CS 591: Formal Methods in Security and Privacy

Formal Proofs for Cryptography

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From the previous class

Symmetric Encryption Schemes

- Our treatment of symmetric encryption schemes is parameterized by three types:

```
type key.     (* encryption keys, key_len bits *)
type text.    (* plaintexts, text_len bits *)
type cipher.  (* ciphertexts - scheme specific *)
```

- An encryption scheme is a *stateless* implementation of this module interface:

```
module type ENC = {
  proc key_gen() : key                  (* key generation *)
  proc enc(k : key, x : text) : cipher  (* encryption *)
  proc dec(k : key, c : cipher) : text  (* decryption *)
}.
```

Scheme Correctness

- An encryption scheme is *correct* if and only if the following procedure returns true with probability 1 for all arguments:

```
module Cor (Enc : ENC) = {
  proc main(x : text) : bool = {
    var k : key; var c : cipher; var y : text;
    k <@ Enc.key_gen();
    c <@ Enc.enc(k, x);
    y <@ Enc.dec(k, c);
    return x = y;
  }
}.
```

- The module `Cor` is parameterized (may be applied to) an arbitrary encryption scheme, `Enc`.

Encryption Oracles

- To define IND-CPA security of encryption schemes, we need the notion of an *encryption oracle*, which both the adversary and IND-CPA game will interact with:

```
module type EO = {
  (* initialization - generates key *)
  proc * init() : unit
  (* encryption by adversary before game's encryption *)
  proc enc_pre(x : text) : cipher
  (* one-time encryption by game *)
  proc genc(x : text) : cipher
  (* encryption by adversary after game's encryption *)
  proc enc_post(x : text) : cipher
}.
```

Standard Encryption Oracle

- Here is the standard encryption oracle, parameterized by an encryption scheme, `Enc`:

```
module EncO (Enc : ENC) : EO = {
  var key : key
  var ctr_pre : int
  var ctr_post : int

  proc init() : unit = {
    key <@ Enc.key_gen();
    ctr_pre <- 0; ctr_post <- 0;
  }
```

Standard Encryption Oracle

```
proc enc_pre(x : text) : cipher = {
  var c : cipher;
  if (ctr_pre < limit_pre) {
    ctr_pre <- ctr_pre + 1;
    c <@ Enc.enc(key, x);
  }
  else {
    c <- ciph_def;  (* default result *)
  }
  return c;
}
```

Standard Encryption Oracle

```
proc genc(x : text) : cipher = {
  var c : cipher;
  c <@ Enc.enc(key, x);
  return c;
}
```

Standard Encryption Oracle

```
  proc enc_post(x : text) : cipher = {
    var c : cipher;
    if (ctr_post < limit_post) {
      ctr_post <- ctr_post + 1;
      c <@ Enc.enc(key, x);
    }
    else {
      c <- ciph_def;  (* default result *)
    }
    return c;
  }
}.
```

Encryption Adversary

- An *encryption adversary* is parameterized by an encryption oracle:

```
module type ADV (EO : EO) = {
  (* choose a pair of plaintexts, x1/x2 *)
  proc * choose() : text * text {EO.enc_pre}

  (* given ciphertext c based on a random boolean b
     (the encryption using EO.genc of x1 if b = true,
      the encryption of x2 if b = false), try to guess b
  *)
  proc guess(c : cipher) : bool {EO.enc_post}
}.
```

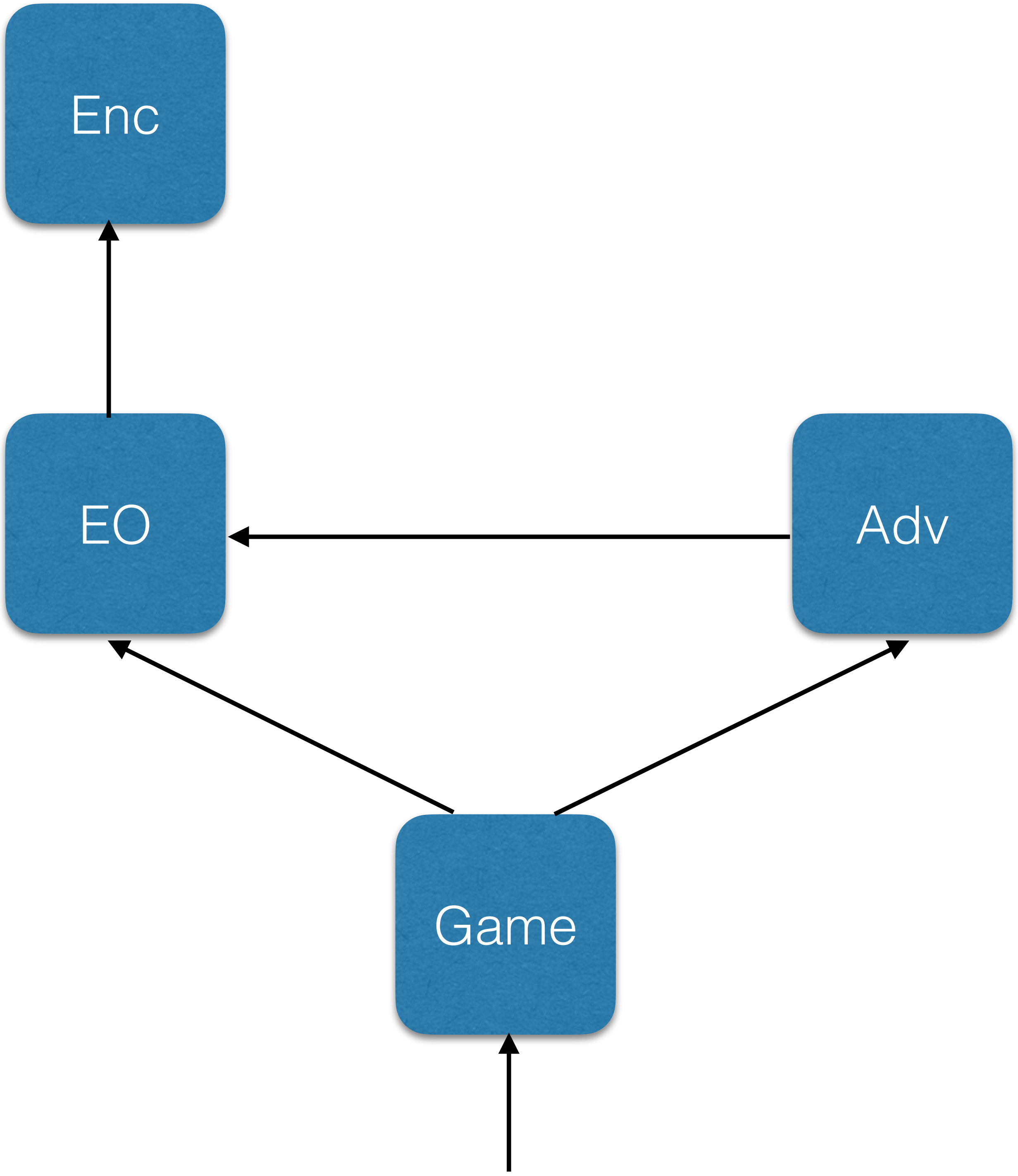
- Adversaries may be probabilistic.

IND-CPA Game

- The IND-CPA Game is parameterized by an encryption scheme and an encryption adversary:

```
module INDCPA (Enc : ENC, Adv : ADV) = {
  module EO = EncO(Enc)           (* make EO from Enc *)
  module A = Adv(EO)              (* connect Adv to EO *)
  proc main() : bool = {
    var b, b' : bool; var x1, x2 : text; var c : cipher;
    EO.init();                    (* initialize EO *)
    (x1, x2) <@ A.choose();       (* let A choose x1/x2 *)
    b <$ {0,1};                   (* choose boolean b *)
    c <@ EO.genc(b ? x1 : x2);    (* encrypt x1 or x2 *)
    b' <@ A.guess(c);             (* let A guess b from c *)
    return b = b';                (* see if A won *)
  }
}.
```

IND-CPA Game



IND-CPA Game

- If the value `b'` that `Adv` returns is independent of the random boolean `b`, then the probability that `Adv` wins the game will be exactly 1/2.
  - E.g., if `Adv` always returns true, it'll win half the time.
- The question is how much better it can do—and we want to prove that it can't do much better than win half the time.
  - But this will depend upon the quality of the encryption scheme.
- An adversary that *wins* with probability greater than 1/2 can be converted into one that *loses* with that probability, and vice versa. When formalizing security, it's convenient to upper-bound the *distance* between the probability of the adversary winning and 1/2.

IND-CPA Security

- In our security theorem for a given encryption scheme `Enc` and adversary Adv, we prove an upper bound on the absolute value of the difference between the probability that `Adv` wins the game and 1/2:

```
`|Pr[INDCPA(Enc, Adv).main() @ &m : res] - 1%r / 2%r|
  <= … Adv …
```

- Ideally, we'd like the upper bound to be 0, so that the probability that `Enc` wins is exactly 1/2, but this won't be possible.

- The upper bound may also be a function of the number of bits `text_len` in `text` and the encryption oracle limits `limit_pre` and `limit_post`.

IND-CPA Security

- Q: Because the adversary can call the encryption oracle with the plaintexts x_1/x_2 it goes on to choose, why isn't it impossible to define a secure scheme?
  - A: Because encryption can (must!) involve randomness.
- Q: What is the rationale for letting the adversary call `enc_pre` and `enc_post` at all?
  - A: It models the possibility that the adversary may be able to influence which plaintexts are encrypted.
- Q: What is the rationale for limiting the number of times `enc_pre` and `enc_post` may be called?
  - A: There will probably be some limit on the adversary's influence on what is encrypted.

Next: Encryption from PRFs

Pseudorandom Functions

- Our pseudorandom function (PRF) is an operator `F` with this type:

```
op F : key -> text -> text.
```

- For each value `k` of type key, `(F k)` is a function from text to text.
- Since key is a bitstring of length `key_len`, then there are at most $2^{\texttt{key\_len}}$ of these functions.
- If we wanted, we could try to spell out the code for `F`, but we choose to keep `F` abstract.
- How do we know if `F` is a “good” PRF?

Pseudorandom Functions

- We will assume that `dtext` (`dkey`) is a sub-distribution on `text` (`key`) that is a distribution (is "lossless"), and where every element of `text` (`key`) has the same non-zero value:

```
op dtext : text distr.
op dkey  : key distr.
```

- A *random function* is a module with the following interface:

```
module type RF = {
  (* initialization *)
  proc * init() : unit
  (* application to a text *)
  proc f(x : text) : text
}.
```

Pseudorandom Functions

- Here is a random function made from our PRF **F**:

```
module PRF : RF = {
  var key : key
  proc init() : unit = {
    key <$ dkey;
  }
  proc f(x : text) : text = {
    var y : text;
    y <- F key x;
    return y;
  }
}.
```

Pseudorandom Functions

- Here is a random function made from true randomness:

```
module TRF : RF = {
  (* mp is a finite map associating texts with texts *)
  var mp : (text, text) fmap
  proc init() : unit = {
    mp <- empty;  (* empty map *)
  }
  proc f(x : text) : text = {
    var y : text;
    if (! x \in mp) {   (* give x a random value in *)
      y <$ dtext;  (* mp if not already in mp's domain *)
      mp.[x] <- y;
    }
   return oget mp.[x];  (* return value of x in mp *)
  }  (* mp.[x] is: None if x is not in mp’s domain, *)
}.   (* and Some z if z is the value of x in mp *)
```

Pseudorandom Functions

- A *random function adversary* is parameterized by a random function module:

```
module type RFA (RF : RF) = {
  proc * main() : bool {RF.f}
}.
```

Pseudorandom Functions

- Here is the random function game:

```
module GRF (RF : RF, RFA : RFA) = {
  module A = RFA(RF)
  proc main() : bool = {
    var b : bool;
    RF.init();
    b <@ A.main();
    return b;
  }
}.
```

- A random function adversary RFA tries to tell the PRF and true random functions apart, by *returning true with different probabilities.*

Pseudorandom Functions

- Our PRF F is “good” if and only if the following is small, whenever RFA is limited in the amount of computation it may do (maybe we say it runs in polynomial time):

```
`|Pr[GRF(PRF, RFA).main() @ &m : res] -
  Pr[GRF(TRF, RFA).main() @ &m : res]|
```

- `RFA` must be limited, because there will typically be many more true random functions than functions of the form `(F k)`, where `k` is a key (there are at most $2^{\texttt{key\_len}}$ such functions).
  - Since `text_len` is the number of bits in `text`, there will be $2^{\texttt{text\_len}}$ `^` $2^{\texttt{text\_len}}$ distinct maps from `text` to `text` (e.g., $2^8 = 256$, $2^8 \wedge 2^8 \sim= 10^{617}$).
  - Thus, with enough running time, `RFA` may be able to tell with reasonable probability if it’s interacting with a PRF random function or a true random function.

Our Symmetric Encryption Scheme

- We construct our encryption scheme `Enc` out of `F`:

```
(+^) : text -> text -> text  (* bitwise exclusive or *)

type cipher = text * text.  (* ciphertexts *)

module Enc : ENC = {
  proc key_gen() : key = {
    var k : key;
    k <$ dkey;
    return k;
  }
```

Our Symmetric Encryption Scheme

```
  proc enc(k : key, x : text) : cipher = {
    var u : text;
    u <$ dtext;
    return (u, x +^ F k u);
  }

  proc dec(k : key, c : cipher) : text = {
    var u, v : text;
    (u, v) <- c;
    return v +^ F k u;
  }
}.
```

Correctness

- Suppose that `enc(k, x)` returns `c` = `(u, x +^ F k u)`, where `u` is randomly chosen.
- Then `dec(k, c)` returns `(x +^ F k u) +^ F k u` = `x`.

Adversarial Attack Strategy

- Before picking its pair of plaintexts, the adversary can call `enc_pre` some number of times with the same argument, `text0` (the bitstring of length `text_len` all of whose bits are `0`).
- This gives us …, $(u_i,\ \texttt{text0 +\^{} F key } u_i)$, …, i.e., …, $(u_i,\ \texttt{F key } u_i)$, …
- Then, when `genc` encrypts one of x_1/x_2, it *may happen* that we get a pair $(u_i,\ x_j\ \texttt{+\^{} F key } u_i)$ for one of them, where u_i appeared in the results of calling `enc_pre`.
- But then

$$\texttt{F key } u_i \texttt{ +\^{} } (x_j \texttt{ +\^{} F key } u_i) = \texttt{text0 +\^{} } x_j = x_j$$

Adversarial Attack Strategy

- Similarly, when calling `enc_post`, before returning its boolean judgement `b` to the game, a collision with the left-side of the cipher text passed from the game to the adversary will allow it to break security.
- Suppose, again, that the adversary repeatedly encrypts `text0` using `enc_pre`, getting …, $(\mathtt{u_i}, \mathtt{F\ key\ u_i})$, …
- Then by *experimenting directly* with `F` with different keys, it may learn enough to guess, with reasonable probability, `key` itself.
- This will enable it to decrypt the cipher text `c` given it by the game, also breaking security.
- Thus we must assume some bounds on how much work the adversary can do (we can't tell if it's running `F`).

IND-CPA Security for Our Scheme

- Our security upper bound

```
`|Pr[INDCPA(Enc, Adv).main() @ &m : res] - 1%r / 2%r|
   <= …
```

will be a function of:

(1) the ability of a random function adversary constructed from `Adv` to tell the PRF random function from the true random function

- this lets us switch in our proof from using `F` to using a true random function

(2) the number of bits `text_len` in `text` and the encryption oracles limits `limit_pre` and `limit_post`

- this quantifies the possibility of collisions in the values of `u`

IND-CPA Security for Our Scheme

- Our security upper bound

```
`|Pr[INDCPA(Enc, Adv).main() @ &m : res] - 1%r / 2%r|
  <= …
```

will be a function of:

(1) the ability of a random function adversary constructed from `Adv` to tell the PRF random function from the true random function; and

(2) the number of bits `text_len` in `text` and the encryption oracles limits `limit_pre` and `limit_post`.

- Q: Why doesn't the upper bound also involve `key_len`, the number of bits in `key`?
  - A: that's part of (1).

IND-CPA Security for Our Scheme

- Later in the course, in lecture and/or lab, we'll survey the proof of IND-CPA security.
- Before then, you can look at all the definitions and the proofs on GitHub:

`https://github.com/alleystoughton/EasyTeach/tree/master/encryption`

If you are interested in doing a course project on the security of cryptographic schemes or protocols, Marco and I can make suggestions