CS 599: Formal Methods in Security and Privacy

Hoare Triples and Hoare Logic

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Programming Language

```
c::= abort
   | skip
   | x:=e
   | c;c
   | if e then c else c
   | while e do c
```

x,y,z,...	program variables
$e_1, e_2, \ldots$	expressions
$c_1, c_2, \ldots$	commands

Specifications - Hoare triple

Precondition
Program
Postcondition

Precondition
(a logical formula)

$$c : P \Rightarrow Q$$

Program

Postcondition
(a logical formula)

Some examples

Precondition

$$x := z + 1 : \{z + 1 > 0\} \Rightarrow \{x > 0\}$$

Postcondition

Is it a good specification?

Some examples

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```
i:=0;
r:=1;
while(i≤k)do
  r:=r * n;
  i:=i + 1
```

Precondition

$$: \{0 \le k\} \Rightarrow \{r = n^k\}$$

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Precondition

$$: \{0 \leq k\} \Rightarrow \{r = n^k\}$$

Postcondition

Is it a good specification? ✗

$$m_{in} = [k = 0, n = 2, i = 0, r = 0]$$

$$m_{out} = [k = 0, n = 2, i = 1, r = 2]$$

Some examples

```
i:=0;
r:=1;
while(i≤k)do
  r:=r * n;
  i:=i + 1
```

Precondition

$$: \{0 < k\} \Rightarrow \{r = n^k\}$$

Postcondition

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i:=0;
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$$m_{in} = [k = 1, n = 2, i = 0, r = 0]$$

$$m_{out} = [k = 1, n = 2, i = 2, r = 4]$$

Some examples

```
i:=0;
r:=1;
while(i<k)do
  r:=r * n;
  i:=i + 1
```

Precondition

$$: \{0 \leq k\} \Rightarrow \{r = n^k\}$$

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Some examples

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i:=0;
r:=1;
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```

Precondition

$$: \{0 \leq k\} \Rightarrow \{r = n^i\}$$

Postcondition

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Some examples

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i:=0;
r:=1;
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Some examples

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i:=0;
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  i:=i + 1
```

Precondition

$$: \{0 < k \wedge k < 0\} \Rightarrow \{r = n^k\}$$

Postcondition

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Some examples

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i:=0;
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$$: \{0 < k \wedge k < 0\} \Rightarrow \{r = n^k\}$$

Postcondition

Is it a good specification?

✓

This is good because there is no memory that satisfies the precondition.

How do we determine the validity of an Hoare triple?

Validity of Hoare triple

Precondition
(a logical formula)

$$c : P \Rightarrow Q$$

Program

Postcondition
(a logical formula)

Validity of Hoare triple

Precondition
(a logical formula)

$$c : P \Rightarrow Q$$

Program

Postcondition
(a logical formula)

We are interested only in inputs that meets P and we want to have outputs satisfying Q.

Validity of Hoare triple

Precondition
(a logical formula)

$$c : P \Rightarrow Q$$

Program

Postcondition
(a logical formula)

We are interested only in inputs that meets P and we want to have outputs satisfying Q.

How shall we formalize this intuition?

Validity of Hoare triple

We say that the triple $\mathtt{c:P\Rightarrow Q}$ is valid

if and only if

for every memory $\mathtt{m}$ such that $\mathtt{P(m)}$

and memory m' such that $\mathtt{\{c\}_m=m'}$

we have $\mathtt{Q(m')}$.

Validity of Hoare triple

We say that the triple `c:P⇒Q` is valid

if and only if

for every memory `m` such that `P(m)`

and memory m' such that $\{c\}_m = m'$

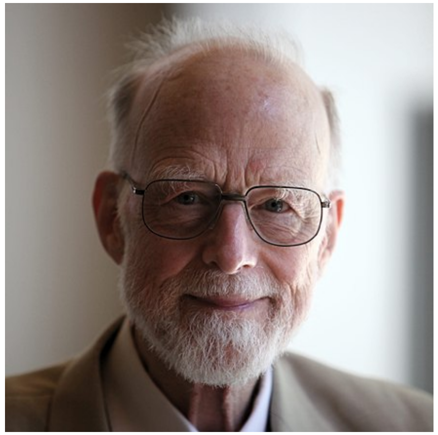
we have `Q(m')`.

Is this condition easy to check?

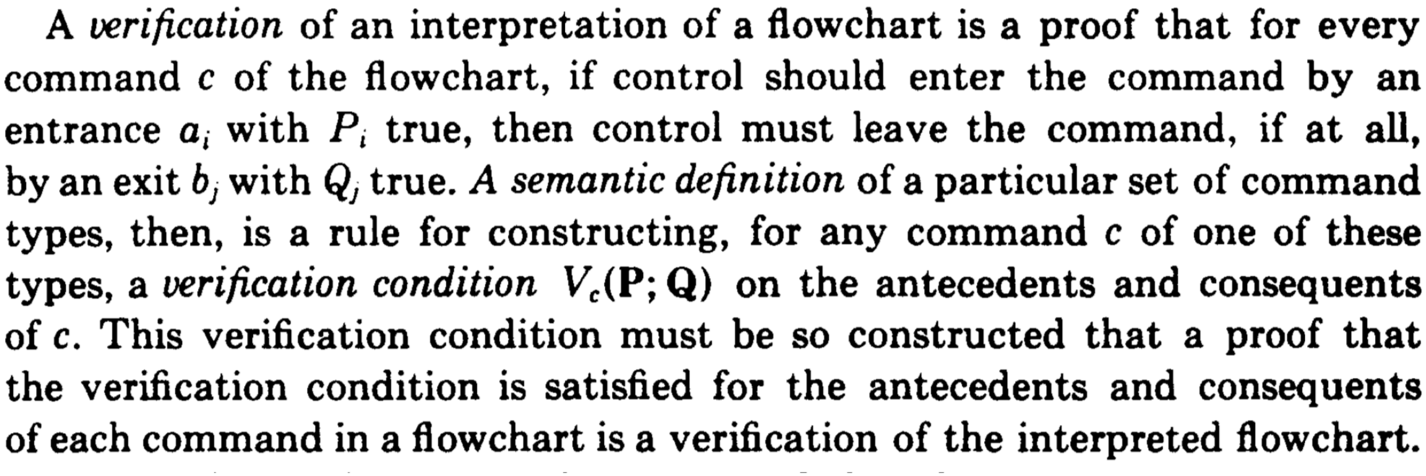
Hoare Logic

Floyd-Hoare reasoning

Robert W Floyd



Tony Hoare



A *verification* of an interpretation of a flowchart is a proof that for every command c of the flowchart, if control should enter the command by an entrance a_i with P_i true, then control must leave the command, if at all, by an exit b_j with Q_j true. *A semantic definition* of a particular set of command types, then, is a rule for constructing, for any command c of one of these types, a *verification condition* $V_c(\mathbf{P}; \mathbf{Q})$ on the antecedents and consequents of c. This verification condition must be so constructed that a proof that the verification condition is satisfied for the antecedents and consequents of each command in a flowchart is a verification of the interpreted flowchart.

Rules of Hoare Logic: Skip

$$\frac{}{\vdash \texttt{skip} \colon P \Rightarrow P}$$

Rules of Hoare Logic: Skip

$$\frac{}{\vdash \texttt{skip}:\ P \Rightarrow P}$$

Is this correct?

Correctness of an axiom

$$\frac{}{\vdash \texttt{c} \;:\; \texttt{P} \Rightarrow \texttt{Q}}$$

We say that an axiom is correct if we can prove the validity of each triple which is an instance of the conclusion.

Correctness of Skip Rule

$$\frac{}{\vdash \texttt{skip: } P \Rightarrow P}$$

To show this rule correct we need to show the validity of the triple `skip: P⇒P`.

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For every `m` such that `P(m)` and m' such that $\{\texttt{skip}\}_m = m'$ we need `P(m')`.

Correctness of Skip Rule

$$\frac{}{\vdash \texttt{skip: } P \Rightarrow P}$$

To show this rule correct we need to show the validity of the triple $\texttt{skip: } P \Rightarrow P$.

For every $\texttt{m}$ such that $\texttt{P(m)}$ and m' such that $\{\texttt{skip}\}_{\texttt{m}} = \texttt{m'}$ we need $\texttt{P(m')}$.

Follow easily by our semantics:
$\{\texttt{skip}\}_{\texttt{m}} = \texttt{m}$

Rules of Hoare Logic: Assignment

$$\frac{}{\vdash \texttt{x:=e}:\ P \Rightarrow P[e/x]}$$

Rules of Hoare Logic: Assignment

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Is this correct?

Some instances

$$x := x + 1 : \{x < 0\} \Rightarrow \{x + 1 < 0\}$$

Is this a valid triple?

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$$x := z + 1 : \{x > 0\} \Rightarrow \{z + 1 > 0\}$$

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Is this a valid triple? ✓

Correctness Assignment Rule

$$\frac{}{\vdash \mathtt{x:=e} \;:\; \mathtt{P[e/x]} \Rightarrow \mathtt{P}}$$

To show this rule correct we need to show the validity $\mathtt{x:=e:P[e/x]} \Rightarrow \mathtt{P}$ for every $\mathtt{x}$, $\mathtt{e}$, $\mathtt{P}$.

Correctness Assignment Rule

$$\frac{}{\vdash \texttt{x:=e} \;:\; \texttt{P[e/x]} \Rightarrow \texttt{P}}$$

To show this rule correct we need to show the validity $\texttt{x:=e:P[e/x]} \Rightarrow \texttt{P}$ for every $\texttt{x,e,P}$.

For every $\texttt{m}$ such that $\texttt{P[e/x](m)}$ and m' such that $\texttt{\{x:=e\}}_{\texttt{m}}\texttt{=m'}$ we need $\texttt{P(m')}$.

Correctness Assignment Rule

$$\frac{}{\vdash \texttt{x:=e} \ : \ \texttt{P[e/x]} \Rightarrow \texttt{P}}$$

To show this rule correct we need to show the validity $\texttt{x:=e:P[e/x]}\Rightarrow\texttt{P}$ for every $\texttt{x}$, $\texttt{e}$, $\texttt{P}$.

For every $\texttt{m}$ such that $\texttt{P[e/x](m)}$ and m' such that $\texttt{\{x:=e\}}_{\texttt{m}}\texttt{=m'}$ we need $\texttt{P(m')}$.

By our semantics: $\texttt{\{x:=e\}}_{\texttt{m}}\texttt{=m[x=\{e\}}_{\texttt{m}}\texttt{]}$ and we can show $\texttt{P[e/x](m)= P(m[x=\{e\}}_{\texttt{m}}\texttt{])}$

Rules of Hoare Logic
Composition

$$\frac{}{\vdash \texttt{c;c'}:\ P \Rightarrow Q}$$

Rules of Hoare Logic
Composition

$$\frac{\vdash \texttt{c} : P \Rightarrow R}{\vdash \texttt{c;c}' : \; P \Rightarrow Q}$$

Rules of Hoare Logic
Composition

$$\frac{\vdash c : P \Rightarrow R \qquad\qquad \vdash c' : R \Rightarrow Q}{\vdash c ; c' : \; P \Rightarrow Q}$$

Rules of Hoare Logic
Composition

$$\frac{\vdash \texttt{c} : P \Rightarrow R \qquad \vdash \texttt{c}' : R \Rightarrow Q}{\vdash \texttt{c};\texttt{c}' : P \Rightarrow Q}$$

Is this correct?

Some Instances

$$\vdash x := z * 2; z := x * 2$$

$$: \{(z * 2) * 2 = 8\} \Rightarrow \{z = 8\}$$

Is this a valid triple?

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Some Instances

How can we prove it?

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$$\frac{}{\vdash x := z * 2 : \{(z * 2) * 2 = 8\} \Rightarrow \{x * 2 = 8\}}$$

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$$\vdash x := z * 2; z := x * 2 : \{(z * 2) * 2 = 8\} \Rightarrow \{z = 8\}$$

Correctness Composition Rule

$$\frac{\vdash \mathtt{c} : \mathtt{P} \Rightarrow \mathtt{R} \qquad \vdash \mathtt{c}' : \mathtt{R} \Rightarrow \mathtt{Q}}{\vdash \mathtt{c} ; \mathtt{c}' : \mathtt{P} \Rightarrow \mathtt{Q}}$$

To show this rule correct we need to show the validity $\mathtt{c} ; \mathtt{c}' : \mathtt{P} \Rightarrow \mathtt{Q}$ for every $\mathtt{c}, \mathtt{c}', \mathtt{P}, \mathtt{Q}$.

Correctness Composition Rule

$$\frac{\vdash \texttt{c}:\texttt{P}\Rightarrow\texttt{R} \qquad \vdash \texttt{c}':\texttt{R}\Rightarrow\texttt{Q}}{\vdash \texttt{c};\texttt{c}':\ \texttt{P}\Rightarrow\texttt{Q}}$$

To show this rule correct we need to show the validity $\texttt{c};\texttt{c}':\texttt{P}\Rightarrow\texttt{Q}$ for every $\texttt{c},\texttt{c}',\texttt{P},\texttt{Q}$.

For every $\texttt{m}$ such that $\texttt{P(m)}$ and m' such that $\{\texttt{c},\texttt{c}'\}_{\texttt{m}}=\texttt{m}'$ we need $Q(\texttt{m}')$.

Correctness Composition Rule

$$\frac{\vdash \texttt{c} : P \Rightarrow R \qquad \vdash \texttt{c}' : R \Rightarrow Q}{\vdash \texttt{c;c}' : P \Rightarrow Q}$$

Correctness Composition Rule

$$\frac{\vdash \texttt{c} : P \Rightarrow R \qquad \vdash \texttt{c'} : R \Rightarrow Q}{\vdash \texttt{c;c'} : P \Rightarrow Q}$$

By our semantics: $\{\texttt{c;c'}\}_{\texttt{m}} = \texttt{m'}$ if and only if there is $\texttt{m''}$ such that $\{\texttt{c}\}_{\texttt{m}} = \texttt{m''}$ and $\{\texttt{c'}\}_{\texttt{m''}} = \texttt{m'}$.

Correctness Composition Rule

$$\frac{\vdash \texttt{c:P}\Rightarrow\texttt{R} \qquad \vdash \texttt{c':R}\Rightarrow\texttt{Q}}{\vdash \texttt{c;c': P}\Rightarrow\texttt{Q}}$$

By our semantics: $\{c;c'\}_m = m'$ if and only if there is m'' such that $\{c\}_m = m''$ and $\{c'\}_{m''} = m'$.

Assuming $c:P\Rightarrow R$ and $c':R\Rightarrow Q$ valid, if $P(m)$ we can show $R(m'')$ and if $R(m'')$ we can show $Q(m')$, hence since we have $P(m)$ we can conclude $Q(m')$.

Correctness Composition Rule

$$\frac{\vdash \texttt{c}:\texttt{P}\Rightarrow\texttt{R} \qquad \vdash \texttt{c}':\texttt{R}\Rightarrow\texttt{Q}}{\vdash \texttt{c;c}':\ \texttt{P}\Rightarrow\texttt{Q}}$$

By our semantics: $\{\texttt{c;c}'\}_{\texttt{m}}=\texttt{m}'$ if and only if there is $\texttt{m}''$ such that $\{\texttt{c}\}_{\texttt{m}}=\texttt{m}''$ and $\{\texttt{c}'\}_{\texttt{m}''}=\texttt{m}'$.

Assuming $\texttt{c}:\texttt{P}\Rightarrow\texttt{R}$ and $\texttt{c}':\texttt{R}\Rightarrow\texttt{Q}$ valid, if $\texttt{P(m)}$ we can show $\texttt{R(m}''\texttt{)}$ and if $\texttt{R(m}''\texttt{)}$ we can show $\texttt{Q(m}'\texttt{)}$, hence since we have $\texttt{P(m)}$ we can conclude $\texttt{Q(m}'\texttt{)}$.

Some examples

$$\vdash x := z * 2; z := x * 2$$

$$: \{z * 4 = 8\} \Rightarrow \{z = 8\}$$

$\boxed{\text{Is this a valid triple?}}$

Some examples

$$\vdash x := z * 2; z := x * 2$$

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Is this a valid triple?

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Some examples

$$\vdash x := z * 2; z := x * 2$$
$$: \{z * 4 = 8\} \Rightarrow \{z = 8\}$$

Is this a valid triple? ✓

Can we prove it with the rules that we have?

Some examples

$$\vdash x := z * 2; z := x * 2$$
$$: \{z * 4 = 8\} \Rightarrow \{z = 8\}$$

Is this a valid triple? ✓

Can we prove it with the rules that we have? ✗

Some Instances

What is the issue?

$$\frac{}{\vdash x := z * 2; z := x * 2 : \{z * 4 = 8\} \Rightarrow \{z = 8\}}$$

Some Instances

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$$\frac{}{\vdash x := z * 2 : \{z * 4 = 8\} \Rightarrow \{x * 2 = 8\}} \qquad \frac{}{\vdash z := x * 2 : \{x * 2 = 8\} \Rightarrow \{z = 8\}}$$

$$\vdash x := z * 2; z := x * 2 : \{z * 4 = 8\} \Rightarrow \{z = 8\}$$

Some Instances

What is the issue?

$$\frac{\dfrac{}{\vdash x := z * 2 : \{z * 4 = 8\} \Rightarrow \{x * 2 = 8\}} \qquad \dfrac{}{\vdash z := x * 2 : \{x * 2 = 8\} \Rightarrow \{z = 8\}}}{\vdash x := z * 2; z := x * 2 : \{z * 4 = 8\} \Rightarrow \{z = 8\}}$$

Rules of Hoare Logic Consequence

$$\frac{P \Rightarrow S \qquad \vdash c : S \Rightarrow R \qquad R \Rightarrow Q}{\vdash c : \; P \Rightarrow Q}$$

Some examples

$$\vdash x := z * 2; z := x * 2$$

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Is this a valid triple?

Some examples

$$\vdash x := z * 2; z := x * 2$$
$$: \{z * 4 = 8\} \Rightarrow \{z = 8\}$$

Is this a valid triple?

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Some examples

$$\vdash x := z * 2; z := x * 2$$
$$: \{z * 4 = 8\} \Rightarrow \{z = 8\}$$

Is this a valid triple? ✓

Can we prove it with the rules that we have?

Some examples

$$\vdash x := z * 2; z := x * 2$$

$$: \{z * 4 = 8\} \Rightarrow \{z = 8\}$$

Is this a valid triple?

✓

Can we prove it with the rules that we have?

✓

Some Instances

$$\dfrac{\dfrac{\{z*4=8\} \Rightarrow \{(z*2)*2=8\} \qquad \dfrac{}{\vdash x := z*2\ \{(z*2)*2=8\} \Rightarrow \{x*2=8\}}}{\vdash x := z*2\!:\ \{z*4=8\} \Rightarrow \{x*2=8\}} \qquad \dfrac{}{\vdash z := x*2\!:\ \{x*2=8\} \Rightarrow \{z=8\}}}{\vdash x := z*2; z := x*2\!:\ \{z*4=8\} \Rightarrow \{z=8\}}$$

Rules of Hoare Logic
If then else

$$\frac{}{\vdash \texttt{if } e \texttt{ then } c_1 \texttt{ else } c_2 : P \Rightarrow Q}$$

Rules of Hoare Logic
If then else

$$\frac{\vdash c_1 : P \Rightarrow Q \qquad \vdash c_2 : P \Rightarrow Q}{\vdash \texttt{if } e \texttt{ then } c_1 \texttt{ else } c_2 : P \Rightarrow Q}$$

Rules of Hoare Logic
If then else

$$\frac{\vdash c_1 : P \Rightarrow Q \qquad\qquad \vdash c_2 : P \Rightarrow Q}{\vdash \texttt{if } e \texttt{ then } c_1 \texttt{ else } c_2 : P \Rightarrow Q}$$

Is this correct?

Some examples

$$\vdash \texttt{if } \texttt{y} = 0 \texttt{ then skip else } x := x + 1; x := x - 1$$

$$: \{x = 1\} \Rightarrow \{x = 1\}$$

Is this a valid triple?

Some examples

$$\vdash \texttt{if y} = 0 \texttt{ then skip else } x := x+1; x := x-1$$
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Is this a valid triple?

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Some examples

$$\vdash \texttt{if y} = \texttt{0 then skip else } x := x + 1; x := x - 1$$

$$: \{x = 1\} \Rightarrow \{x = 1\}$$

Is this a valid triple? ✓

Can we prove it with the rules that we have?

Some examples

$$\vdash \texttt{if } \texttt{y} = 0 \texttt{ then skip else } x := x + 1; x := x - 1$$
$$: \{x = 1\} \Rightarrow \{x = 1\}$$

Is this a valid triple?

Can we prove it with the rules that we have?

Some Instances

$$\dfrac{\dfrac{}{\vdash \texttt{skip}: \{x = 1\} \Rightarrow \{x = 1\}} \qquad \dfrac{\vdots}{\vdash x := x + 1; x := x - 1 : \{x = 1\} \Rightarrow \{x = 1\}}}{\vdash \texttt{if } \texttt{y} = \texttt{0} \texttt{ then } \texttt{skip} \texttt{ else } x := x + 1; x := x - 1 \quad : \{x = 1\} \Rightarrow \{x = 1\}}$$

Rules of Hoare Logic
If then else

$$\frac{\vdash \texttt{c}_1 : P \Rightarrow Q \qquad\qquad \vdash \texttt{c}_2 : P \Rightarrow Q}{\vdash \texttt{if e then c}_1 \texttt{ else c}_2 \; : \; P \Rightarrow Q}$$

Rules of Hoare Logic
If then else

$$\frac{\vdash c_1 : P \Rightarrow Q \qquad\qquad \vdash c_2 : P \Rightarrow Q}{\vdash \texttt{if e then } c_1 \texttt{ else } c_2 : P \Rightarrow Q}$$

Is this strong enough?

Some examples

$$\vdash \texttt{if false then skip else } x = x + 1$$
$$: \{x = 0\} \Rightarrow \{x = 1\}$$

Is this a valid triple?

Some examples

$$\vdash \texttt{if false then skip else } x = x + 1$$
$$: \{x = 0\} \Rightarrow \{x = 1\}$$

Is this a valid triple? ✓

Some examples

$$\vdash \texttt{if false then skip else } x = x + 1$$
$$: \{x = 0\} \Rightarrow \{x = 1\}$$

Is this a valid triple? ✓

Can we prove it with the rules that we have?

Some examples

$$\vdash \texttt{if false then skip else } x = x + 1$$
$$: \{x = 0\} \Rightarrow \{x = 1\}$$

Is this a valid triple? ✓

Can we prove it with the rules that we have? ✗

Rules of Hoare Logic
If then else

$$\frac{\vdash c_1 : e \land P \Rightarrow Q \qquad \vdash c_2 : \neg e \land P \Rightarrow Q}{\vdash \texttt{if } e \texttt{ then } c_1 \texttt{ else } c_2 : P \Rightarrow Q}$$

Is this correct?

Rules of Hoare Logic
If then else

$$\frac{\vdash c_1 : e \wedge P \Rightarrow Q \qquad \vdash c_2 : \neg e \wedge P \Rightarrow Q}{\vdash \texttt{if } e \texttt{ then } c_1 \texttt{ else } c_2 \; : \; P \Rightarrow Q}$$

Is this correct?

Homework

Rules of Hoare Logic: Abort

$$\frac{}{\vdash \texttt{Abort: } ? \Rightarrow ?}$$

Rules of Hoare Logic: Abort

$$\frac{}{\vdash \texttt{Abort: } ? \Rightarrow ?}$$

What can be a good specification?

Validity of Hoare triple

We say that the triple $\texttt{c:P}\Rightarrow\texttt{Q}$ is valid

if and only if

for every memory $\texttt{m}$ such that $\texttt{P(m)}$

and memory m' such that $\{\texttt{c}\}_{\texttt{m}}=\texttt{m'}$

we have $\texttt{Q(m')}$.

Rules of Hoare Logic: Abort

$$\frac{}{\vdash \texttt{Abort} : P \Rightarrow Q}$$

Rules of Hoare Logic: Abort

$$\vdash \texttt{Abort:P} \Rightarrow \texttt{Q}$$

To show this rule correct we need to show the validity $\texttt{Abort:P} \Rightarrow \texttt{Q}$ for every $\texttt{P}$, $\texttt{Q}$.

Rules of Hoare Logic: Abort

$$\vdash \texttt{Abort}:P \Rightarrow Q$$

To show this rule correct we need to show the validity $\texttt{Abort}:P \Rightarrow Q$ for every P, Q.

For every m such that $P(m)$ and m' such that $\{\texttt{Abort}\}_m = m'$ we need $Q(m')$.

Rules of Hoare Logic: Abort

$$\vdash \texttt{Abort}:P \Rightarrow Q$$

To show this rule correct we need to show the validity $\texttt{Abort}:P \Rightarrow Q$ for every P, Q.

For every m such that $P(m)$ and m' such that $\{\texttt{Abort}\}_m = m'$ we need Q(m').

Vacuously True

Rules of Hoare Logic
While

$$\frac{}{\vdash \texttt{while e do c : ??}}$$

Rules of Hoare Logic
While

$$\frac{P \Rightarrow \neg e}{\vdash \texttt{while e do c} : P \Rightarrow P}$$

Rules of Hoare Logic
While

$$\frac{P \Rightarrow e \qquad\qquad \vdash c : P \Rightarrow P}{\vdash \texttt{while e do c} : P \Rightarrow P}$$

Rules of Hoare Logic
While

$$\frac{\vdash c : e \wedge P \Rightarrow P}{\vdash \texttt{while } e \texttt{ do } c : P \Rightarrow P \wedge \neg e}$$

Invariant

An example

$$\vdash \texttt{while } x = 0 \texttt{ do } x := x + 1$$
$$: \{x = 1\} \Rightarrow \{x = 1\}$$

How can we derive this?

An example

$$\vdash \texttt{while } x = 0 \texttt{ do } x := x + 1$$
$$: \{x = 1\} \Rightarrow \{x = 1\}$$

$\boxed{\text{What can be a good Invariant?}}$

An example

$$\vdash \texttt{while}\ x = 0\ \texttt{do}\ x := x + 1$$
$$: \{x = 1\} \Rightarrow \{x = 1\}$$

What can be a good Invariant?

$$Inv = \{x = 1\}$$

An example

$$\frac{}{\vdash \texttt{while}\ x = 0\ \texttt{do}\ x := x + 1 : \{x = 1\} \Rightarrow \{x = 1\}}$$

An example

$$\frac{\vdash \texttt{while}\ x = 0\ \texttt{do}\ x := x + 1 : \{x = 1\} \Rightarrow \{x = 1 \land x \neq 0\} \qquad x = 1 \land x \neq 0 \Rightarrow x = 1}{\vdash \texttt{while}\ x = 0\ \texttt{do}\ x := x + 1 : \{x = 1\} \Rightarrow \{x = 1\}}$$

An example

$$\dfrac{\dfrac{x = 1 \wedge x = 0 \Rightarrow x + 1 = 1 \qquad \vdash x := x + 1 : \{x + 1 = 1\} \Rightarrow \{x = 1\}}{\vdash x := x + 1 : \{x = 1 \wedge x = 0\} \Rightarrow \{x = 1\}}}{\vdash \texttt{while}\ x = 0\ \texttt{do}\ x := x + 1 : \{x = 1\} \Rightarrow \{x = 1 \wedge x \neq 0\} \qquad x = 1 \wedge x \neq 0 \Rightarrow x = 1}$$

$$\overline{\vdash \texttt{while}\ x = 0\ \texttt{do}\ x := x + 1 : \{x = 1\} \Rightarrow \{x = 1\}}$$

An example

$$
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\qquad x = 1 \land x \neq 0 \Rightarrow x = 1
}
{\vdash \mathtt{while}\ x = 0\ \mathtt{do}\ x := x + 1 : \{x = 1\} \Rightarrow \{x = 1\}}
$$

Another example

$\vdash$

```
x:=3;
y:=1;
while x > 1 do
  y := y+1;
  x := x-1;
```

$: \{true\} \Rightarrow \{y = 3\}$

How can we derive this?

Another example

$\vdash$

```
x:=3;
y:=1;
while x > 1 do
  y := y+1;
  x := x-1;
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$: \{true\} \Rightarrow \{y = 3\}$

What can be a good Invariant?

Another example

$\vdash$

```
x:=3;
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while x > 1 do
  y := y+1;
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```

$: \{true\} \Rightarrow \{y = 3\}$

$\boxed{\text{What can be a good Invariant?}}$

$$\mathtt{Inv} = \{y = 4 - x \wedge x \geq 1\}$$

How do we know that these are the right rules?

Soundness

If we can derive $\vdash \texttt{c} : \texttt{P} \Rightarrow \texttt{Q}$ through the rules of the logic, then the triple $\texttt{c} : \texttt{P} \Rightarrow \texttt{Q}$ is valid.

Are the rules we presented sound?

Completeness

If a triple $\mathtt{c} : \mathtt{P} \Rightarrow \mathtt{Q}$ is valid, then we can derive $\vdash \mathtt{c} : \mathtt{P} \Rightarrow \mathtt{Q}$ through the rules of the logic.

Are the rules we presented complete?

Relative Completeness

$$\frac{\mathtt{P} \Rightarrow \mathtt{S} \qquad \vdash \mathtt{c} : \mathtt{S} \Rightarrow \mathtt{R} \qquad \mathtt{R} \Rightarrow \mathtt{Q}}{\vdash \mathtt{c} : \mathtt{P} \Rightarrow \mathtt{Q}}$$

Relative Completeness

$$\frac{P \Rightarrow S \qquad \vdash c : S \Rightarrow R \qquad R \Rightarrow Q}{\vdash c : \; P \Rightarrow Q}$$

If a triple $c : Pre \Rightarrow Post$ is valid, and we have an oracle to derive all the true statements of the form $P \Rightarrow S$ and of the form $R \Rightarrow Q$,which we can use in applications of the conseq rule, then we can derive $\vdash c : Pre \Rightarrow Post$ through the rules of the logic.