mu_2$,
2) the support of μ_L and μ_R is contained in $\mathtt{R}$. That is, if $\mu_{\mathtt{L}}\mathtt{(a,b)>0,then\ (a,b)}\in\mathtt{R}$, $\mathtt{and}$ if $\mu_{\mathtt{R}}\mathtt{(a,b)>0,then\ (a,b)}\in\mathtt{R}$.
3) $\Delta(\mu_{L,}\mu_R) \leq \delta$

Probabilistic Relational Hoare Logic

Skip

$$\frac{}{\vdash_0 \texttt{skip} \sim \texttt{skip} : P \Rightarrow P}$$

Probabilistic Relational Hoare Logic

Composition

$$\frac{\vdash_{\delta_1} c_1 \sim c_2 : P \Rightarrow R \qquad \vdash_{\delta_2} c_1' \sim c_2' : R \Rightarrow S}{\vdash_{\delta_1+\delta_2} c_1;c_1' \sim c_2;c_2' : P \Rightarrow S}$$

Probabilistic Relational Hoare Logic

A specific rule for PRG

$\vdash_{2\text{^-}n}$ `x`$_1$ `:=$ Uniform({0,1}`^{n}`) ~`

`x`$_2$ `:=$ PRG(Uniform({0,1}`^{k}`))`

`: True` $\Rightarrow$ `x`$_1$`<1>=x`$_2$`<2>`

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We can apply the PRG rule, the composition rule, and the assignment rule and prove:

$$\vdash_{2^{-n}} \texttt{OneTimePad} \sim \texttt{StreamCipher} : m\langle 1\rangle = m\langle 2\rangle \Rightarrow c\langle 1\rangle = c\langle 2\rangle$$

Differential Privacy

Data

Releasing the mean of Some Data

```
Mean(d : private data) : public real
   i:=0;
   s:=0;
   while (i<size(d))
      s:=s + d[i]
      i:=i+1;
   return (s/i)
```

Privacy-preserving data analysis?

We want to release some information to a data analyst and protect the privacy of the individuals contributing their data.

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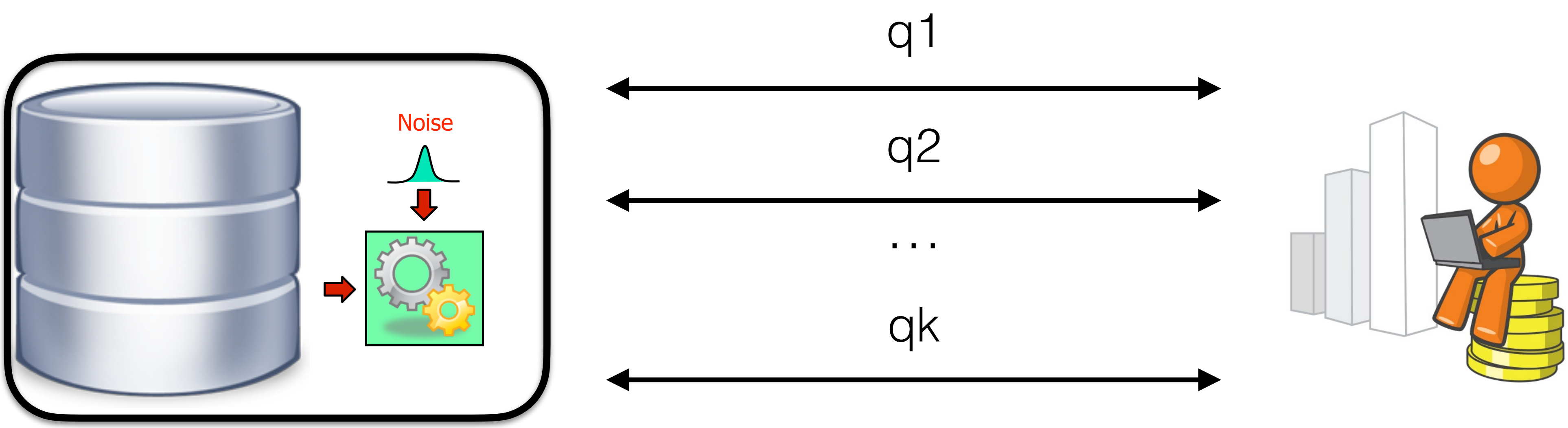
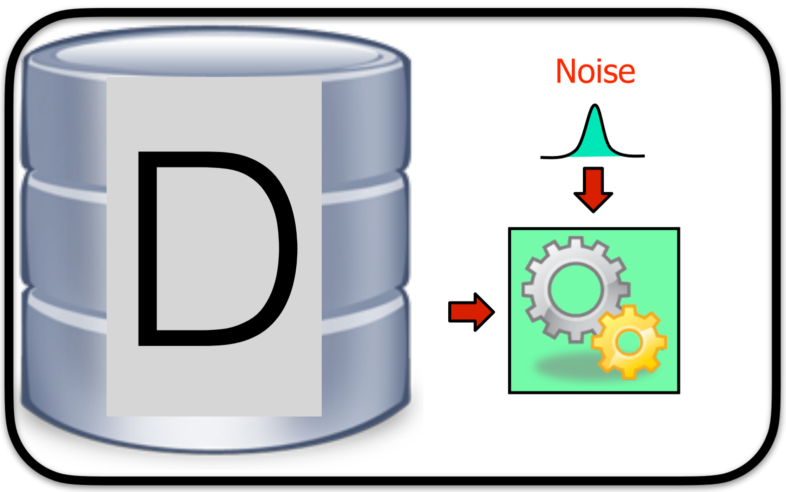


Data analyst

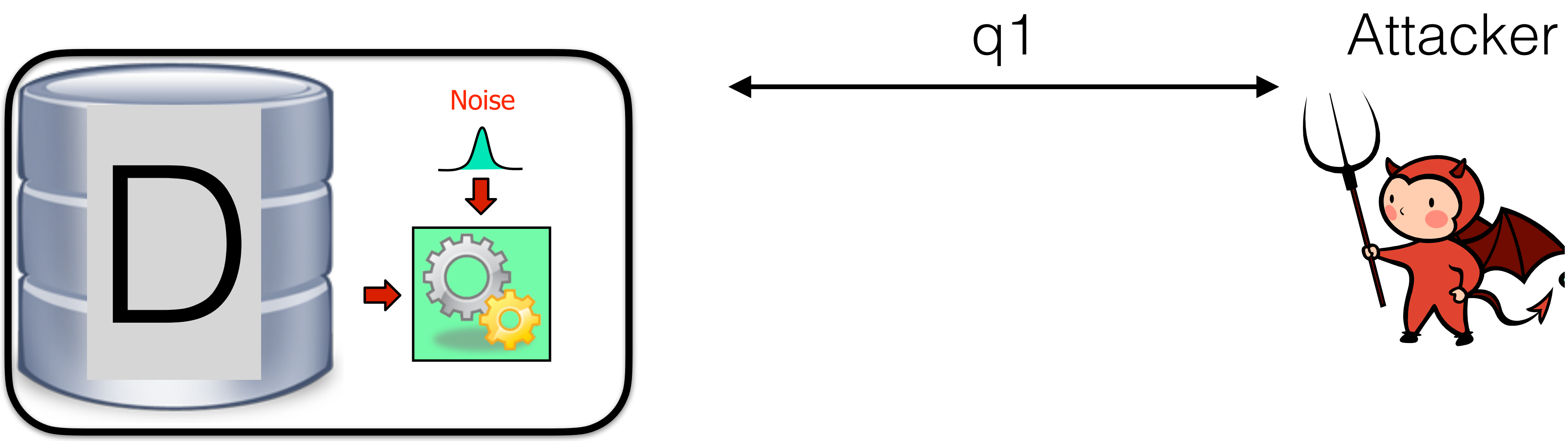
Fundamental Law of Information Reconstruction

The release of too many overly accurate statistics permits reconstruction attacks.

Reconstruction attack



Reconstruction attack



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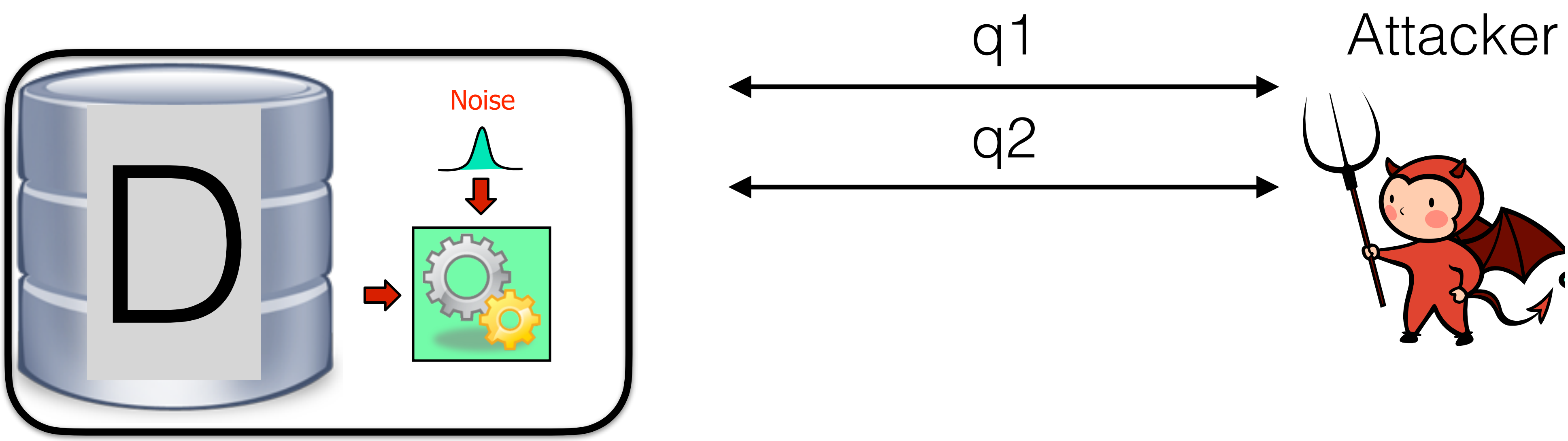

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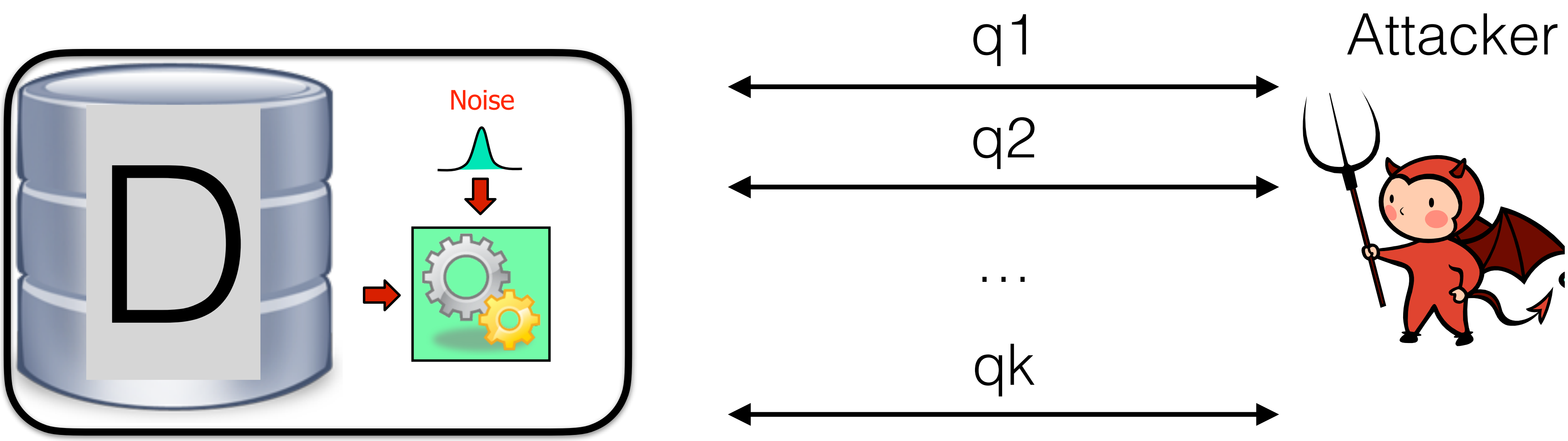

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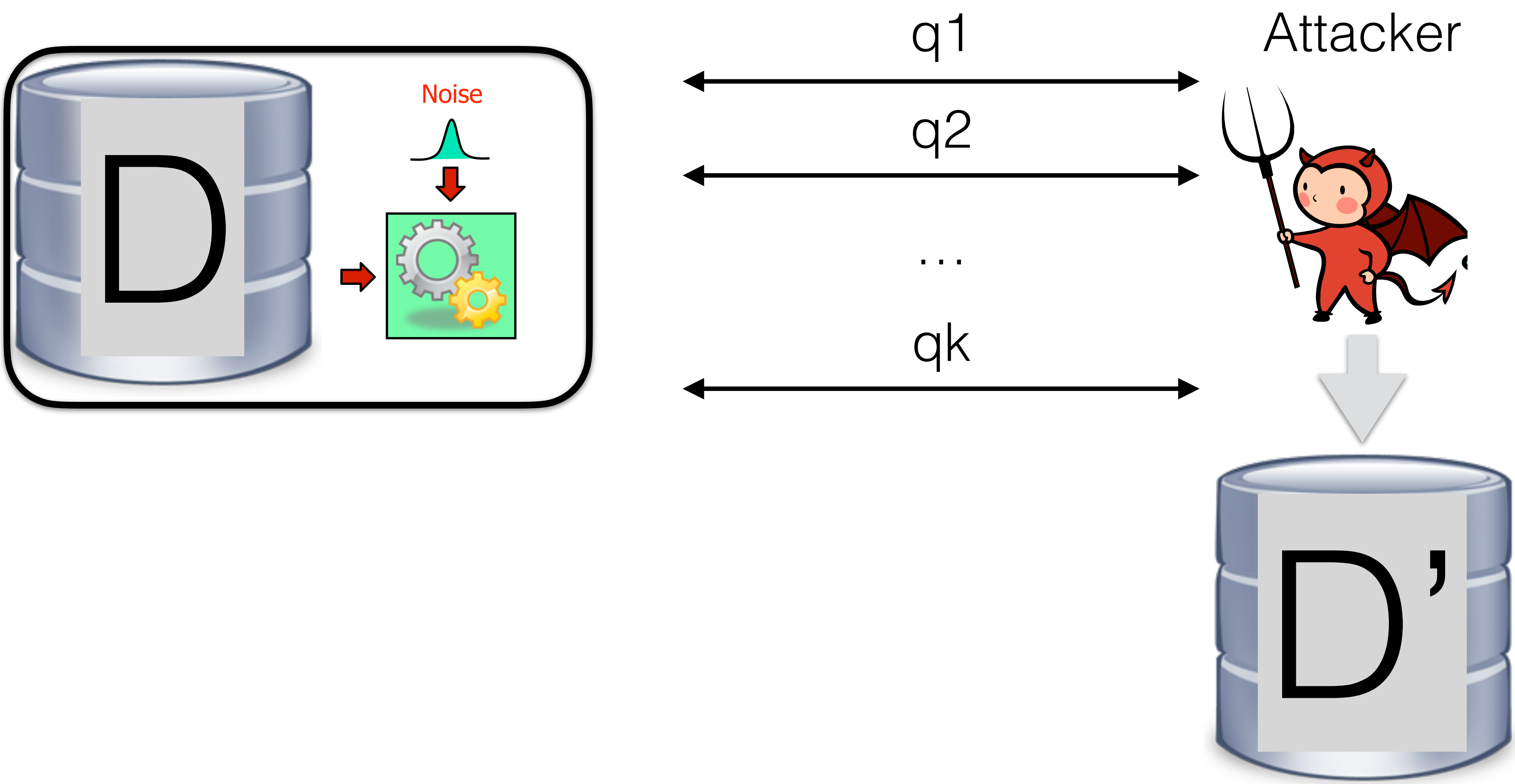

Reconstruction attack

We say that the attacker wins if

$$d(D, D') \sim 0$$

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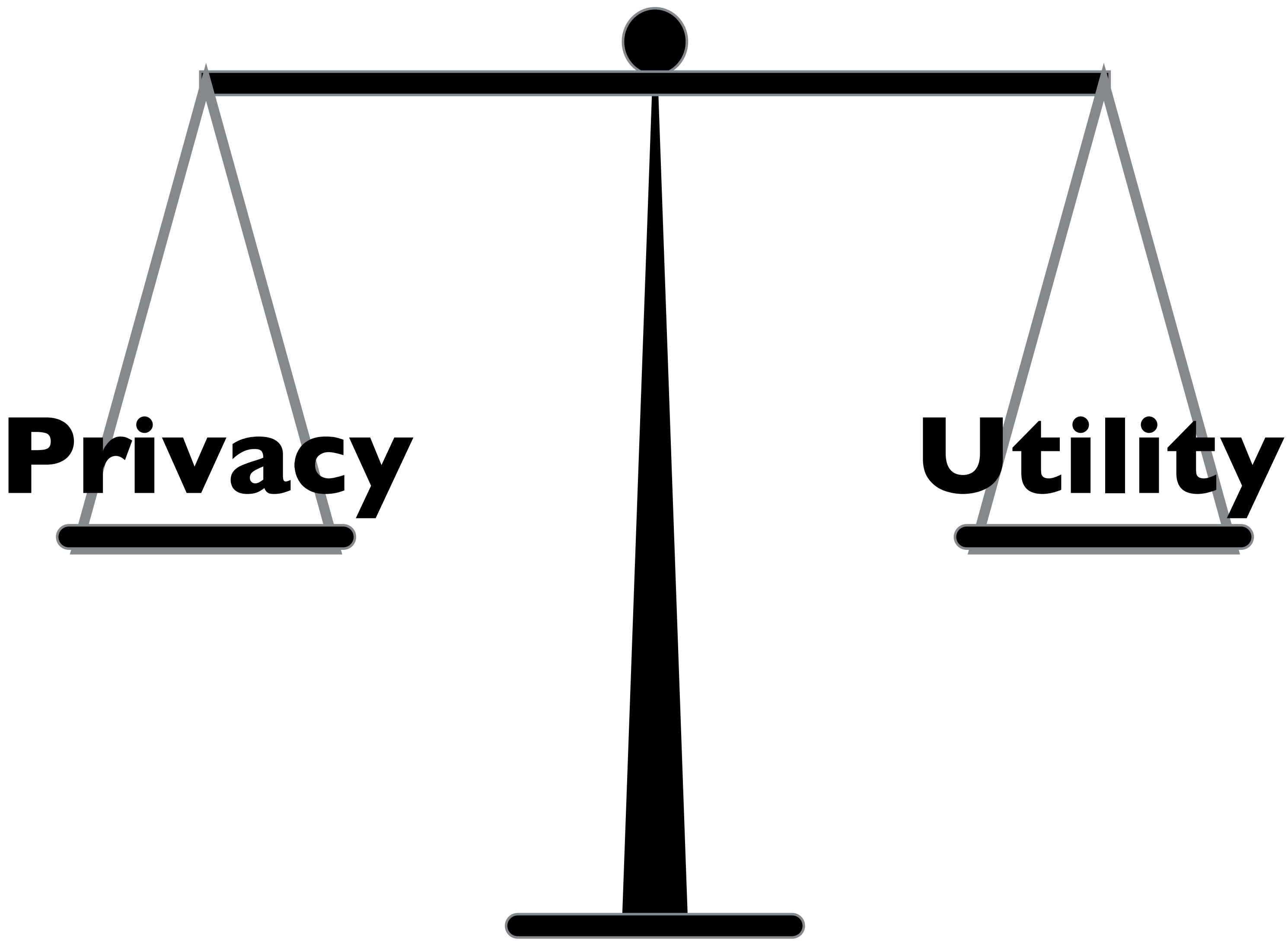
In this class case we can use Hamming distance

Privacy vs Utility



Quantitative notions of Privacy

- The impossibility results discussed above suggest a quantitative notion of privacy,
- a notion where the privacy loss depends on the number of queries that are allowed,
- and on the accuracy with which we answer them.

Differential privacy:
understanding the mathematical and computational meaning of this trade-off.

[Dwork, McSherry, Nissim, Smith, TCC06]

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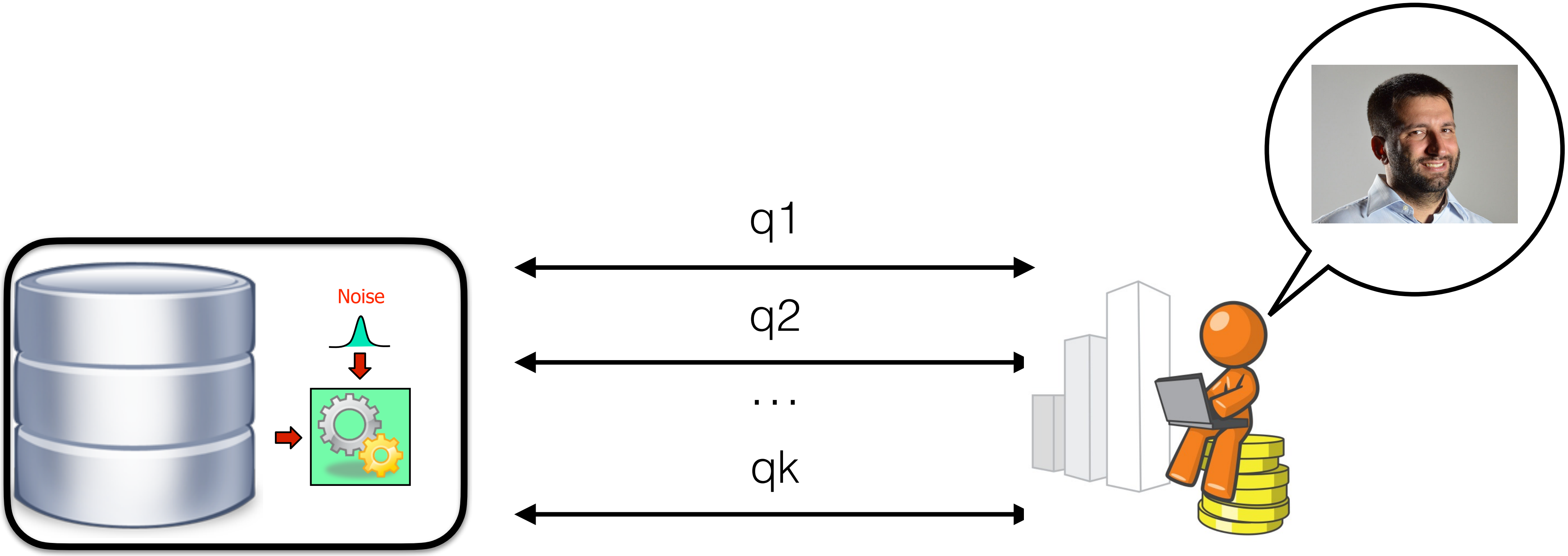

Privacy-preserving data analysis?

Prior Knowledge

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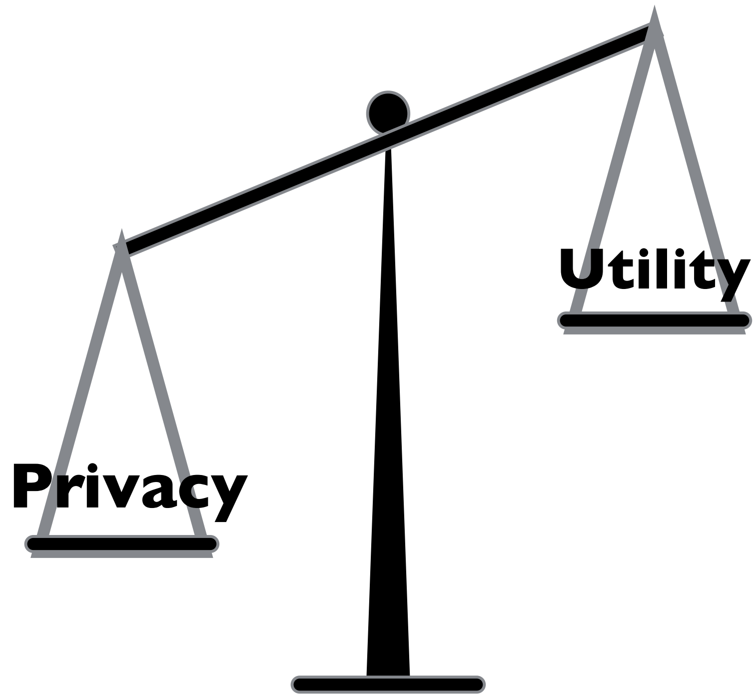
Posterior Knowledge

Privacy-preserving data analysis?

Privacy-preserving data analysis?

Question: What is the problem with this requirement?

Privacy-preserving data analysis?



If nothing can be learned about an individual, then nothing at all can be learned at all!

[DworkNaor10]

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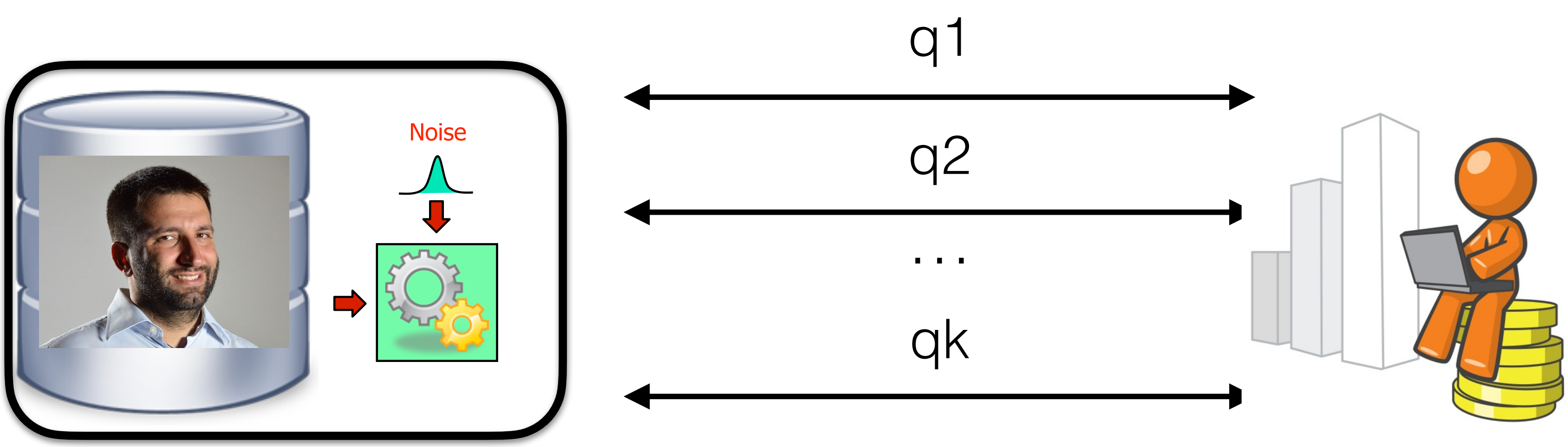

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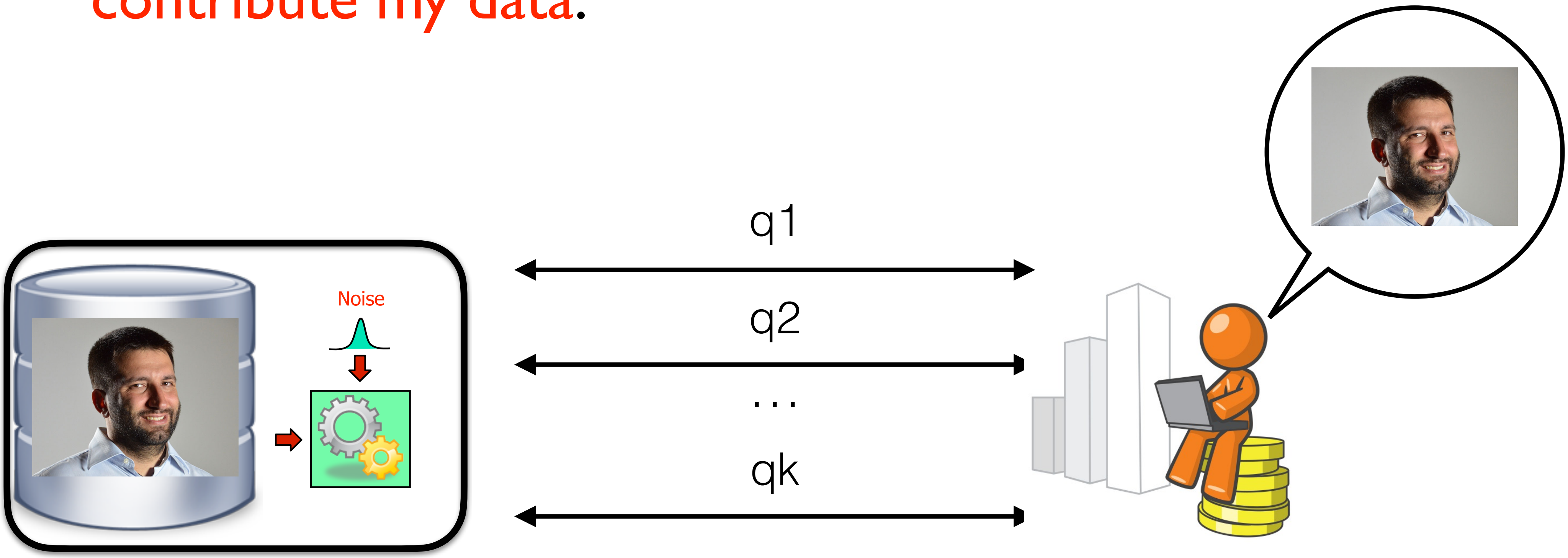

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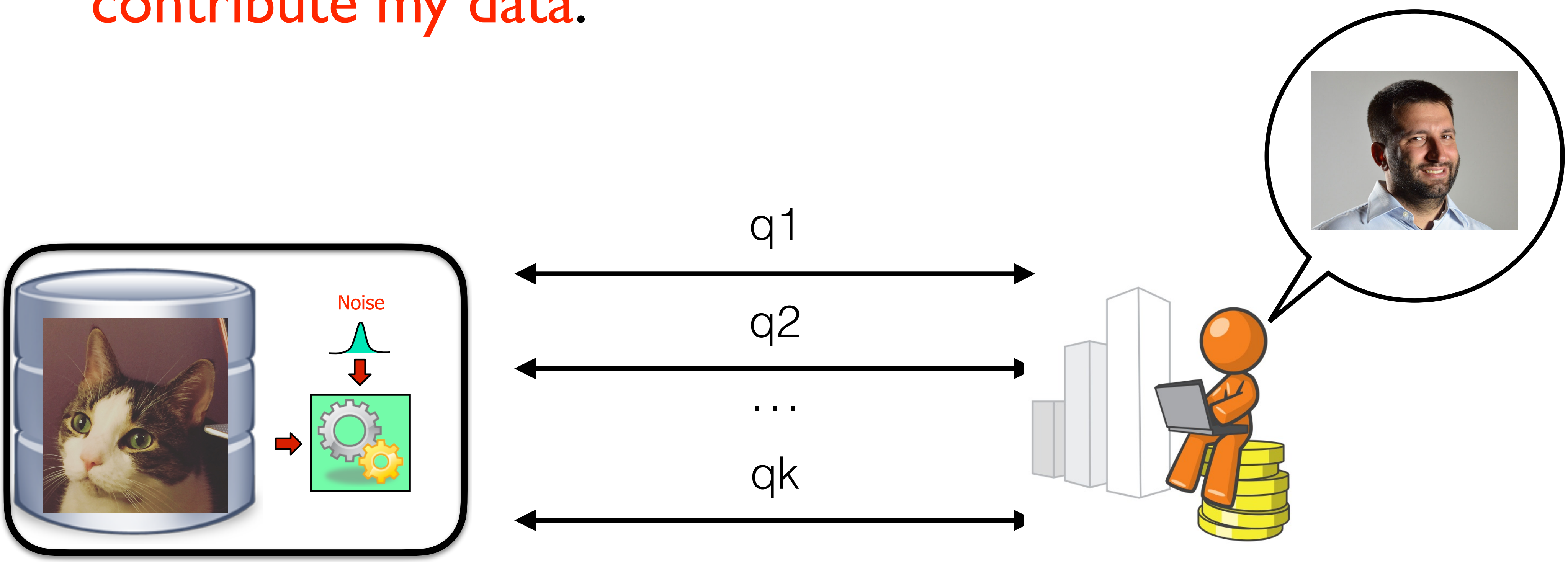


Adjacent databases

- We can formalize the concept of contributing my data or not in terms of a notion of distance between datasets.
- Given two datasets D, D'$\in$DB, their distance is defined as:

$$D\Delta D' = |\{k \leq n \mid D(k) \neq D'(k)\}|$$

- We will call two datasets adjacent when $D\Delta D' = 1$ and we will write $D \sim D'$.

Privacy Loss

In general we can think about the following quantity as the privacy loss incurred by observing r on the databases b and b'.

$$L_{b,b'}(r) = \log \frac{\Pr[Q(b)=r]}{\Pr[Q(b')=r]}$$

(ε,δ)-Differential Privacy

Definition

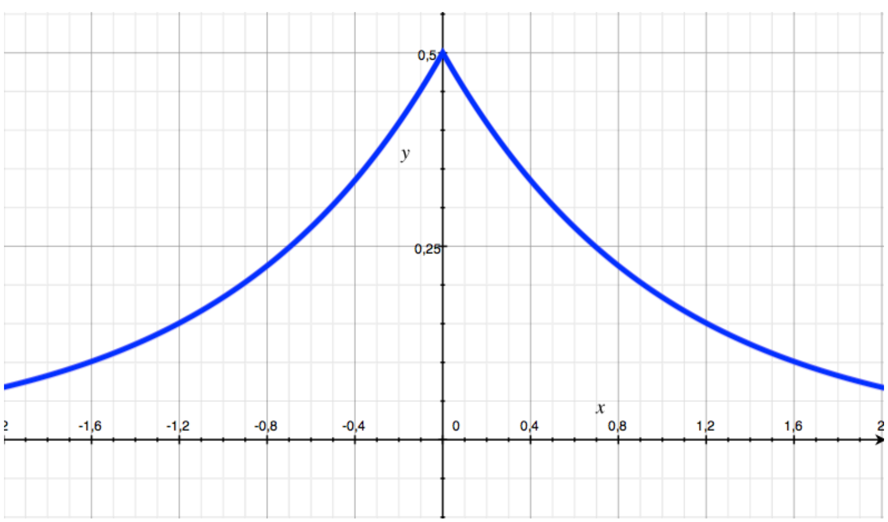
Given $\varepsilon,\delta \geq 0$, a probabilistic query $Q: X^n \rightarrow R$ is (ε,δ)-differentially private iff for all adjacent database b_1, b_2 and for every $S \subseteq R$:

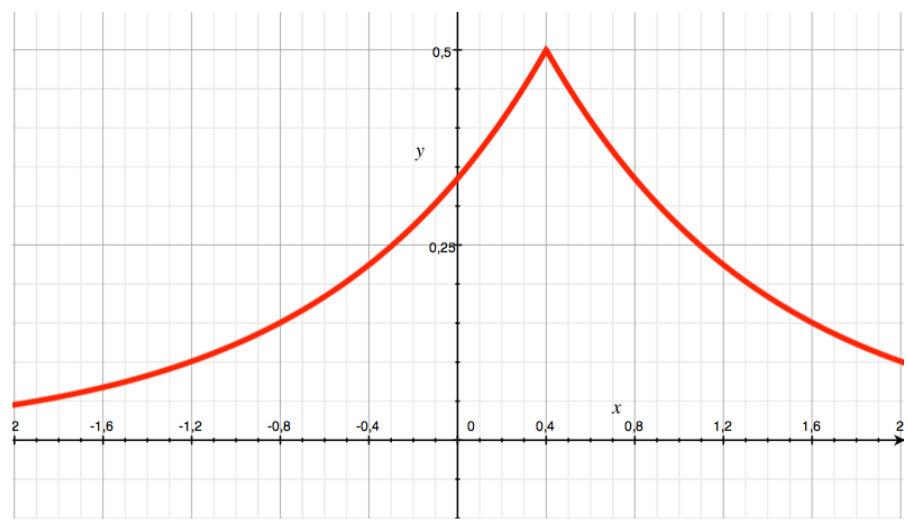
$$\Pr[Q(b_1) \in S] \leq \exp(\varepsilon)\Pr[Q(b_2) \in S] + \delta$$

Differential Privacy

Q : db => R   probabilistic

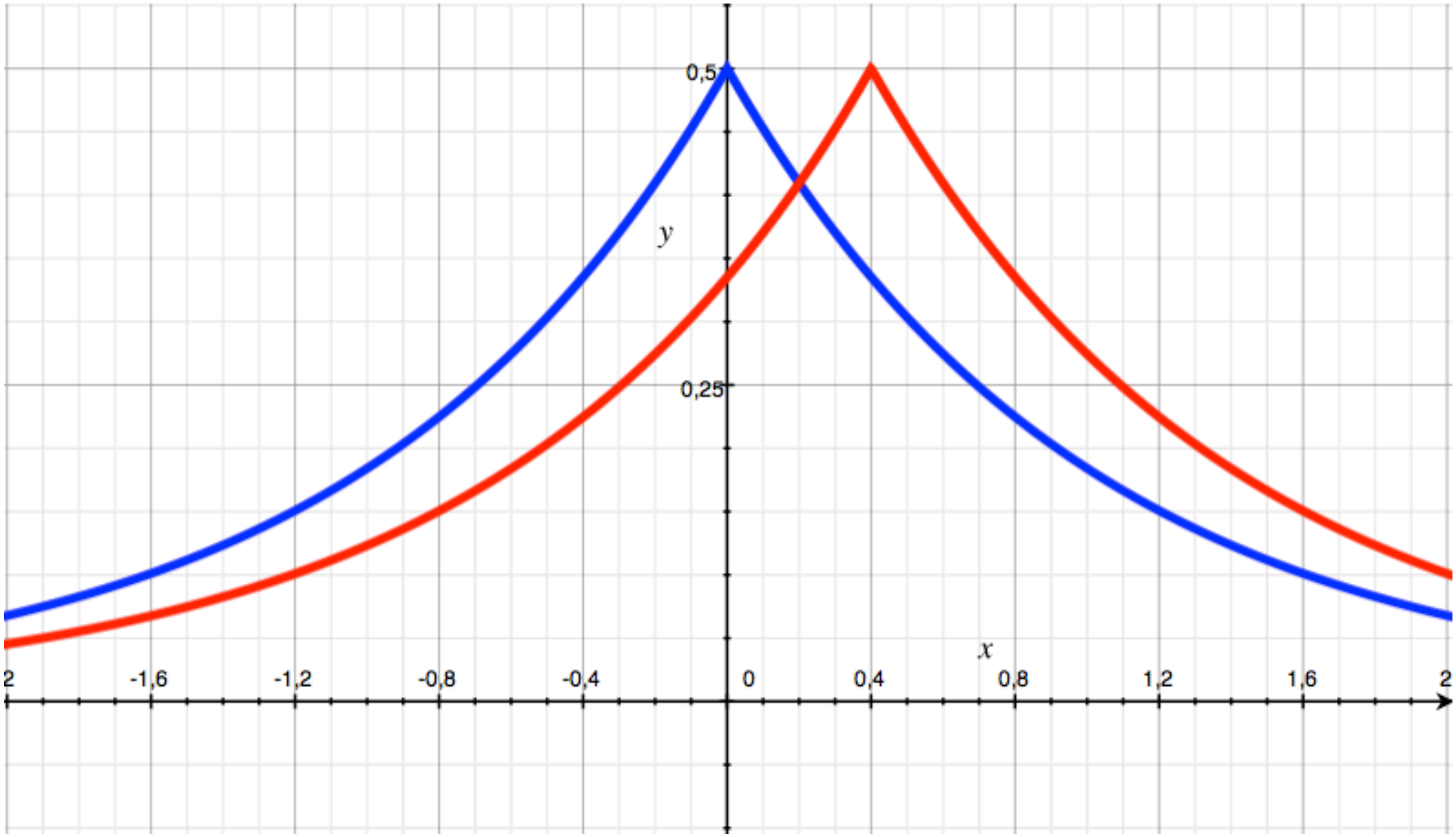
$Q(b \cup \{x\})$ $Q(b \cup \{y\})$





Differential Privacy

$d(Q(b \cup \{x\}), Q(b \cup \{y\})) \leq \varepsilon$ with probability $1-\delta$



(ε,δ)-Differential Privacy

$$\left| \log \frac{\Pr[Q(b_1)=r]}{\Pr[Q(b_2)=r]} \right| \leq \varepsilon \quad \text{with probability } 1-\delta$$