CS 599: Formal Methods in Security and Privacy

Proofs of Protocol Security in Real/Ideal Paradigm

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Real/Ideal Paradigm

- We proved IND-CPA (indistinguishability under chosen plaintext attack) security of a symmetric encryption scheme built from a pseudorandom function plus randomness
- Now, we're going to consider a proof in the Real/Ideal Paradigm of the security of a three party cryptographic protocol
- In the real/ideal paradigm there are two games:
  - A "real" game based on how the actual protocol works
  - An "ideal" game that is secure by construction
- In the security proof we show that an Adversary can't distinguish the two games—or can only distinguish them with negligible probability

Private Count Retrieval Protocol

- The Private Count Retrieval (PCR) Protocol involves three parties:
  - a Server, which holds a database
  - a Client, which makes queries about the database
  - an *untrusted* Third Party (TP), which mediates between the Server and Client
- A database is one-dimensional: it consists of a list of elements
- Each query is also an element, and is a request for the count of the number of times it occurs in the database

Private Count Retrieval Protocol

- For example, suppose the database is `[0; 2; 0; 4; 2]`.
- If the query is `0`, the answer is:
  - `2`
- If the query is `4`, the answer is: `1`
- If the query is `3`, the answer is:
  - `0`

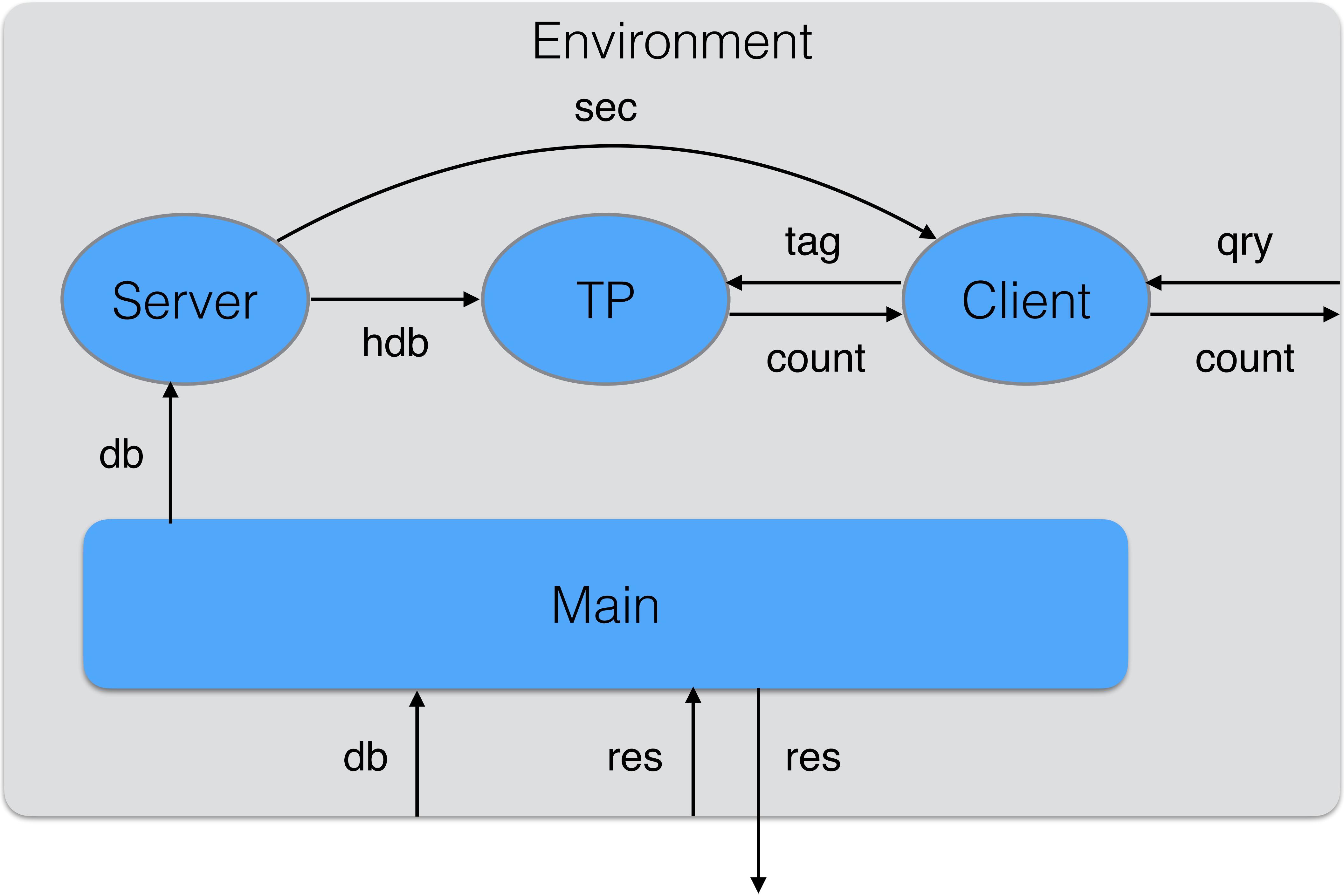
Security Goals for PCR

- Informally, the goal is for:
    - Client to only learn the counts for its queries, not anything else about the database (we'll limit how many queries it can make)
    - Server to learn nothing about the queries made by the Client other than the number of queries that were made
    - TP to learn nothing about the database and queries other than certain element *patterns*

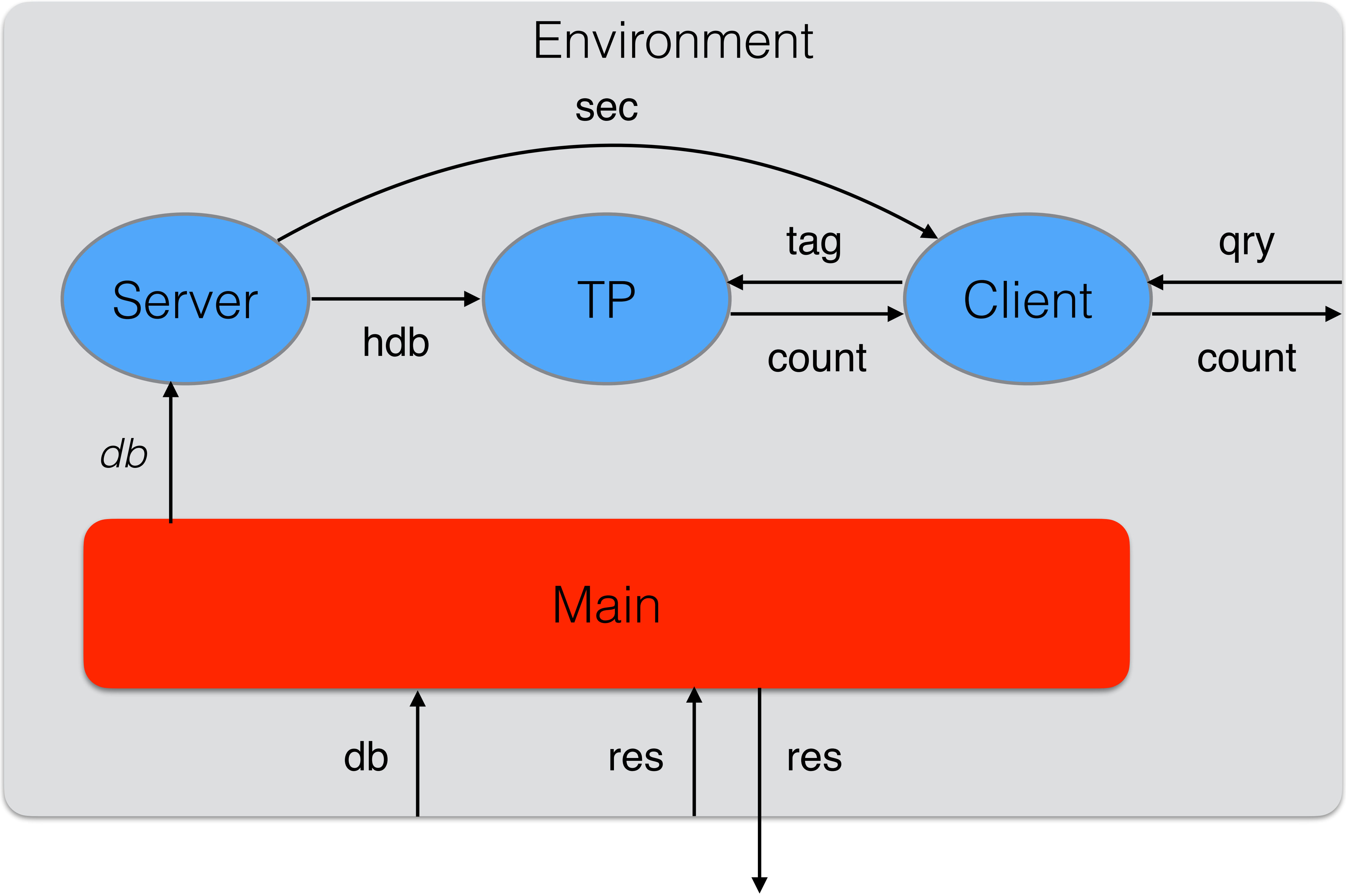
Hashing

- The PCR protocol makes use of *hashing*, a process transforming a value of some type into a bit string of a fixed length
  - When distinct inputs are hashed, it should be very unlikely that the resulting bit strings are equal
  - Given a bit string, it should be hard to find an input that hashes to it
- In an implementation, we might use a member of the SHA family of hash functions
- But in our proofs, we'll model hashing via a *random oracle*
  - Like the true random function of the IND-CPA example, but directly accessible to the adversary.

PCR Protocol Operation

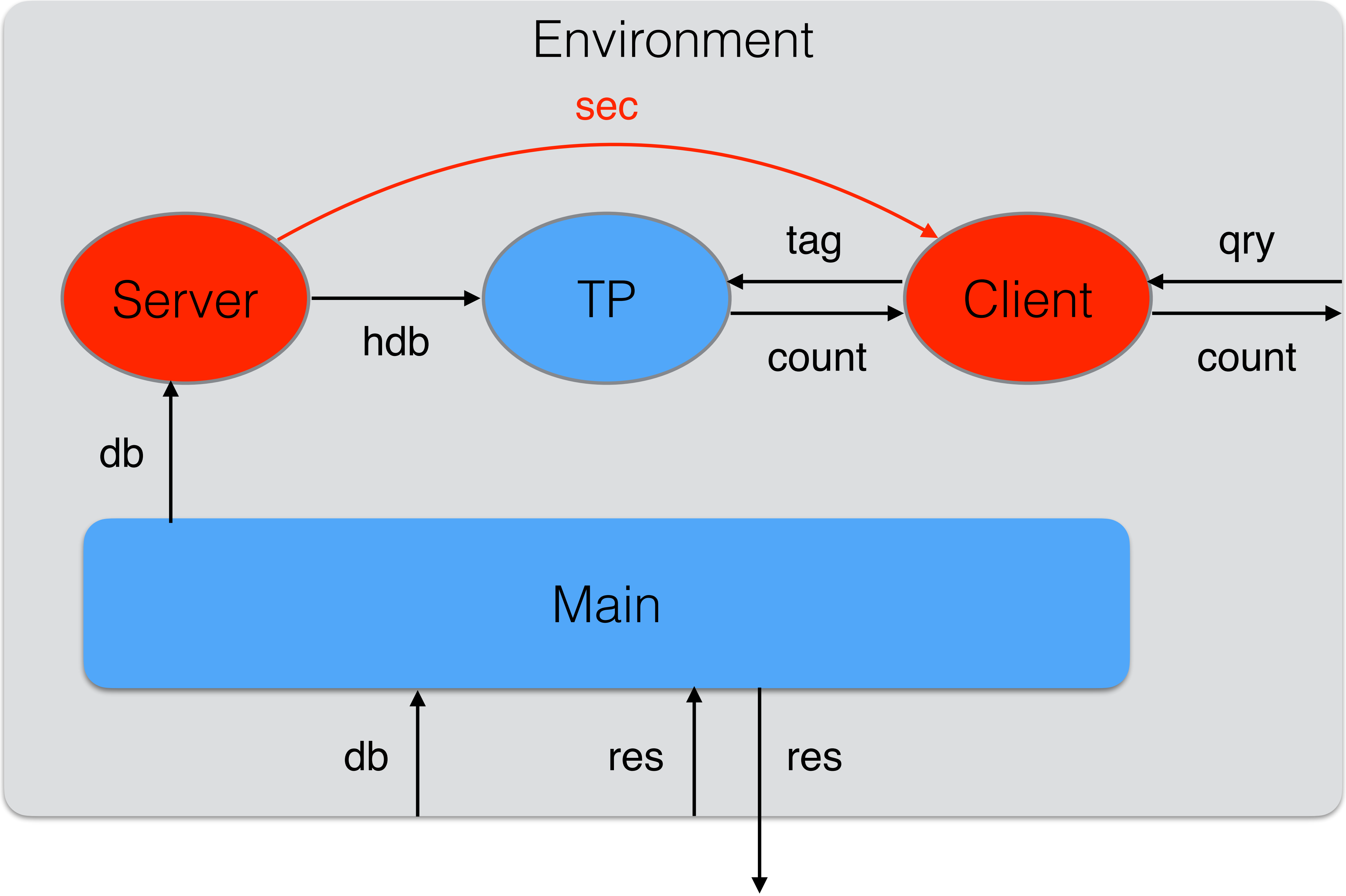


PCR Protocol Operation

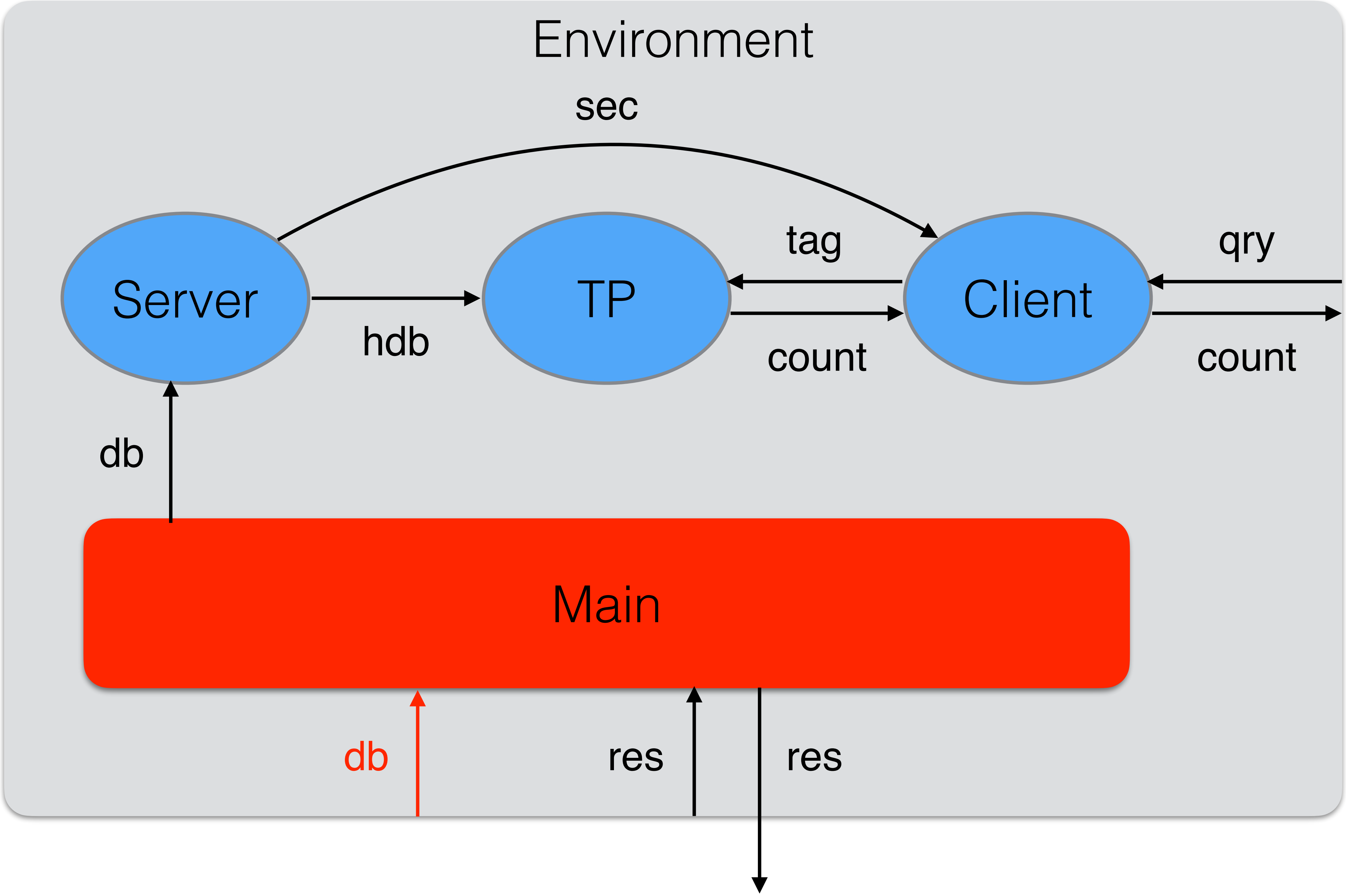


PCR Protocol Operation

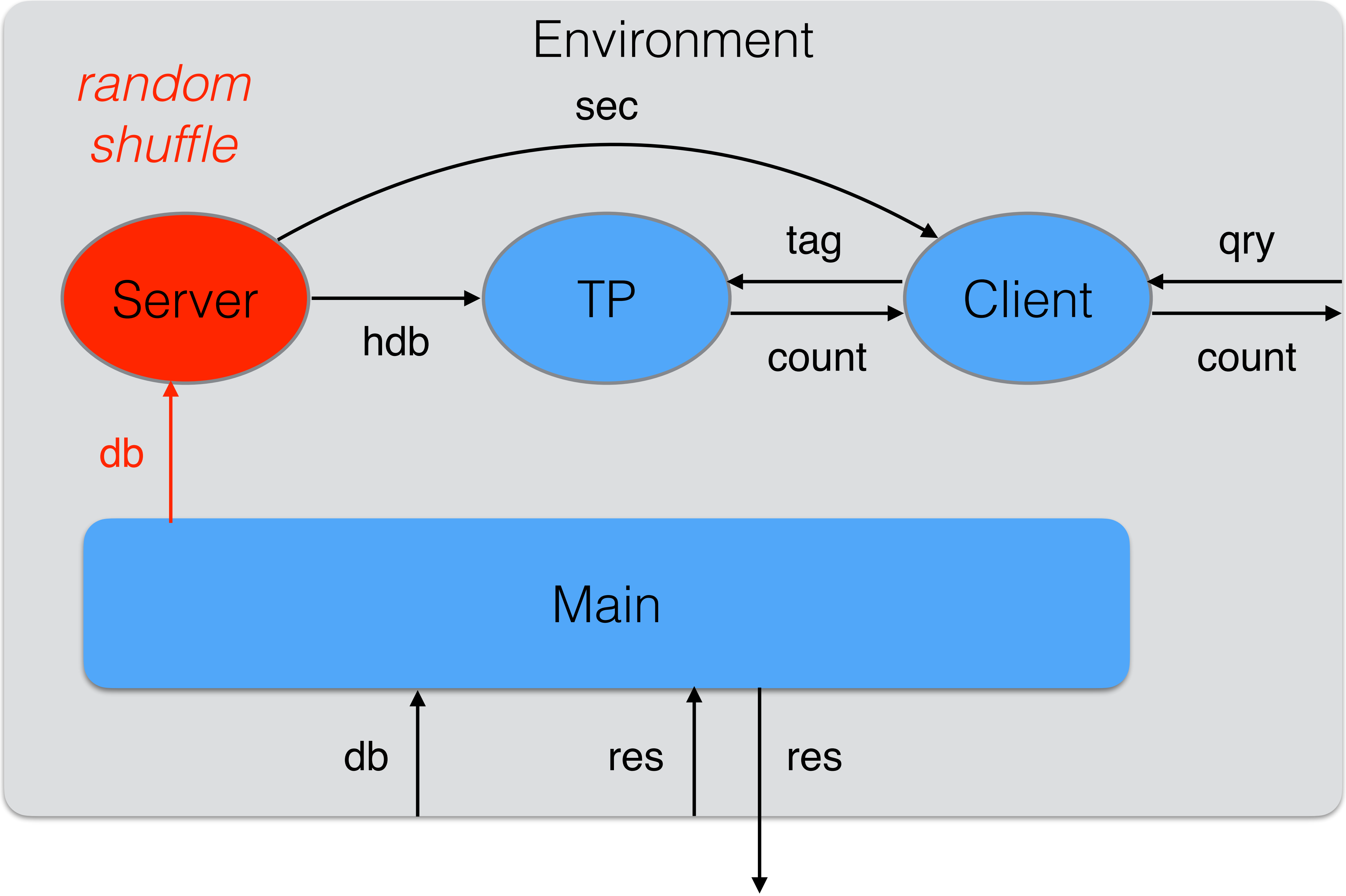
secrets are bit strings of length **sec_len**



PCR Protocol Operation

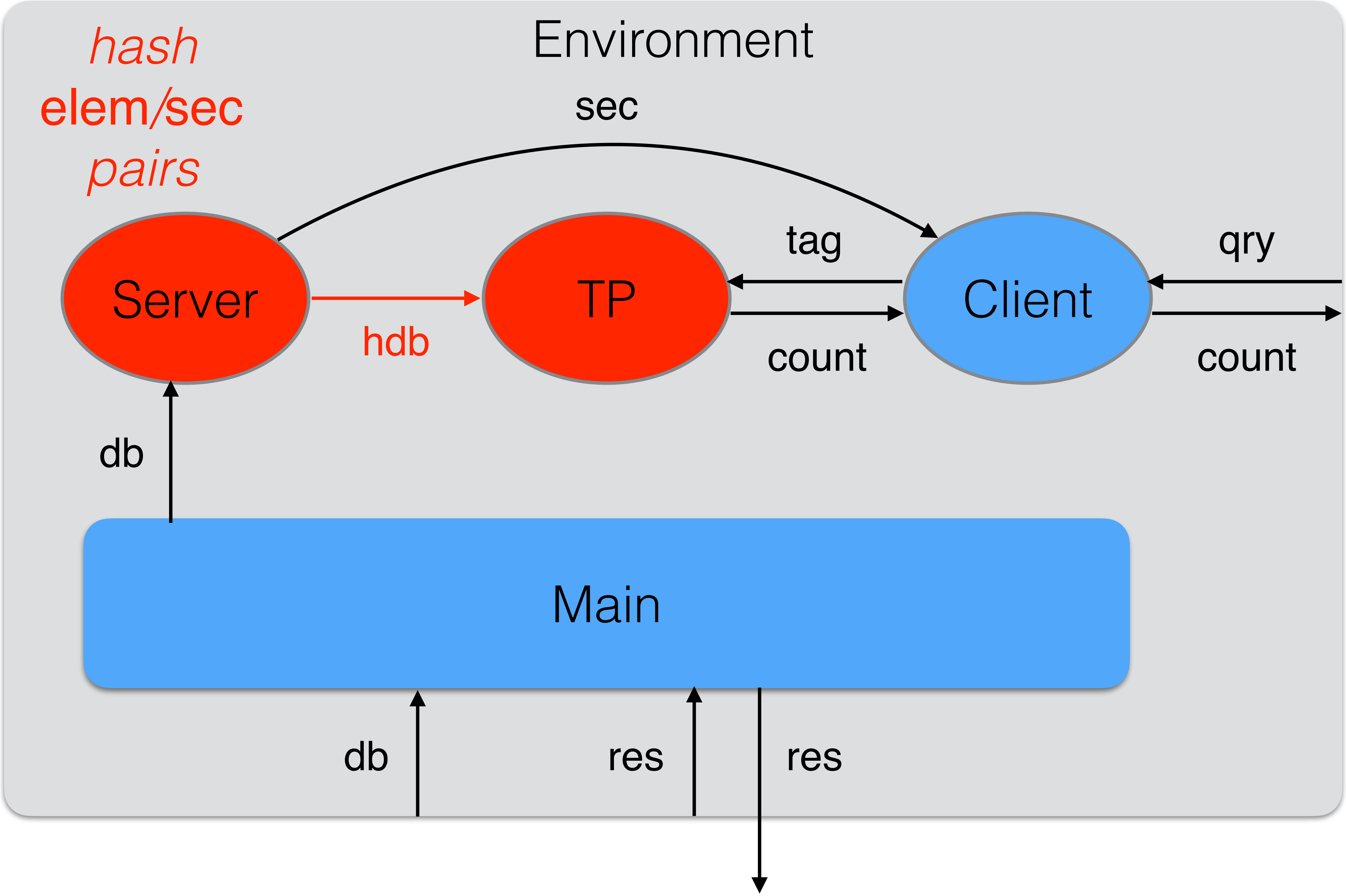


PCR Protocol Operation

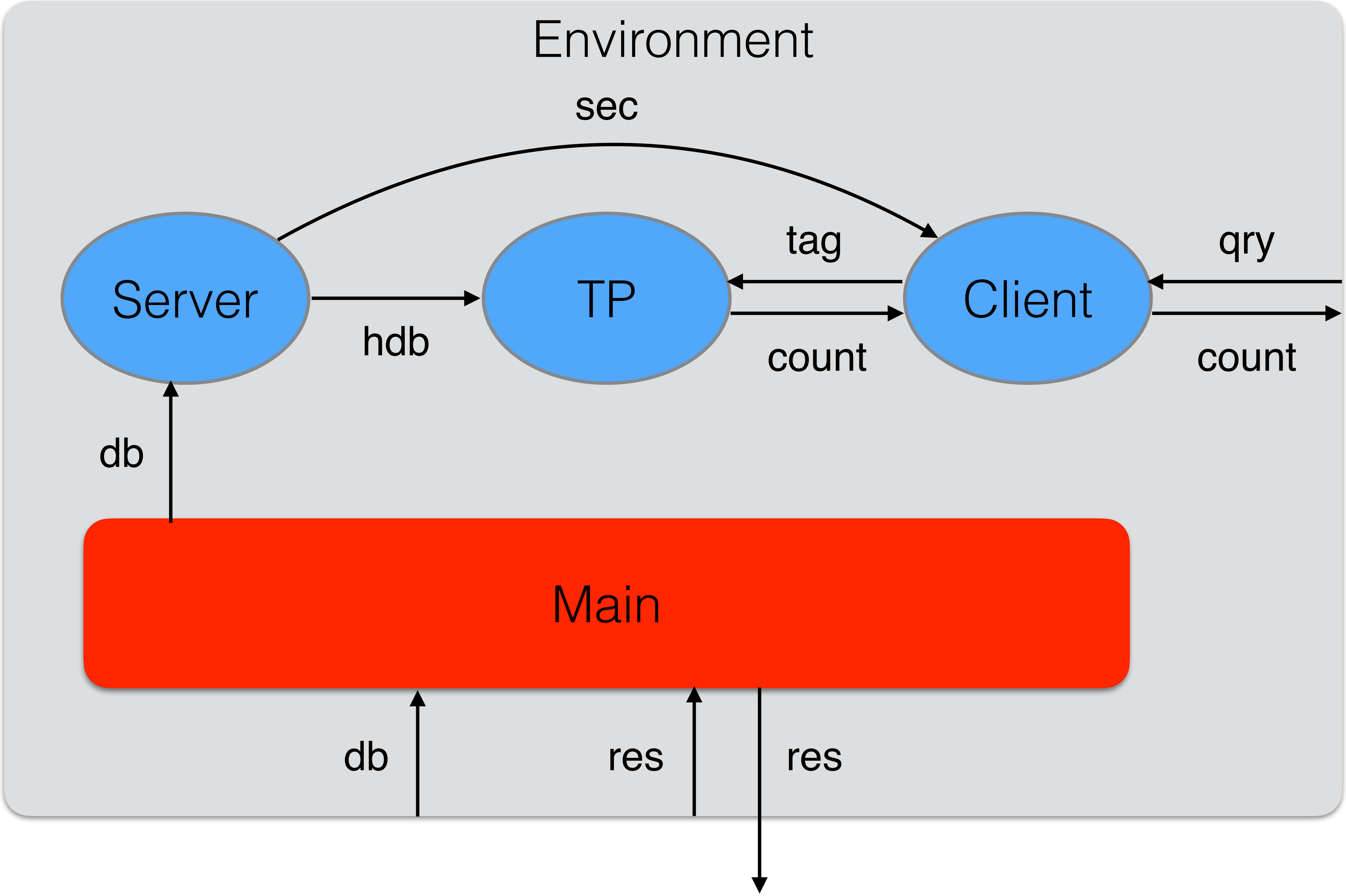


PCR Protocol Operation

tags are bit strings of length **tag_len**



PCR Protocol Operation



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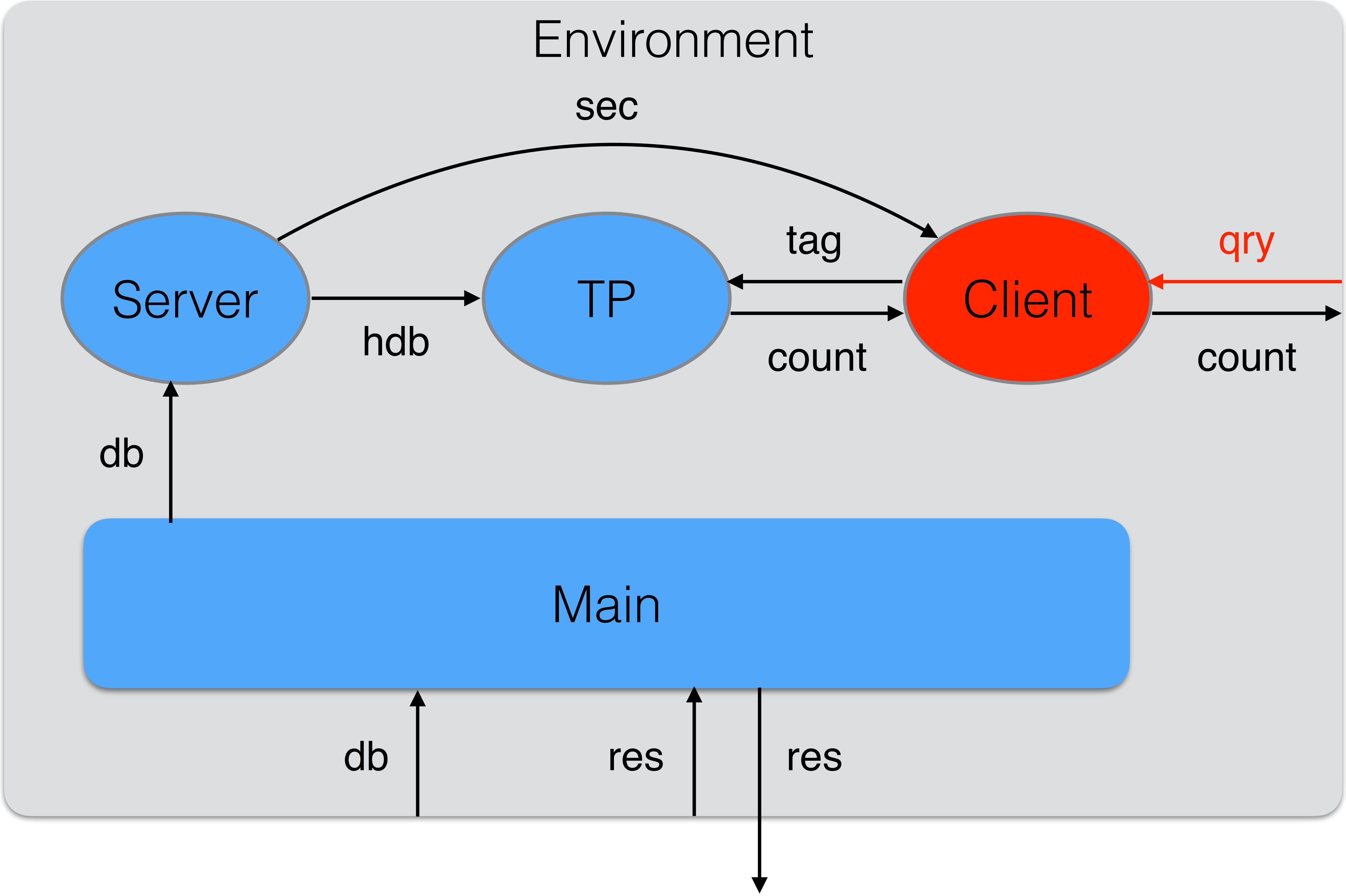

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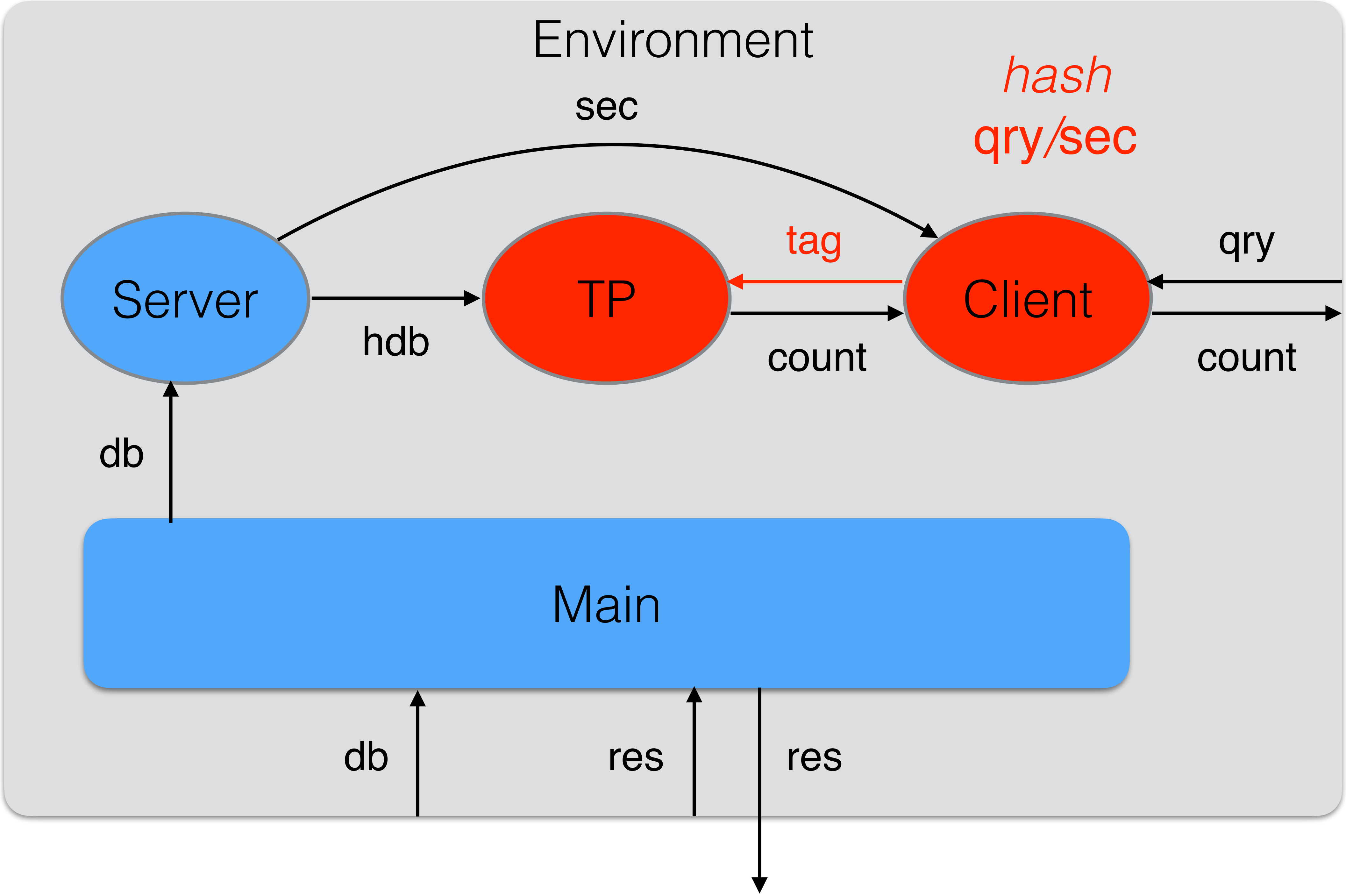

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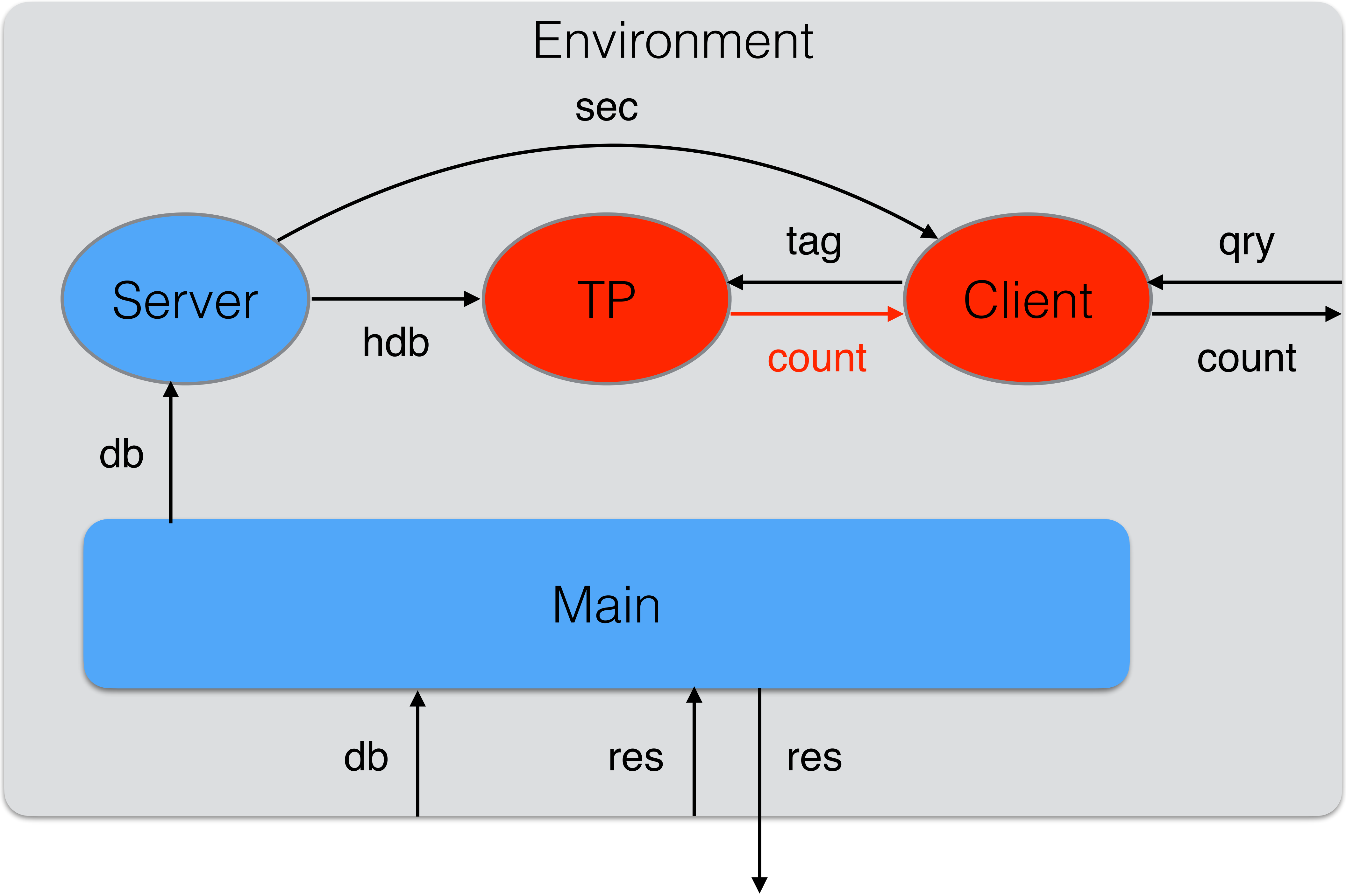

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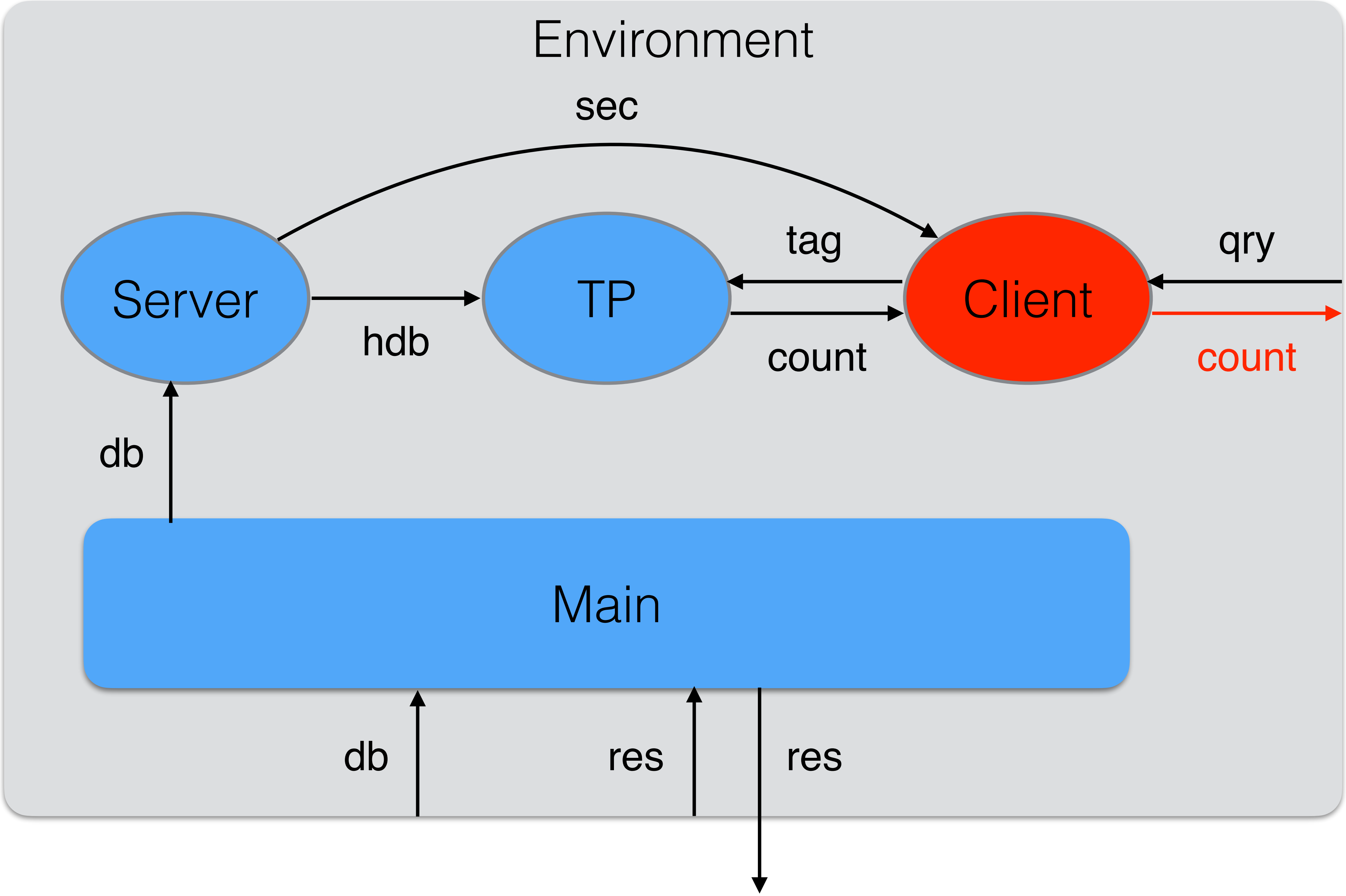

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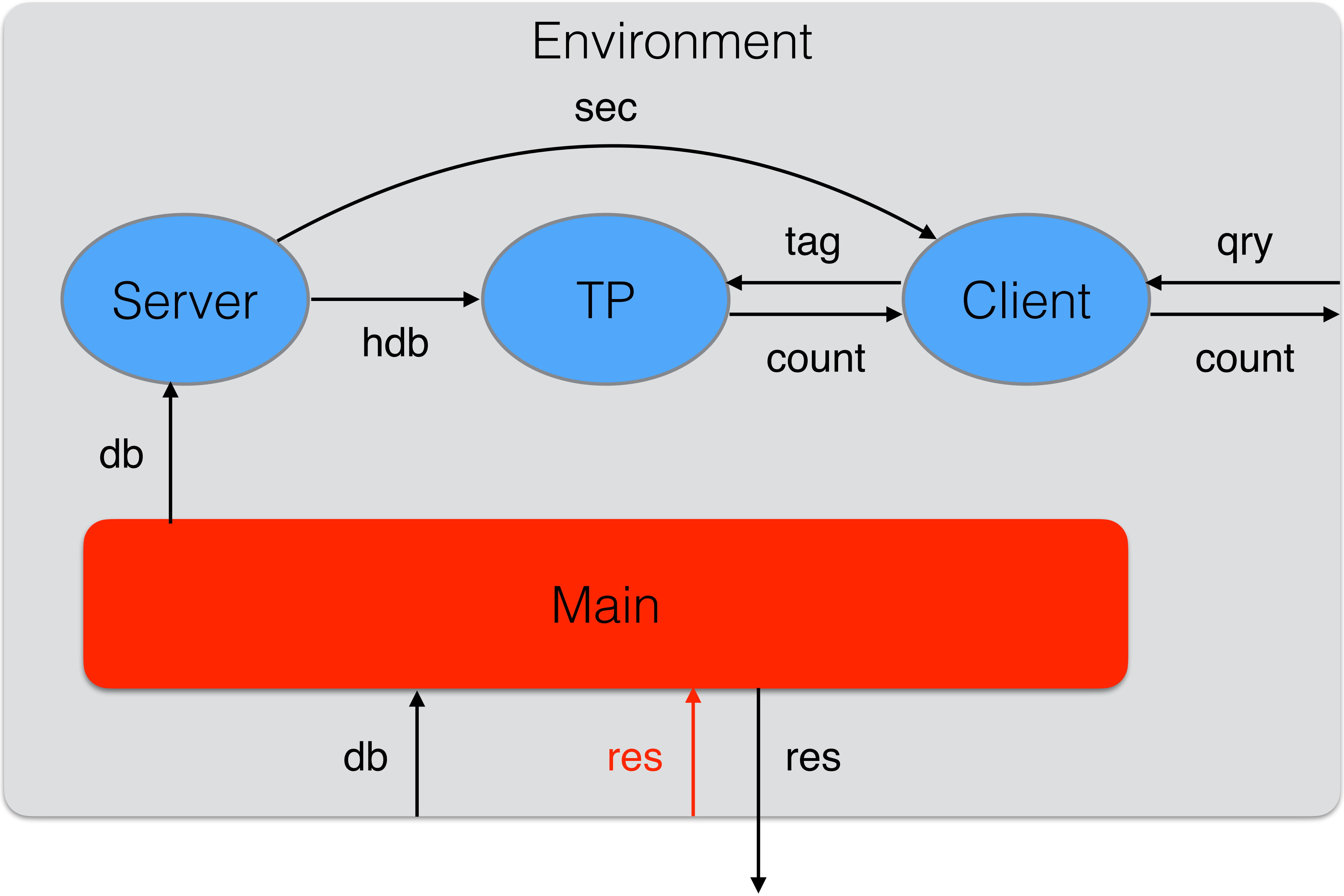

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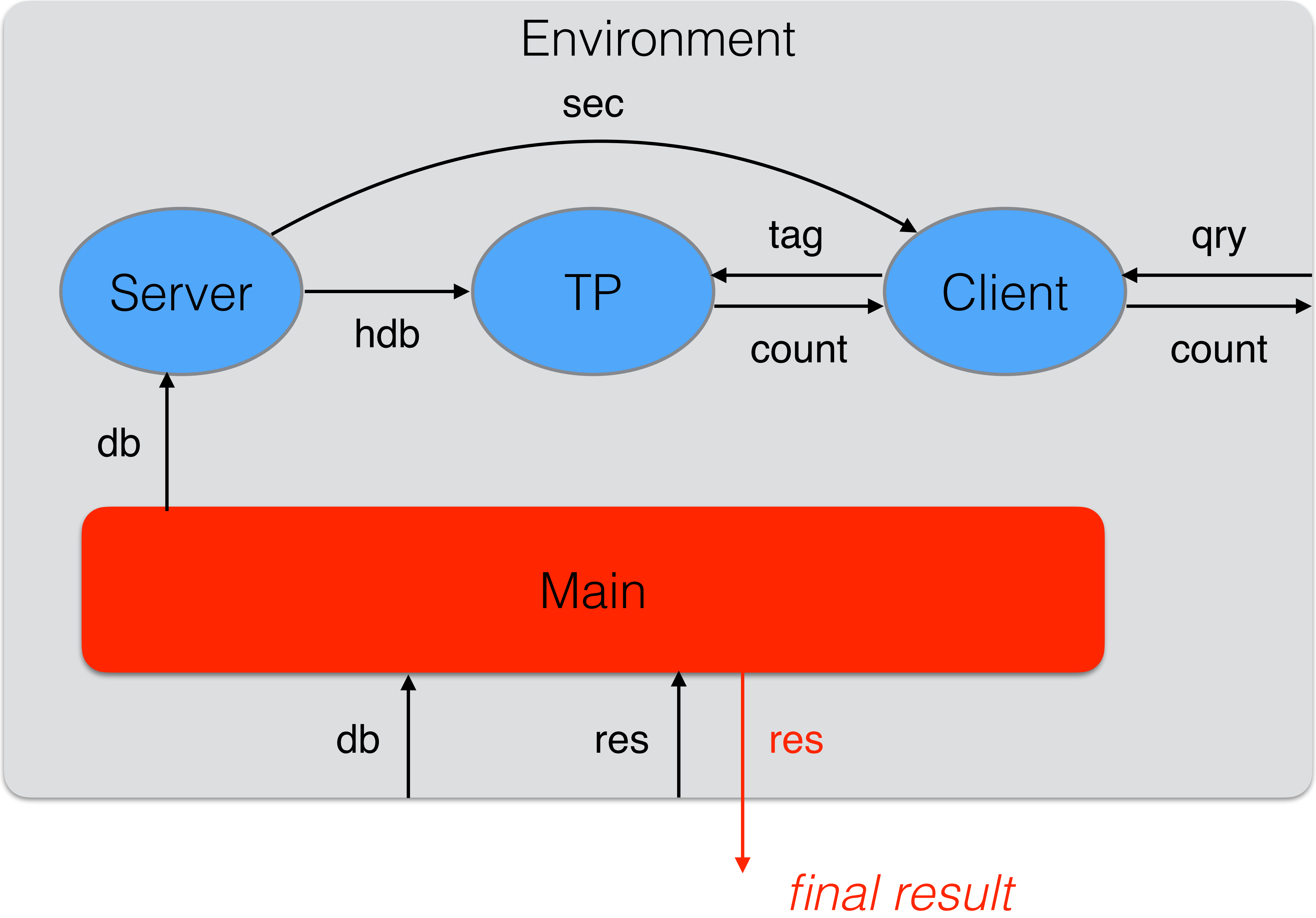


Protocol Example

- E.g., suppose the original database was `[0; 1; 1; 2]` and the queries are `1`, `2` and `3`
- The Server’s shuffled database might be `[1; 0; 2; 1]`
- TP will get a hashed database [t_2; t_1; t_3; t_2] and hash tags t_2, t_3 and t_4, and so will return to Client counts `2`, `1` and `0` (assuming no hash collisions)

EasyCrypt Code

- On GitHub you can find:
  - All the EasyCrypt definitions and proofs
  - A link to a conference paper about PCR and its proofs
    - Joint work with Mayank Varia

`https://github.com/alleystoughton/PCR`

Elements, Secrets and Hashing in EasyCrypt

- Elements (type `elem`) may be anything
- Secrets (type `sec`) are bits strings of length `sec_len`
- Hash tags (type `tag`) are bit strings of length `tag_len`
- Hashing is done using a random oracle in which element/secret pairs are hashed to hash tags

Random Oracle Theory `RandomOracle`

```
module type OR = {
  proc init() : unit
  proc hash(inp : input) : output
}.

module Or : OR = {
  var mp : (input, output) fmap

  proc init() : unit = {
    mp <- empty;
  }

  proc hash(inp : input) : output = {
    if (! dom mp inp) {
      mp.[inp] <$ output_distr;
    }
    return oget mp.[inp];
  }
}.
```

Random Oracle

```
clone RandomOracle as RO with
  type input <- elem * sec,
  op input_default <- (elem_default, zeros_sec),
  op output_len <- tag_len,
  type output <- tag,
  op output_default <- zeros_tag,
  op output_distr <- tag_distr
proof *.
(* realization *) … (* end *)
```

Thus `RO.Or` with module type `RO.OR` is the random oracle

Random Shuffling

```
module Shuffle = {
  proc shuffle(xs : elem list) : elem list = {
    var ys : elem list; var i : int;
    ys <- [];
    while (0 < size xs) {
      i <$ [0 .. size xs - 1];
      ys <- ys ++ [nth elem_default xs i];
      xs <- trim xs i;
    }
    return ys;
  }
}.
```

each of the (`size xs`)! reorderings of `xs` are equally possible (because of duplicates, some of these reorderings may be the same)

PCR Protocol

```
type db = elem list. type hdb = tag list.

…
type server_view = server_view_elem list.
type tp_view = tp_view_elem list.
type client_view = client_view_elem list.

module type ENV = {
  proc init_and_get_db() : db option
  proc get_qry() : elem option
  proc put_qry_count(cnt : int) : unit
  proc final() : bool
}.
```

Each party has a *view* variable that records everything it sees

PCR Protocol

```
module Protocol (Env : ENV) = {
  module Or = RO.Or
  …
  proc main() : bool = {
    var db_opt : db option; var b : bool;
    init_views(); Or.init();
    server_gen_sec(); client_get_sec();
    db_opt <@ Env.init_and_get_db();
    if (db_opt <> None) {
      server_hash_db(oget db_opt);
      tp_get_hdb();
      client_loop();
    }
    b <@ Env.final();
    return b;
  }
}.
```

PCR Protocol

```
proc client_loop() : unit = {
  var cnt : int; var tag : tag;
  var qry_opt : elem option;
  var not_done : bool <- true;
  while (not_done) {
    qry_opt <@ Env.get_qry();
    cv <- cv ++ [cv_got_qry qry_opt];
    if (qry_opt = None) {
      not_done <- false;
    } else {
      tag <@ Or.hash((oget qry_opt, client_sec));
      cnt <@ tp_count_tag(tag);
      cv <- cv ++
            [cv_query_count(oget qry_opt, tag, cnt)];
      Env.put_qry_count(cnt);
    }
  }
}
```

Adversarial Model

- We are modeling what is called *semi-honest* or *honest-but curious* security
- In this model, the Adversary is given access to a given protocol party's *view*—the party's data—but it is not allowed to modify that data
- The Adversary is also given access to the **`hash`** procedure of the random oracle — this is different from having access to its map
- The Real and Ideal games for each protocol party are parameterized by the Adversary
  - The Adversary tries to learn more from the protocol's view plus the **`hash`** procedure's view of the random oracle than it *should*
- At the end of the games, the Adversary returns a boolean judgement, trying to make the probability it returns `true` be as different as possible in the Real and Ideal games

Real Games

- The Real Games for the Server, Third Party and Client are formed as specializations of `Protocol`
- For a given party, we define the module type `ADV` of Adversaries for that party
    - In calls to the Adversary, the party's current view is supplied
- The Real Game `GReal` is
    - parameterized by `Adv : ADV`
    - defined by giving `Protocol` an environment `Env` made out of `Adv`

Example: Adversary for Server

```
module type ADV(O : RO.OR) = {
  proc init_and_get_db(view : server_view) :
    db option {O.hash}
  proc get_qry(view : server_view) : elem option {O.hash}
  proc qry_done(view : server_view) : unit {O.hash}
  proc final(view : server_view) : bool {O.hash}
}.
```

- Adversary can do hashing when deciding which database and queries to choose
- Queries are chosen one by one — *adaptively*
- `qry_done` is called with server view, which does not include the count for the query
- Each time the Adversary is called, it can do hashing to try to increase its knowledge

Example: Real Game for Server

```
module GReal(Adv : ADV) = {
  module Or = RO.Or
  module A  = Adv(Or)

  module Env : ENV = {
    proc init_and_get_db() : db option = {
      var db_opt : db option;
      db_opt <@ A.init_and_get_db(Protocol.sv);
      return db_opt;
    }

    proc get_qry() : elem option = {
      var qry_opt : elem option;
      qry_opt <@ A.get_qry(Protocol.sv);
      return qry_opt;
    }

    proc put_qry_count(cnt : int) : unit = {
      A.qry_done(Protocol.sv);
    }
```

Real Game for Server

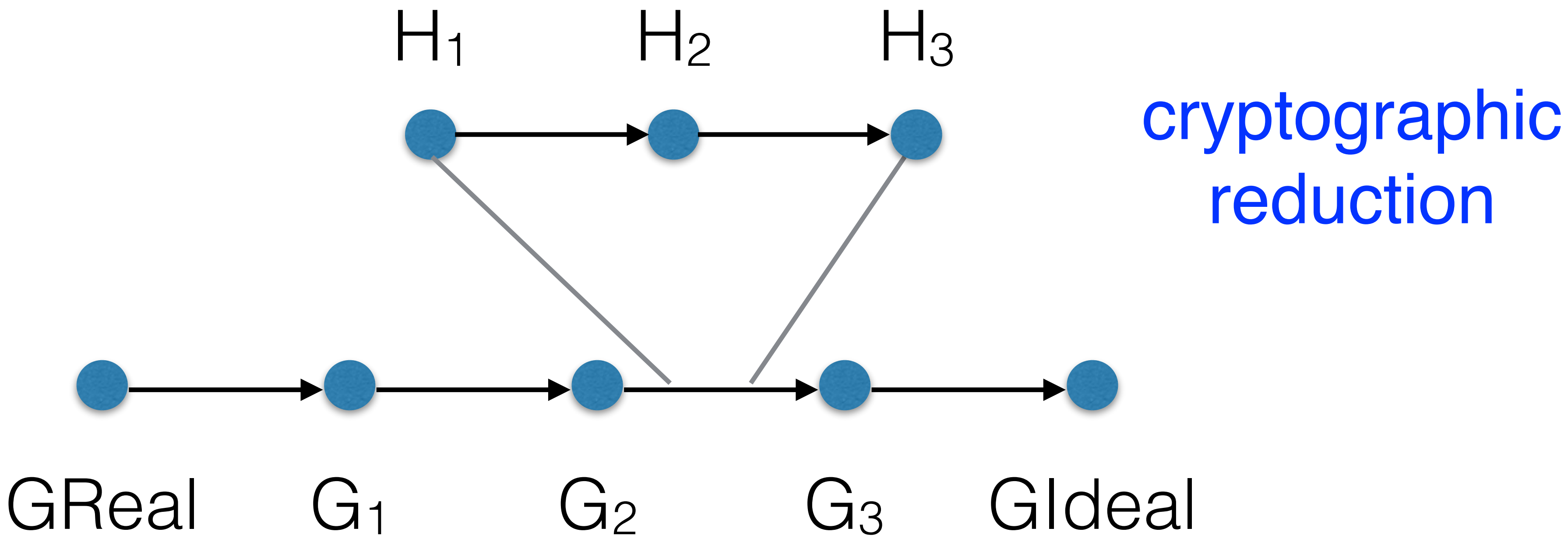
```
    proc final() : bool = {
      var b : bool;
      b <@ A.final(Protocol.sv);
      return b;
    }
  }

  proc main() : bool = {
    var b : bool;
    b <@ Protocol(Env).main();
    return b;
  }
}.
```

Ideal Games

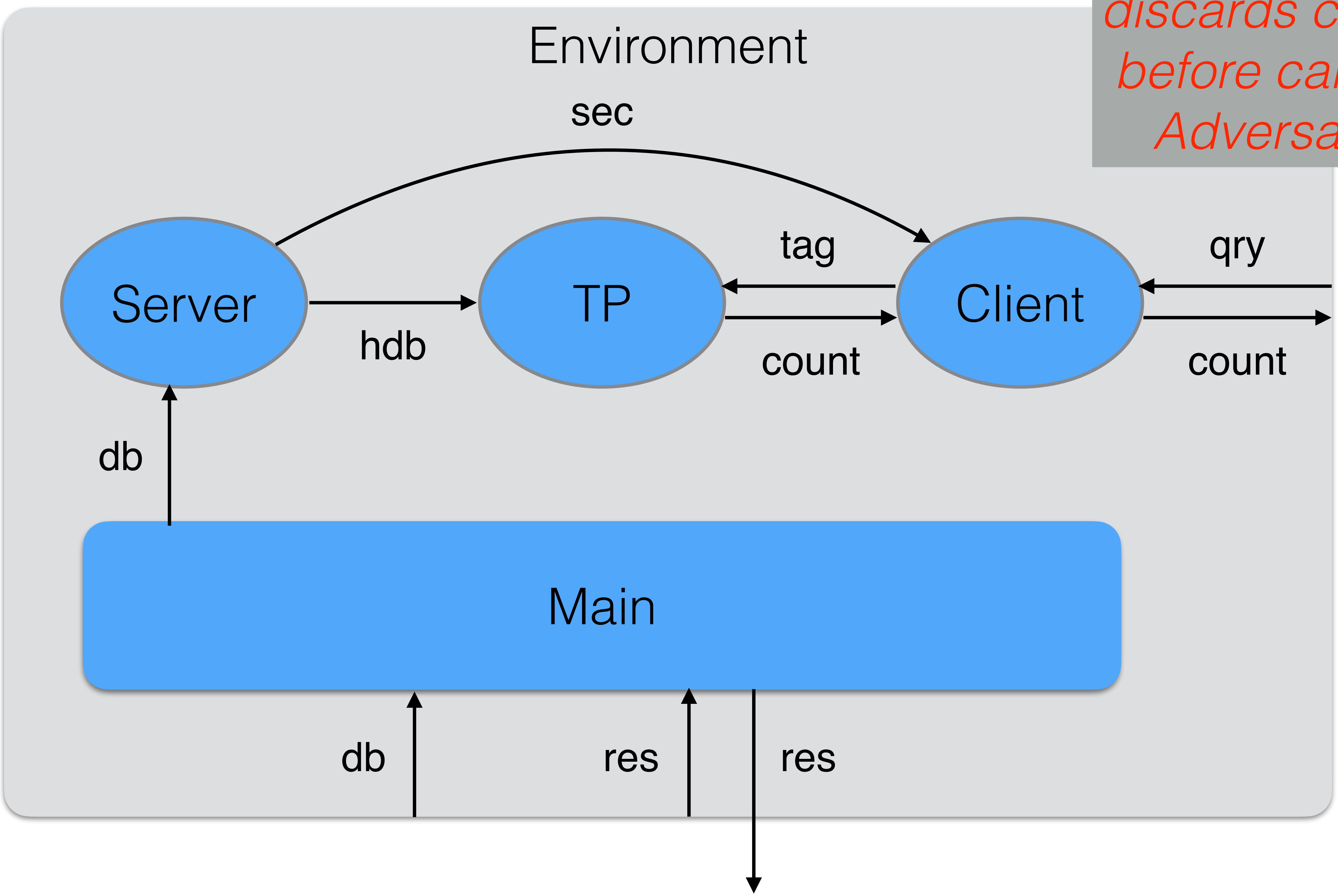
- A party’s Ideal Game is also parameterized by a Simulator (in addition to the Adversary)
- Simulator’s job is to convince the Adversary it’s interacting with the real game: it must simulate the party’s view and the hashing function’s view of the random oracle state
- Because we are working information-theoretically, when assessing the information leakage from the Ideal Game to the Simulator (and thus Adversary), we don’t have to scrutinize its Simulator
    - It can’t learn more about the database or queries by brute force computation
- In fact, in our EasyCrypt security theorems, the Simulators are existentially quantified

Two Dimensional Sequences of Games

- When proving security against a protocol party, we use EasyCrypt's pRHL and ambient logics to connect the party's Real and Ideal Games via a sequence of games
  - Upper bound on distance between source and target games is sum of intermediate transitions' upper bounds
- We can prove a game transition using a previously proved sequence of games



Reminder: Real Game for Server

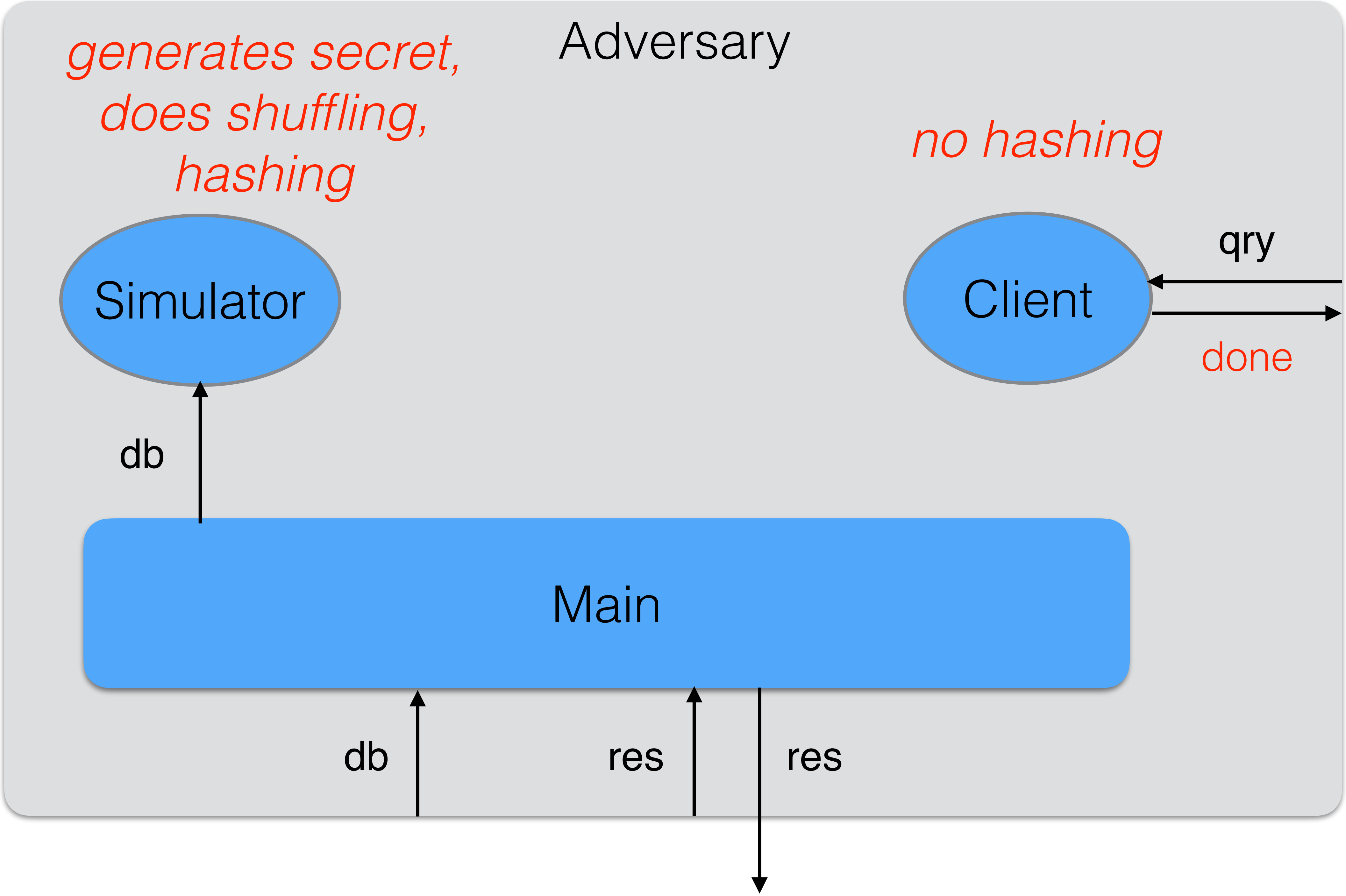


Environment discards count before calling Adversary

Real Game for Server

- What (if anything) can the Server learn about the queries and their counts?
- We formalize this by asking what can be learned from the Server views that are passed to the Adversary — plus the ability to run the `hash` procedure of the random oracle
  - We can think that each time the Adversary is called, the Server is woken up
- To answer and prove this, we need to formalize an Ideal Game

Ideal Game for Server



Adversary

generates secret, does shuffling, hashing

no hashing

Ideal Game for Server

- The Simulator doesn’t directly learn anything about the queries, and so the Server views it simulates can’t convey anything about them either
- And the query loop doesn’t modify the random oracle, so experimentation with the random oracle won’t learn anything either
- But because the Server is woken up each iteration of the query loop, the Server does learn the number of queries

Proof of Security Against Server

- We are able to prove perfect security: Real/Ideal games equally likely to return true:

```
lemma GReal_GIdeal :
  exists (Sim <: SIM{-GReal, -GIdeal}),
  forall (Adv <: ADV{-GReal, -GIdeal, -Sim}) &m,
  Pr[GReal(Adv).main() @ &m : res] =
  Pr[GIdeal(Adv, Sim).main() @ &m : res].
```

- The only challenge is dealing with the redundant hashing performed by the Client in the Real but not the Ideal Game
- We remove it using a variation of a technique due to Benjamin Grégoire

Redundant Hashing

```
module type HASHING = {
  proc hash(inp : input) : output
  proc rhash(inp : input) : unit
}.

module type HASHING_ADV(H : HASHING) = {
  proc main() : bool {H.hash H.rhash}
}.
```

Two implementations of `HASHING`, both built from a random oracle `O`:

- `NonOptHashing` (``non optimized hashing''), in which `rhash` hashes its input, but discards the result
- `OptHashing` (``optimized hashing’'), where `rhash` does nothing

Redundant Hashing

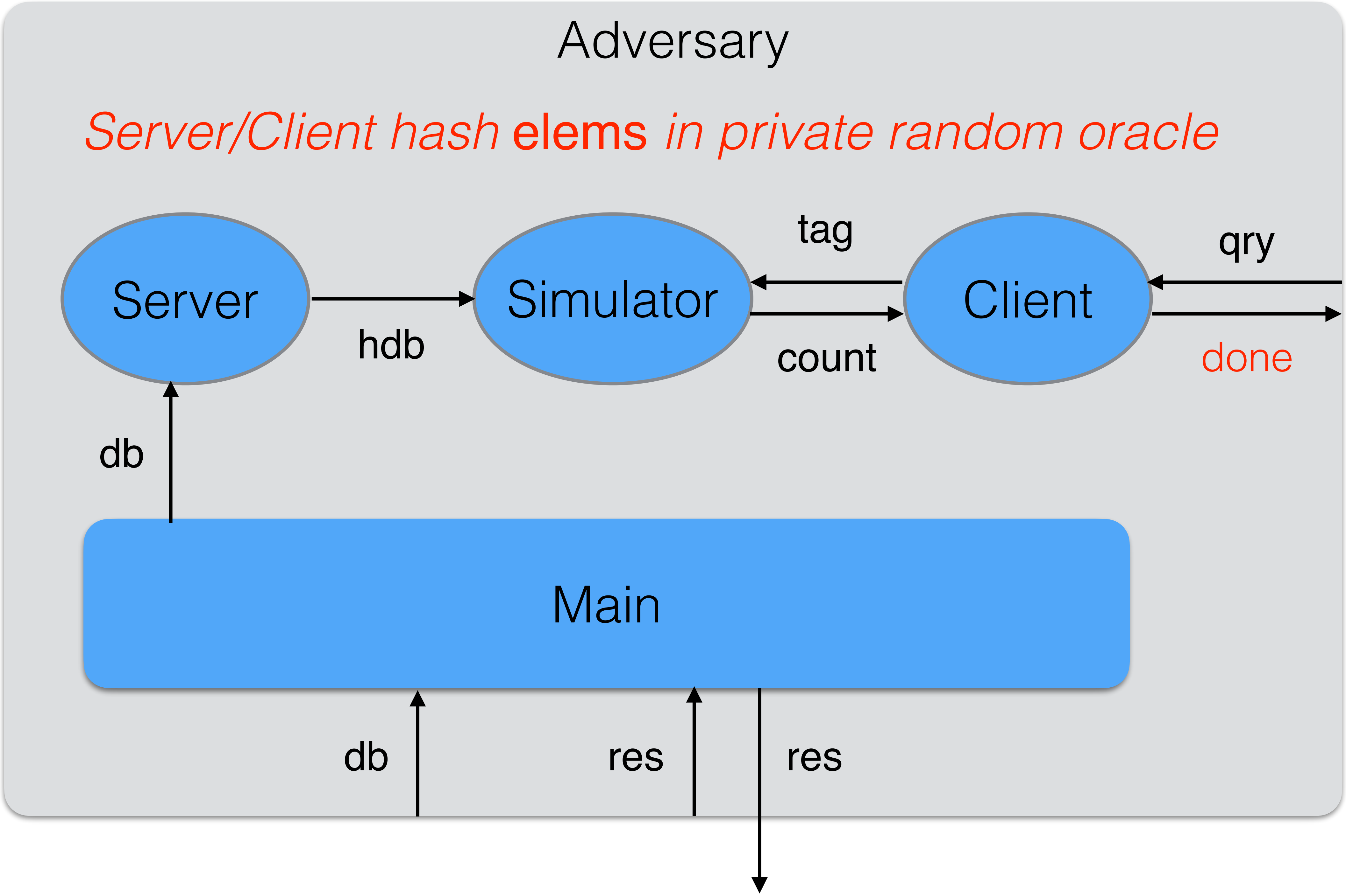
```
module GNonOptHashing(HashAdv : HASHING_ADV) = {
  module H  = NonOptHashing(Or)
  module HA = HashAdv(H)
  proc main() : bool = {
    var b : bool;
    Or.init(); b <@ HA.main();
    return b;
  }
}.

module GOptHashing(HashAdv : HASHING_ADV) = {
  module H  = OptHashing(Or)
  module HA = HashAdv(H)
  proc main() : bool = {
    var b : bool;
    Or.init(); b <@ HA.main();
    return b;
  }
}.
```

Redundant Hashing

```
lemma GNonOptHashing_GOptHashing
        (HashAdv <: HASHING_ADV{Or}) &m :
  Pr[GNonOptHashing(HashAdv).main() @ &m : res] =
  Pr[GOptHashing(HashAdv).main() @ &m : res].
```

- Proof intuition: redundant hashing can be put off until it’s superseded by `hash` or no longer necessary
  - Proof uses EasyCrypt’s `eager` tactics
- To use in Server proof, we define a concrete adversary `HashAdv` in such a way that the left side of the gap in the sequence of games proof can be connected with `GNonOptHashing(HashAdv)`, and `GOptHashing(HashAdv)` can be connected with the right side of the gap

Ideal Game for Third Party



Ideal Game for Third Party

- The Adversary is invoked with the TP’s view when the database and queries are requested by the game and client loop
- In the Ideal Game, Adversary only learns *patterns*, not anything more about the database and queries
  - It doesn’t have access to the private random oracle used by Server/Client
  - So even though the database and queries *were* used to derive the hashed database `[t₁; …; tₙ]` and query tags `s₁`, …, `sₘ`, these tags were all randomly (but consistently) chosen, and so convey no information about the particular elements
  - And the Server’s random shuffling means it doesn’t learn anything about the order of the database

Security Against Third Party

- E.g., suppose the original database was `[0; 1; 1; 2]`
- The Server's shuffled database might be `[1; 0; 2; 1]`
- In the Real Game, TP will get a hashed database $[t_2, t_1, t_3, t_2]$, where `t1 = hash(0, sec)`, `t2 = hash(1, sec)` and `t3 = hash(2, sec)` — for the shared Server/Client `sec`
- In the Ideal Game, TP will get a hashed database with the same pattern, $[s_2; s_1; s_3; s_2]$, but where the s_i have no connection with `hash` or `sec`
- In order to tell the games apart, we can prove it has to *guess* `sec`, i.e., call `hash` with a pair whose second component is `sec`

Security Against Third Party

- To try to differentiate the games, the Adversary can pick a database with a large number of distinct elements, where each element appears a different number of times (e.g., `[0; 1; 1; 2; 2; 2; …]`).

- When given (in TP's view) the hashed database that was created in the Real or Ideal Game from shuffling the database and then hashing its elements (either paired with `sec` in the random oracle, or in the private random oracle), it can (assuming no hash collisions) match the resulting tags `t` with their elements `e`.

- Given a particular `(e, t)` pair, it can search for a `sec'` such that hashing `(e, sec')` results in `t`. When it finds one, it can check that the rest of the hashed database is consistent with `sec'`. Otherwise it can try another choice of `sec'`.

Security Against Third Party

- This process is guaranteed to succeed in the Real Game, it’s highly unlikely to succeed in the Ideal Game
- In any event, if the Adversary never calls the random oracle with a pair whose second component is `sec`, we can prove it will fail to distinguish the Real and Ideal Games

Proof of Security Against Third Party

- To obtain a security theorem, we must limit (`limit`) the number of *distinct* inputs the Adversary may hash
    - The Server and Client are unrestricted
- We use a cryptographic reduction to bridge the Real and Ideal Games — one proved with up-to-bad reasoning, and so — that makes us assume the Adversary's procedures are lossless (always terminating), and prove that the Client Loop always terminates
    - When we form **GReal** and **GIdeal**, we terminate the Client Loop after **qrys_max** steps (in `GReal`, by returning `None` from the environment's `get_qry` procedure)

Proof of Security Against Third Party

- Here is the relevant part of the Environment for `GReal`:

```
module Env : ENV = {
  var qrys_ctr : int
  ...
  proc get_qry() : elem option = {
    var qry_opt : elem option;
    qry_opt <@ A.get_qry(Protocol.tpv);
    if (qry_opt <> None) {
      if (qrys_ctr < qrys_max) { qrys_ctr <- qrys_ctr + 1; }
      else { qry_opt <- None; }
    }
    return qry_opt;
  }
}
```

Third Party Proof

- We reduce security against TP to the security of a new abstraction, “Secrecy Random Oracles”
  - They offer *limited* (**limit**) hashing of element/secret pairs (what Adversary does), as well as *unlimited* hashing of elements (what Server and Client do)
  - “Dependent” implementation with single map, where hashing an element is same as hashing pair of it and **sec** — connection with Real Game
  - “Independent” implementation with separate maps — connection with Ideal Game
  - We prove that a Secrecy Adversary can only tell the games involving the two implementations apart if it does limited hashing of a pair whose second component is `sec`

Third Party Proof

- The Secrecy Random Oracles proof is carried out using up-to-bad reasoning
- As long as the Secrecy Adversary doesn't do limited hashing with a pair with right side `sec` (the "bad" event), we can maintain an invariant:
    - keeping the non-`sec`-part of the map of the dependent implementation in sync with the non-`sec`-part of the `elem * sec` map of the independent implementation; and
    - keeping the `sec`-part of the map of the dependent implementation in sync with the `elem` map of the independent implementation

Third Party Proof

- We reduce the upper-bounding of the probability of the bad event holding to a lemma about another new abstraction, "Secret Guessing Oracles"
  - It gives the adversary limited (`limit`) number of chances to guess `sec`
  - EasyCrypt's pHL is used to upper bound the probability of the adversary winning by

$$\texttt{limit} / 2^{\texttt{sec\_len}}$$

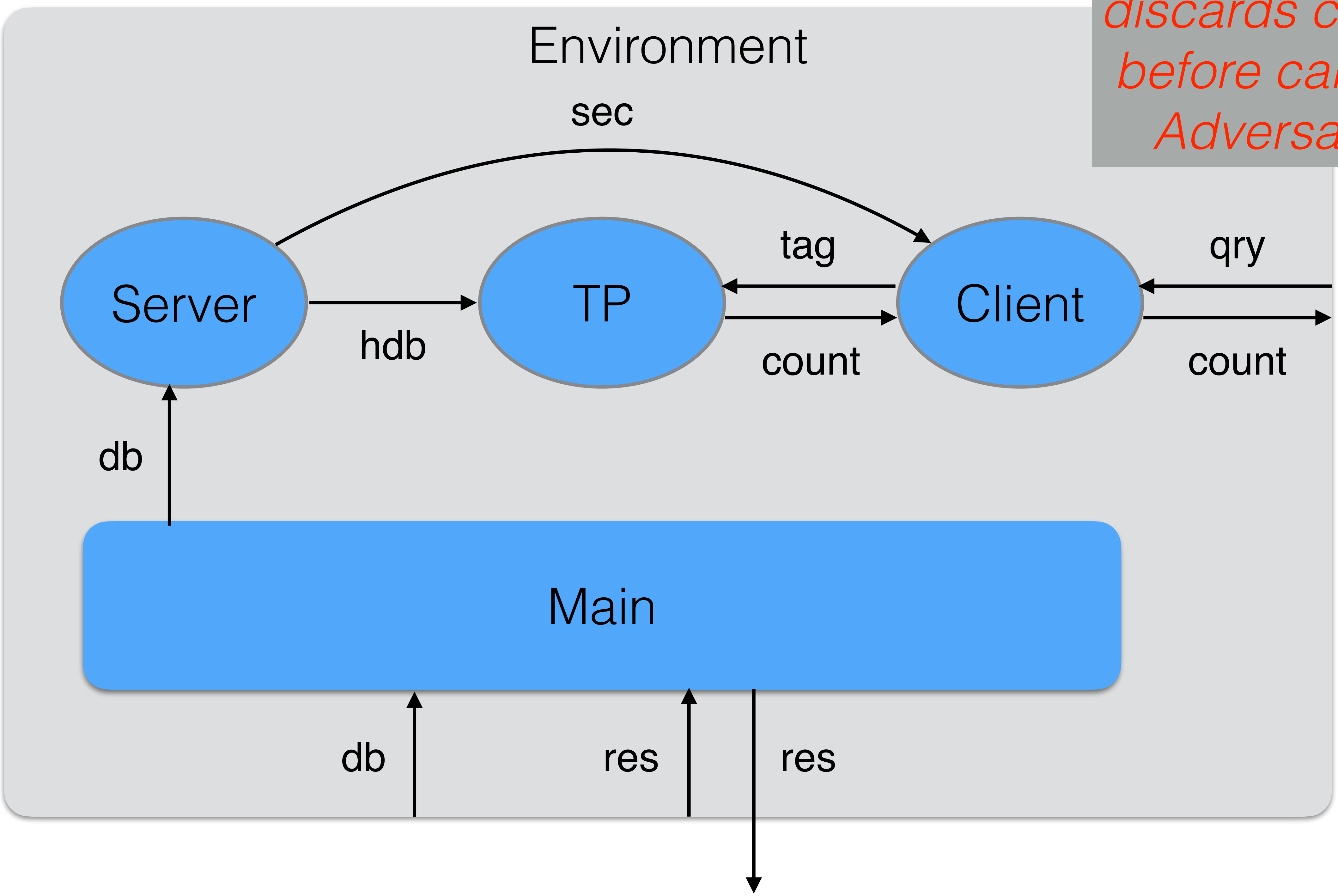
- Both the Secrecy Random Oracles and Secret Guessing Oracles definitions and proofs are packaged up into reusable theories

Third Party Proof

- The theorem for security against the TP upper-bounds the distance between the probabilities of the Real and Ideal Games returning `true` by

$$\texttt{limit} / 2^{\texttt{sec\_len}}$$

Reminder: Real Game for Client

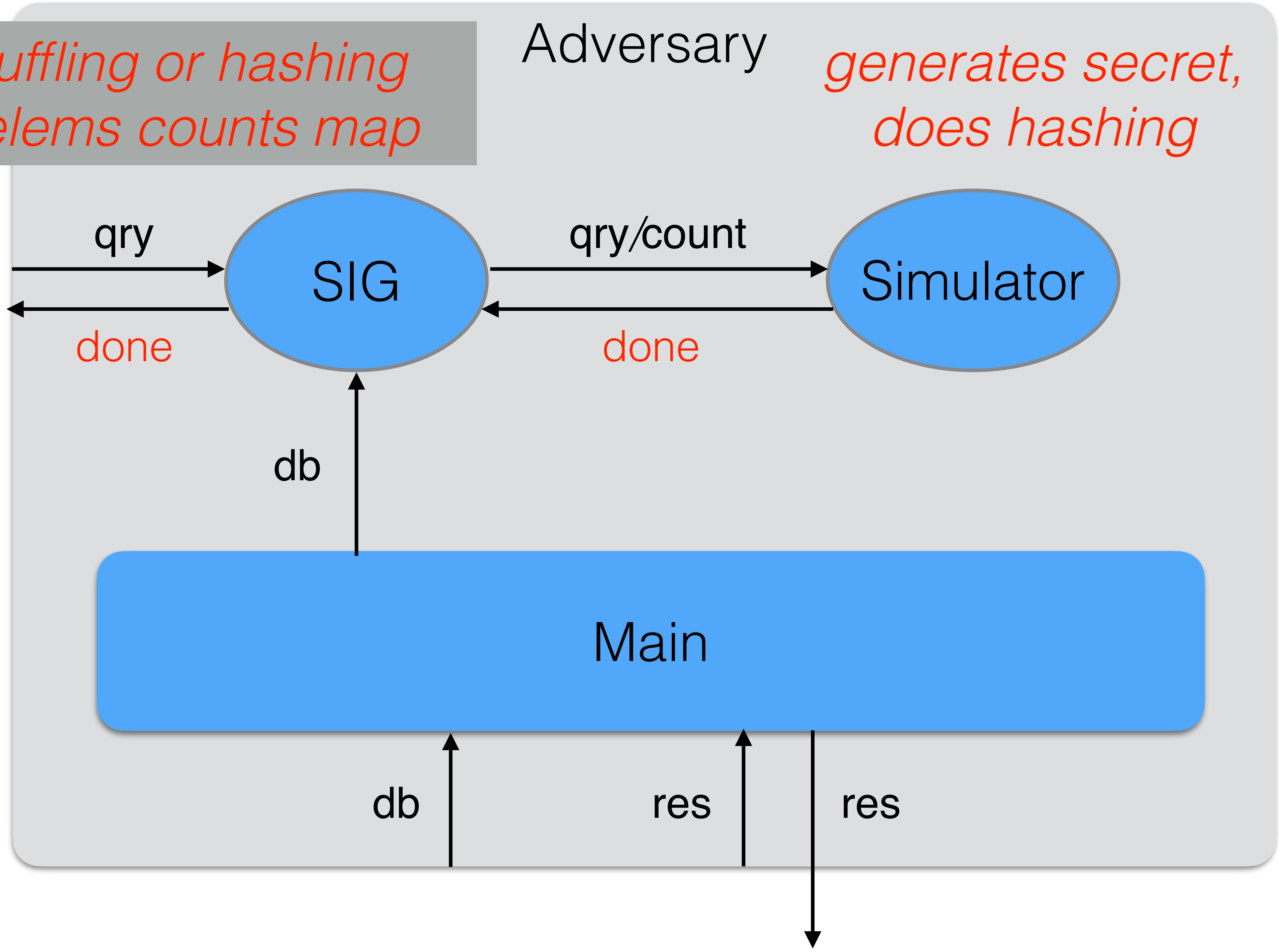


Environment discards count before calling Adversary

SIG = Simulator's Interface to Game

Ideal Game for Client

no shuffling or hashing uses elems counts map



Adversary

generates secret, does hashing

qry

SIG

qry/count

Simulator

done

done

db

Main

db

res

res

Proof of Security Against Client

- The Adversary can distinguish the Real and Ideal Games by *causing* or *forcing* a hash collision
  - If it can find distinct `elem` and `elem’` such that `(elem, sec)` and `(elem’, sec)` hash to the same hash tag, `tag`, then it can let `db = [elem]` and the only query be `elem’`
    - In Real Game, count will be
      - 1
    - In Ideal Game, count will be
      - 0
  - It can let `db` be a list of *distinct* elements of greater length than number of distinct hash tags, and work through that same list of elements as queries

Proof of Security Against Client

- Thus we must impose a hashing budget on the Adversary — not just on the hashing it does directly, *but also on the hashing it makes Server and Client do*:
  - `adv_budget` — distinct hashing done by Adversary
  - `db_uniqs_max` — maximum number of distinct elements in database
  - `qrys_max` — maximum number of queries
- `budget = adv_budget + db_uniqs_max + qrys_max`
- If Adversary doesn't respect budget, we terminate game early (we terminate the Client Loop after `qrys_max` steps)
- Because the proof uses up-to-bad reasoning, we need that Adversary is always terminating and Client Loop terminates

Proof of Security Against Client

- We have Budgeted Random Oracles, which provide:
    - *separate* budgeted hashing functions for the Adversary, Server and Client
        - set a flag when over budget, but keep working
        - for Adversary and Server, only distinct inputs matter, but for Client its the number of hashes
    - ordinary (unrestricted) hashing (which the Adversary uses before making its final judgement)
- There are two implementations of budgeted random oracles:
    - a “collision-possible” one in which hash collisions may occur
    - a “collision-free-while-within-budget” one in which hash collisions don’t happen if only budgeted hashing is done and the individual budgets are respected

Proof of Security Against Client

- Each move back and forth between the collision-possible and collision-free-while-within-budget versions incurs a penalty of

$$(\texttt{budget} * (\texttt{budget} - 1)) / 2^{\texttt{tag\_len} + 1}$$

- This is proved using up-to-bad reasoning, where the “bad” event is when a collision occurs
- EasyCrypt’s failure event lemma and pHL are used to bound the probability that failure occurs
- The proof is packaged into a reusable theory

Client Proof

- Move to collision-possible budgeted random oracle
- Move to collision-free-while-within-budget random oracle
- Use complex relational invariant to switch to Server, TP and Client using an elements counts map instead of hashed database (but Server still does hashing)
- Switch back to collision-possible budgeted random oracle
- Switch back to ordinary random oracle (Adversary still subjected to budget)
- Get rid of Server’s hashing, which is now seen to be redundant
- Show that computing elements counts map works out same without first shuffling database
- Final refactoring

Client Proof

- Theorem for security against the Client upper bounds the distance between the probabilities of the Real and Ideal Games returning **true** by

  $$(\texttt{budget} * (\texttt{budget} - 1)) / 2^{\texttt{tag\_len}}$$

  which is two times

  $$(\texttt{budget} * (\texttt{budget} - 1)) / 2^{\texttt{tag\_len} + 1}$$

Summary/Lessons Learned

- Size of EasyCrypt formalization:
  - About **380 lines** of theorem statements and relevant definitions (random oracles, protocol definition, etc.)
  - About **5,275 lines** of proof (which one can trust EasyCrypt to check)
- Two-dimensional game structure very useful
- Formalizing Protocol once — parameterized by Environment — and then specializing to Real Games works well
- Because we work information-theoretically, Simulators are existentially quantified (so part of proof, not specification)
- Removing redundant hashing was crucial, and our version of Grégoire’s technique was proved once and used twice

Summary/Lessons Learned

- Use of budgeted random oracles in Client proof let us do the hard step of the proof without worrying about hash collisions
- EasyCrypt made it easy to obtain concrete upper bounds in terms of game parameters on the distances between real and ideal games

Discussion

- Q: In the PCR Protocol, does the Client always get correct counts for its queries?
  - A: Not in the highly unlikely event of hash collisions
- Q: Why do we let the Adversary choose the database and queries?
  - A: This models how it may have inside information about what elements (e.g., people’s names) are likely to appear in the database or in queries
    - E.g., TP, when analyzing the tags it sees, might guess that a tag appearing numerous times corresponds to “Alice”, based on knowledge of an organization. But it won’t be able to confirm that guess.

Discussion

- Q: Is it realistic to assume two parties can communicate, without the other one eavesdropping?
  - A: Yes. The Adversary works on behalf of a given party, and has no special access to the network

Discussion

- Q: Are the restrictions we place on the Adversary realistic?
  - A: Server:
    - No restrictions
  - A: TP:
    - Limit on distinct hashes
  - A: Client:
    - Budget for Adversary’s distinct hashing
    - Budget on number of distinct elements in database
    - Budget on number of queries

in reality, the Adversary doesn’t choose the database or queries

Questions?